

# lyryx with Open Texts

# LINEAR ALGEBRA with Applications

## Open Edition



**ADAPTABLE | ACCESSIBLE | AFFORDABLE**

Adapted for

**Emory University**

**Math 221**

**Linear Algebra**

Sections 1 & 2

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April 15, 2021

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Course page

[http://math.emory.edu/~lchen41/teaching/2021\\_Spring\\_Math221](http://math.emory.edu/~lchen41/teaching/2021_Spring_Math221)

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## Supplementary Exercises for Chapter 4

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**Exercise 4.1** Suppose that  $\mathbf{u}$  and  $\mathbf{v}$  are nonzero vectors. If  $\mathbf{u}$  and  $\mathbf{v}$  are not parallel, and  $a\mathbf{u} + b\mathbf{v} = a_1\mathbf{u} + b_1\mathbf{v}$ , show that  $a = a_1$  and  $b = b_1$ .

**Exercise 4.2** Consider a triangle with vertices  $A$ ,  $B$ , and  $C$ . Let  $E$  and  $F$  be the midpoints of sides  $AB$  and  $AC$ , respectively, and let the medians  $EC$  and  $FB$  meet at  $O$ . Write  $\vec{EO} = s\vec{EC}$  and  $\vec{FO} = t\vec{FB}$ , where  $s$  and  $t$  are scalars. Show that  $s = t = \frac{1}{3}$  by expressing  $\vec{AO}$  two ways in the form  $a\vec{EO} + b\vec{AC}$ , and applying Exercise 4.1. Conclude that the medians of a triangle meet at the point on each that is one-third of the way from the midpoint to the vertex (and so are concurrent).

**Exercise 4.3** A river flows at 1 km/h and a swimmer moves at 2 km/h (relative to the water). At what angle must he swim to go straight across? What is his resulting speed?

**Exercise 4.4** A wind is blowing from the south at 75 knots, and an airplane flies heading east at 100 knots. Find the resulting velocity of the airplane.

125 knots in a direction  $\theta$  degrees east of north, where  $\cos \theta = 0.6$  ( $\theta = 53^\circ$  or 0.93 radians).

**Exercise 4.5** An airplane pilot flies at 300 km/h in a direction  $30^\circ$  south of east. The wind is blowing from the south at 150 km/h.

- Find the resulting direction and speed of the airplane.
- Find the speed of the airplane if the wind is from the west (at 150 km/h).

**Exercise 4.6** A rescue boat has a top speed of 13 knots. The captain wants to go due east as fast as possible in water with a current of 5 knots due south. Find the velocity vector  $\mathbf{v} = (x, y)$  that she must achieve, assuming the  $x$  and  $y$  axes point east and north, respectively, and find her resulting speed.

(12, 5). Actual speed 12 knots.

**Exercise 4.7** A boat goes 12 knots heading north. The current is 5 knots from the west. In what direction does the boat actually move and at what speed?

**Exercise 4.8** Show that the distance from a point  $A$  (with vector  $\mathbf{a}$ ) to the plane with vector equation  $\mathbf{n} \cdot \mathbf{p} = d$  is  $\frac{1}{\|\mathbf{n}\|} |\mathbf{n} \cdot \mathbf{a} - d|$ .

**Exercise 4.9** If two distinct points lie in a plane, show that the line through these points is contained in the plane.

**Exercise 4.10** The line through a vertex of a triangle, perpendicular to the opposite side, is called an **altitude** of the triangle. Show that the three altitudes of any triangle are concurrent. (The intersection of the altitudes is called the **orthocentre** of the triangle.) [*Hint*: If  $P$  is the intersection of two of the altitudes, show that the line through  $P$  and the remaining vertex is perpendicular to the remaining side.]

