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LINEAR ALGEBRA
with Applications

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Adapted for

Emory University

Math 221
Linear Algebra
Sections **1** & **2**
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Contents

5.2 Independence and Dimension

Some spanning sets are better than others. If $U = \text{span}\{\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_k\}$ is a subspace of $\mathbb{R}^n$, then every vector in U can be written as a linear combination of the $\mathbf{x}_i$ in at least one way. Our interest here is in spanning sets where each vector in U has a *exactly one* representation as a linear combination of these vectors.

Linear Independence

Given $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_k$ in $\mathbb{R}^n$, suppose that two linear combinations are equal:

$$r_1\mathbf{x}_1 + r_2\mathbf{x}_2 + \cdots + r_k\mathbf{x}_k = s_1\mathbf{x}_1 + s_2\mathbf{x}_2 + \cdots + s_k\mathbf{x}_k$$

We are looking for a condition on the set $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_k\}$ of vectors that guarantees that this representation is *unique*; that is, $r_i = s_i$ for each i. Taking all terms to the left side gives

$$(r_1 - s_1)\mathbf{x}_1 + (r_2 - s_2)\mathbf{x}_2 + \cdots + (r_k - s_k)\mathbf{x}_k = \mathbf{0}$$

so the required condition is that this equation forces all the coefficients $r_i - s_i$ to be zero.

> ### Definition 5.3 Linear Independence in $\mathbb{R}^n$
>
> *With this in mind, we call a set $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_k\}$ of vectors **linearly independent** (or simply **independent**) if it satisfies the following condition:*
>
> $$\text{If } t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k = \mathbf{0} \text{ then } t_1 = t_2 = \cdots = t_k = 0$$

We record the result of the above discussion for reference.

> ### Theorem 5.2.1
>
> *If $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_k\}$ is an independent set of vectors in $\mathbb{R}^n$, then every vector in $\text{span}\{\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_k\}$ has a **unique** representation as a linear combination of the $\mathbf{x}_i$.*

It is useful to state the definition of independence in different language. Let us say that a linear combination **vanishes** if it equals the zero vector, and call a linear combination **trivial** if every coefficient is zero. Then the definition of independence can be compactly stated as follows:

> A set of vectors is independent if and only if the only linear combination that vanishes is the trivial one.

Hence we have a procedure for checking that a set of vectors is independent:

Independence Test

To verify that a set $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_k\}$ of vectors in $\mathbb{R}^n$ is independent, proceed as follows:

1. *Set a linear combination equal to zero: $t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k = \mathbf{0}$.*

2. *Show that $t_i = 0$ for each i (that is, the linear combination is trivial).*

Of course, if some nontrivial linear combination vanishes, the vectors are not independent.

Example 5.2.1

Determine whether $\{(1,\ 0,\ -2,\ 5),\ (2,\ 1,\ 0,\ -1),\ (1,\ 1,\ 2,\ 1)\}$ is independent in $\mathbb{R}^4$.

Solution. Suppose a linear combination vanishes:

$$r(1,\ 0,\ -2,\ 5) + s(2,\ 1,\ 0,\ -1) + t(1,\ 1,\ 2,\ 1) = (0,\ 0,\ 0,\ 0)$$

Equating corresponding entries gives a system of four equations:

$$r + 2s + t = 0,\ \ s + t = 0,\ \ -2r + 2t = 0,\ \ \text{and } 5r - s + t = 0$$

The only solution is the trivial one $r = s = t = 0$ (verify), so these vectors are independent by the independence test.

Example 5.2.2

Show that the standard basis $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \ldots,\ \mathbf{e}_n\}$ of $\mathbb{R}^n$ is independent.

Solution. The components of $t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + \cdots + t_n\mathbf{e}_n$ are $t_1,\ t_2,\ \ldots,\ t_n$ (see the discussion preceding Example 5.1.6) So the linear combination vanishes if and only if each $t_i = 0$. Hence the independence test applies.

Example 5.2.3

If $\{\mathbf{x},\ \mathbf{y}\}$ is independent, show that $\{2\mathbf{x} + 3\mathbf{y},\ \mathbf{x} - 5\mathbf{y}\}$ is also independent.

Solution. If $s(2\mathbf{x} + 3\mathbf{y}) + t(\mathbf{x} - 5\mathbf{y}) = \mathbf{0}$, collect terms to get $(2s + t)\mathbf{x} + (3s - 5t)\mathbf{y} = \mathbf{0}$. Since $\{\mathbf{x},\ \mathbf{y}\}$ is independent this combination must be trivial; that is, $2s + t = 0$ and $3s - 5t = 0$. These equations have only the trivial solution $s = t = 0$, as required.

Example 5.2.4

Show that the zero vector in $\mathbb{R}^n$ does not belong to any independent set.

Solution. No set $\{\mathbf{0}, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ of vectors is independent because we have a vanishing, nontrivial linear combination $1 \cdot \mathbf{0} + 0\mathbf{x}_1 + 0\mathbf{x}_2 + \cdots + 0\mathbf{x}_k = \mathbf{0}$.

Example 5.2.5

Given $\mathbf{x}$ in $\mathbb{R}^n$, show that $\{\mathbf{x}\}$ is independent if and only if $\mathbf{x} \neq \mathbf{0}$.

Solution. A vanishing linear combination from $\{\mathbf{x}\}$ takes the form $t\mathbf{x} = \mathbf{0}$, t in $\mathbb{R}$. This implies that $t = 0$ because $\mathbf{x} \neq \mathbf{0}$.

The next example will be needed later.

Example 5.2.6

Show that the nonzero rows of a row-echelon matrix R are independent.

Solution. We illustrate the case with 3 leading 1s; the general case is analogous. Suppose R has the form $R = \begin{bmatrix} 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ where $*$ indicates a nonspecified number. Let R_1, R_2, and R_3 denote the nonzero rows of R. If $t_1 R_1 + t_2 R_2 + t_3 R_3 = 0$ we show that $t_1 = 0$, then $t_2 = 0$, and finally $t_3 = 0$. The condition $t_1 R_1 + t_2 R_2 + t_3 R_3 = 0$ becomes

$$(0, \, t_1, \, *, \, *, \, *, \, *) + (0, \, 0, \, 0, \, t_2, \, *, \, *) + (0, \, 0, \, 0, \, 0, \, t_3, \, *) = (0, \, 0, \, 0, \, 0, \, 0, \, 0)$$

Equating second entries show that $t_1 = 0$, so the condition becomes $t_2 R_2 + t_3 R_3 = 0$. Now the same argument shows that $t_2 = 0$. Finally, this gives $t_3 R_3 = 0$ and we obtain $t_3 = 0$.

A set of vectors in $\mathbb{R}^n$ is called **linearly dependent** (or simply **dependent**) if it is *not* linearly independent, equivalently if some nontrivial linear combination vanishes.

Example 5.2.7

If $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors in $\mathbb{R}^3$, show that $\{\mathbf{v}, \mathbf{w}\}$ is dependent if and only if $\mathbf{v}$ and $\mathbf{w}$ are parallel.

Solution. If $\mathbf{v}$ and $\mathbf{w}$ are parallel, then one is a scalar multiple of the other (Theorem 4.1.4), say $\mathbf{v} = a\mathbf{w}$ for some scalar a. Then the nontrivial linear combination $\mathbf{v} - a\mathbf{w} = \mathbf{0}$ vanishes, so $\{\mathbf{v}, \mathbf{w}\}$ is dependent.
Conversely, if $\{\mathbf{v}, \mathbf{w}\}$ is dependent, let $s\mathbf{v} + t\mathbf{w} = \mathbf{0}$ be nontrivial, say $s \neq 0$. Then $\mathbf{v} = -\frac{t}{s}\mathbf{w}$ so $\mathbf{v}$ and $\mathbf{w}$ are parallel (by Theorem 4.1.4). A similar argument works if $t \neq 0$.

With this we can give a geometric description of what it means for a set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ in $\mathbb{R}^3$ to be independent. Note that this requirement means that $\{\mathbf{v}, \mathbf{w}\}$ is also independent ($a\mathbf{v} + b\mathbf{w} = \mathbf{0}$

means that $0\mathbf{u} + a\mathbf{v} + b\mathbf{w} = \mathbf{0}$), so $M = \text{span}\{\mathbf{v}, \mathbf{w}\}$ is the plane containing $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{0}$ (see the discussion preceding Example 5.1.4). So we assume that $\{\mathbf{v}, \mathbf{w}\}$ is independent in the following example.

Example 5.2.8

$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ independent

$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ not independent
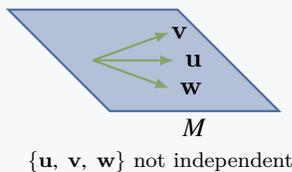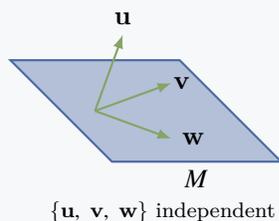
Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be nonzero vectors in $\mathbb{R}^3$ where $\{\mathbf{v}, \mathbf{w}\}$ independent. Show that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is independent if and only if $\mathbf{u}$ is not in the plane $M = \text{span}\{\mathbf{v}, \mathbf{w}\}$. This is illustrated in the diagrams.

Solution. If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is independent, suppose $\mathbf{u}$ is in the plane $M = \text{span}\{\mathbf{v}, \mathbf{w}\}$, say $\mathbf{u} = a\mathbf{v} + b\mathbf{w}$, where a and b are in $\mathbb{R}$. Then $1\mathbf{u} - a\mathbf{v} - b\mathbf{w} = \mathbf{0}$, contradicting the independence of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

On the other hand, suppose that $\mathbf{u}$ is not in M; we must show that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is independent. If $r\mathbf{u} + s\mathbf{v} + t\mathbf{w} = \mathbf{0}$ where r, s, and t are in $\mathbb{R}^3$, then $r = 0$ since otherwise $\mathbf{u} = -\frac{s}{r}\mathbf{v} + \frac{-t}{r}\mathbf{w}$ is in M. But then $s\mathbf{v} + t\mathbf{w} = \mathbf{0}$, so $s = t = 0$ by our assumption.

This shows that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is independent, as required.

By the inverse theorem, the following conditions are equivalent for an $n \times n$ matrix A:

1. A *is invertible.*

2. *If* $A\mathbf{x} = \mathbf{0}$ *where* $\mathbf{x}$ *is in* $\mathbb{R}^n$, *then* $\mathbf{x} = \mathbf{0}$.

3. $A\mathbf{x} = \mathbf{b}$ *has a solution* $\mathbf{x}$ *for every vector* $\mathbf{b}$ *in* $\mathbb{R}^n$.

While condition 1 makes no sense if A is not square, conditions 2 and 3 are meaningful for any matrix A and, in fact, are related to independence and spanning. Indeed, if $\mathbf{c}_1$, $\mathbf{c}_2$, ..., $\mathbf{c}_n$ are the columns of A, and if we write $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then

$$A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n$$

by Definition 2.5. Hence the definitions of independence and spanning show, respectively, that condition 2 is equivalent to the independence of $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ and condition 3 is equivalent to the requirement that $\text{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\} = \mathbb{R}^m$. This discussion is summarized in the following theorem:

Theorem 5.2.2

If A is an $m \times n$ matrix, let $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ denote the columns of A.

1. $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ *is independent in* $\mathbb{R}^m$ *if and only if* $A\mathbf{x} = \mathbf{0}$, $\mathbf{x}$ *in* $\mathbb{R}^n$, *implies* $\mathbf{x} = \mathbf{0}$.

> 2. $\mathbb{R}^m = \text{span}\{c_1, c_2, \ldots, c_n\}$ if and only if $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ for every vector $\mathbf{b}$ in $\mathbb{R}^m$.

For a *square* matrix A, Theorem 5.2.2 characterizes the invertibility of A in terms of the spanning and independence of its columns (see the discussion preceding Theorem 5.2.2). It is important to be able to discuss these notions for *rows*. If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are $1 \times n$ rows, we define $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ to be the set of all linear combinations of the $\mathbf{x}_i$ (as matrices), and we say that $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is linearly independent if the only vanishing linear combination is the trivial one (that is, if $\{\mathbf{x}_1^T, \mathbf{x}_2^T, \ldots, \mathbf{x}_k^T\}$ is independent in $\mathbb{R}^n$, as the reader can verify).[6]

Theorem 5.2.3

The following are equivalent for an $n \times n$ matrix A:

1. *A is invertible.*

2. *The columns of A are linearly independent.*

3. *The columns of A span $\mathbb{R}^n$.*

4. *The rows of A are linearly independent.*

5. *The rows of A span the set of all $1 \times n$ rows.*

Proof. Let $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ denote the columns of A.

$(1) \Leftrightarrow (2)$. By Theorem 2.4.5, A is invertible if and only if $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$; this holds if and only if $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ is independent by Theorem 5.2.2.

$(1) \Leftrightarrow (3)$. Again by Theorem 2.4.5, A is invertible if and only if $A\mathbf{x} = \mathbf{b}$ has a solution for every column B in $\mathbb{R}^n$; this holds if and only if $\text{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\} = \mathbb{R}^n$ by Theorem 5.2.2.

$(1) \Leftrightarrow (4)$. The matrix A is invertible if and only if A^T is invertible (by Corollary 2.4.1 to Theorem 2.4.4); this in turn holds if and only if A^T has independent columns (by $(1) \Leftrightarrow (2)$); finally, this last statement holds if and only if A has independent rows (because the rows of A are the transposes of the columns of A^T).

$(1) \Leftrightarrow (5)$. The proof is similar to $(1) \Leftrightarrow (4)$. □

Example 5.2.9

Show that $S = \{(2, -2, 5), (-3, 1, 1), (2, 7, -4)\}$ is independent in $\mathbb{R}^3$.

Solution. Consider the matrix $A = \begin{bmatrix} 2 & -2 & 5 \\ -3 & 1 & 1 \\ 2 & 7 & -4 \end{bmatrix}$ with the vectors in S as its rows. A routine computation shows that $\det A = -117 \neq 0$, so A is invertible. Hence S is independent by Theorem 5.2.3. Note that Theorem 5.2.3 also shows that $\mathbb{R}^3 = \text{span } S$.

---

[6] It is best to view columns and rows as just two different *notations* for ordered n-tuples. This discussion will become redundant in Chapter 6 where we define the general notion of a vector space.

Dimension

It is common geometrical language to say that $\mathbb{R}^3$ is 3-dimensional, that planes are 2-dimensional and that lines are 1-dimensional. The next theorem is a basic tool for clarifying this idea of "dimension". Its importance is difficult to exaggerate.

> **Theorem 5.2.4: Fundamental Theorem**
>
> Let U be a subspace of $\mathbb{R}^n$. If U is spanned by m vectors, and if U contains k linearly independent vectors, then $k \leq m$.

This proof is given in Theorem 6.3.2 in much greater generality.

> **Definition 5.4 Basis of $\mathbb{R}^n$**
>
> If U is a subspace of $\mathbb{R}^n$, a set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ of vectors in U is called a **basis** of U if it satisfies the following two conditions:
>
> 1. $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ is linearly independent.
>
> 2. $U = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$.

The most remarkable result about bases[7] is:

> **Theorem 5.2.5: Invariance Theorem**
>
> If $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k\}$ are bases of a subspace U of $\mathbb{R}^n$, then $m = k$.

Proof. We have $k \leq m$ by the fundamental theorem because $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ spans U, and $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k\}$ is independent. Similarly, by interchanging $\mathbf{x}$'s and $\mathbf{y}$'s we get $m \leq k$. Hence $m = k$. □

The invariance theorem guarantees that there is no ambiguity in the following definition:

> **Definition 5.5 Dimension of a Subspace of $\mathbb{R}^n$**
>
> If U is a subspace of $\mathbb{R}^n$ and $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ is any basis of U, the number, m, of vectors in the basis is called the **dimension** of U, denoted
>
> $$\dim U = m$$

The importance of the invariance theorem is that the dimension of U can be determined by counting the number of vectors in *any* basis.[8]

---

[7] The plural of "basis" is "bases".

[8] We will show in Theorem 5.2.6 that every subspace of $\mathbb{R}^n$ does indeed *have* a basis.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ denote the standard basis of $\mathbb{R}^n$, that is the set of columns of the identity matrix. Then $\mathbb{R}^n = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ by Example 5.1.6, and $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is independent by Example 5.2.2. Hence it is indeed a basis of $\mathbb{R}^n$ in the present terminology, and we have

---

Example 5.2.10

$\dim(\mathbb{R}^n) = n$ and $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis.

This agrees with our geometric sense that $\mathbb{R}^2$ is two-dimensional and $\mathbb{R}^3$ is three-dimensional. It also says that $\mathbb{R}^1 = \mathbb{R}$ is one-dimensional, and $\{1\}$ is a basis. Returning to subspaces of $\mathbb{R}^n$, we define

$$\dim\{\mathbf{0}\} = 0$$

This amounts to saying $\{\mathbf{0}\}$ has a basis containing *no* vectors. This makes sense because $\mathbf{0}$ cannot belong to *any* independent set (Example 5.2.4).

Example 5.2.11

Let $U = \left\{ \begin{bmatrix} r \\ s \\ r \end{bmatrix} \mid r, s \text{ in } \mathbb{R} \right\}$. Show that U is a subspace of $\mathbb{R}^3$, find a basis, and calculate $\dim U$.

Solution. Clearly, $\begin{bmatrix} r \\ s \\ r \end{bmatrix} = r\mathbf{u} + s\mathbf{v}$ where $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. It follows that $U = \text{span}\{\mathbf{u}, \mathbf{v}\}$, and hence that U is a subspace of $\mathbb{R}^3$. Moreover, if $r\mathbf{u} + s\mathbf{v} = \mathbf{0}$, then $\begin{bmatrix} r \\ s \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so $r = s = 0$. Hence $\{\mathbf{u}, \mathbf{v}\}$ is independent, and so a **basis** of U. This means $\dim U = 2$.

Example 5.2.12

Let $B = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be a basis of $\mathbb{R}^n$. If A is an invertible $n \times n$ matrix, then $D = \{A\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_n\}$ is also a basis of $\mathbb{R}^n$.

Solution. Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. Then $A^{-1}\mathbf{x}$ is in $\mathbb{R}^n$ so, since B is a basis, we have $A^{-1}\mathbf{x} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_n\mathbf{x}_n$ for t_i in $\mathbb{R}$. Left multiplication by A gives $\mathbf{x} = t_1(A\mathbf{x}_1) + t_2(A\mathbf{x}_2) + \cdots + t_n(A\mathbf{x}_n)$, and it follows that D spans $\mathbb{R}^n$. To show independence, let $s_1(A\mathbf{x}_1) + s_2(A\mathbf{x}_2) + \cdots + s_n(A\mathbf{x}_n) = \mathbf{0}$, where the s_i are in $\mathbb{R}$. Then $A(s_1\mathbf{x}_1 + s_2\mathbf{x}_2 + \cdots + s_n\mathbf{x}_n) = \mathbf{0}$ so left multiplication by A^{-1} gives $s_1\mathbf{x}_1 + s_2\mathbf{x}_2 + \cdots + s_n\mathbf{x}_n = \mathbf{0}$. Now the independence of B shows that each $s_i = 0$, and so proves the independence of D. Hence D is a basis of $\mathbb{R}^n$.

While we have found bases in many subspaces of $\mathbb{R}^n$, we have not yet shown that *every* subspace *has* a basis. This is part of the next theorem, the proof of which is deferred to Section 6.4 (Theorem 6.4.1) where it will be proved in more generality.

Theorem 5.2.6

Let $U \neq \{\mathbf{0}\}$ be a subspace of $\mathbb{R}^n$. Then:

1. U has a basis and $\dim U \leq n$.

2. Any independent set in U can be enlarged (by adding vectors from the standard basis) to a basis of U.

3. Any spanning set for U can be cut down (by deleting vectors) to a basis of U.

Example 5.2.13

Find a basis of $\mathbb{R}^4$ containing $S = \{\mathbf{u}, \mathbf{v}\}$ where $\mathbf{u} = (0, 1, 2, 3)$ and $\mathbf{v} = (2, -1, 0, 1)$.

Solution. By Theorem 5.2.6 we can find such a basis by adding vectors from the standard basis of $\mathbb{R}^4$ to S. If we try $\mathbf{e}_1 = (1, 0, 0, 0)$, we find easily that $\{\mathbf{e}_1, \mathbf{u}, \mathbf{v}\}$ is independent. Now add another vector from the standard basis, say $\mathbf{e}_2$.
Again we find that $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{u}, \mathbf{v}\}$ is independent. Since B has $4 = \dim \mathbb{R}^4$ vectors, then B must span $\mathbb{R}^4$ by Theorem 5.2.7 below (or simply verify it directly). Hence B is a basis of $\mathbb{R}^4$.

Theorem 5.2.6 has a number of useful consequences. Here is the first.

Theorem 5.2.7

Let U be a subspace of $\mathbb{R}^n$ where $\dim U = m$ and let $B = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ be a set of m vectors in U. Then B is independent if and only if B spans U.

Proof. Suppose B is independent. If B does not span U then, by Theorem 5.2.6, B can be enlarged to a basis of U containing more than m vectors. This contradicts the invariance theorem because $\dim U = m$, so B spans U. Conversely, if B spans U but is not independent, then B can be cut down to a basis of U containing fewer than m vectors, again a contradiction. So B is independent, as required. ☐

As we saw in Example 5.2.13, Theorem 5.2.7 is a "labour-saving" result. It asserts that, given a subspace U of dimension m and a set B of exactly m vectors in U, to prove that B is a basis of U it suffices to show either that B spans U or that B is independent. It is not necessary to verify both properties.

Theorem 5.2.8

Let $U \subseteq W$ be subspaces of $\mathbb{R}^n$. Then:

1. $\dim U \leq \dim W$.

2. If $\dim U = \dim W$, then $U = W$.

Proof. Write $\dim W = k$, and let B be a basis of U.

1. If $\dim U > k$, then B is an independent set in W containing more than k vectors, contradicting the fundamental theorem. So $\dim U \leq k = \dim W$.

2. If $\dim U = k$, then B is an independent set in W containing $k = \dim W$ vectors, so B spans W by Theorem 5.2.7. Hence $W = \text{span } B = U$, proving (2). ☐

It follows from Theorem 5.2.8 that if U is a subspace of $\mathbb{R}^n$, then $\dim U$ is one of the integers 0, 1, 2, …, n, and that:
$$\dim U = 0 \quad \text{if and only if} \quad U = \{\mathbf{0}\},$$
$$\dim U = n \quad \text{if and only if} \quad U = \mathbb{R}^n$$

The other subspaces of $\mathbb{R}^n$ are called **proper**. The following example uses Theorem 5.2.8 to show that the proper subspaces of $\mathbb{R}^2$ are the lines through the origin, while the proper subspaces of $\mathbb{R}^3$ are the lines and planes through the origin.

Example 5.2.14

1. If U is a subspace of $\mathbb{R}^2$ or $\mathbb{R}^3$, then $\dim U = 1$ if and only if U is a line through the origin.

2. If U is a subspace of $\mathbb{R}^3$, then $\dim U = 2$ if and only if U is a plane through the origin.

Proof.

1. Since $\dim U = 1$, let $\{\mathbf{u}\}$ be a basis of U. Then $U = \text{span}\,\{\mathbf{u}\} = \{t\mathbf{u} \,|\, t \text{ in } \mathbb{R}\}$, so U is the line through the origin with direction vector $\mathbf{u}$. Conversely each line L with direction vector $\mathbf{d} \neq \mathbf{0}$ has the form $L = \{t\mathbf{d} \,|\, t \text{ in } \mathbb{R}\}$. Hence $\{\mathbf{d}\}$ is a basis of U, so U has dimension 1.

2. If $U \subseteq \mathbb{R}^3$ has dimension 2, let $\{\mathbf{v}, \mathbf{w}\}$ be a basis of U. Then $\mathbf{v}$ and $\mathbf{w}$ are not parallel (by Example 5.2.7) so $\mathbf{n} = \mathbf{v} \times \mathbf{w} \neq \mathbf{0}$. Let $P = \{\mathbf{x} \text{ in } \mathbb{R}^3 \,|\, \mathbf{n} \cdot \mathbf{x} = 0\}$ denote the plane through the origin with normal $\mathbf{n}$. Then P is a subspace of $\mathbb{R}^3$ (Example 5.1.1) and both $\mathbf{v}$ and $\mathbf{w}$ lie in P (they are orthogonal to $\mathbf{n}$), so $U = \text{span}\,\{\mathbf{v}, \mathbf{w}\} \subseteq P$ by Theorem 5.1.1. Hence
$$U \subseteq P \subseteq \mathbb{R}^3$$

Since $\dim U = 2$ and $\dim (\mathbb{R}^3) = 3$, it follows from Theorem 5.2.8 that $\dim P = 2$ or 3, whence $P = U$ or $\mathbb{R}^3$. But $P \neq \mathbb{R}^3$ (for example, $\mathbf{n}$ is not in P) and so $U = P$ is a plane through the origin.

Conversely, if U is a plane through the origin, then $\dim U = 0$, 1, 2, or 3 by Theorem 5.2.8. But $\dim U \neq 0$ or 3 because $U \neq \{\mathbf{0}\}$ and $U \neq \mathbb{R}^3$, and $\dim U \neq 1$ by (1). So $\dim U = 2$. ☐

Note that this proof shows that if $\mathbf{v}$ and $\mathbf{w}$ are nonzero, nonparallel vectors in $\mathbb{R}^3$, then $\text{span}\,\{\mathbf{v}, \mathbf{w}\}$ is the plane with normal $\mathbf{n} = \mathbf{v} \times \mathbf{w}$. We gave a geometrical verification of this fact in Section 5.1.

Exercises for 5.2

In Exercises 5.2.1-5.2.6 we write vectors $\mathbb{R}^n$ as rows.

Exercise 5.2.1 Which of the following subsets are independent? Support your answer.

a. $\{(1, -1, 0), (3, 2, -1), (3, 5, -2)\}$ in $\mathbb{R}^3$

b. $\{(1, 1, 1), (1, -1, 1), (0, 0, 1)\}$ in $\mathbb{R}^3$

c. $\{(1, -1, 1, -1), (2, 0, 1, 0), (0, -2, 1, -2)\}$ in $\mathbb{R}^4$

d. $\{(1, 1, 0, 0), (1, 0, 1, 0), (0, 0, 1, 1), (0, 1, 0, 1)\}$ in $\mathbb{R}^4$

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b. Yes. If $r\begin{bmatrix}1\\1\\1\end{bmatrix} + s\begin{bmatrix}1\\1\\1\end{bmatrix} + t\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$,
then $r+s = 0$, $r-s = 0$, and $r+s+t = 0$. These equations give $r = s = t = 0$.

d. No. Indeed: $\begin{bmatrix}1\\1\\0\\0\end{bmatrix} - \begin{bmatrix}1\\0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \begin{bmatrix}0\\1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$.

Exercise 5.2.2 Let $\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}\}$ be an independent set in $\mathbb{R}^n$. Which of the following sets is independent? Support your answer.

a. $\{\mathbf{x} - \mathbf{y}, \mathbf{y} - \mathbf{z}, \mathbf{z} - \mathbf{x}\}$

b. $\{\mathbf{x} + \mathbf{y}, \mathbf{y} + \mathbf{z}, \mathbf{z} + \mathbf{x}\}$

c. $\{\mathbf{x} - \mathbf{y}, \mathbf{y} - \mathbf{z}, \mathbf{z} - \mathbf{w}, \mathbf{w} - \mathbf{x}\}$

d. $\{\mathbf{x} + \mathbf{y}, \mathbf{y} + \mathbf{z}, \mathbf{z} + \mathbf{w}, \mathbf{w} + \mathbf{x}\}$

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b. Yes. If $r(\mathbf{x} + \mathbf{y}) + s(\mathbf{y} + \mathbf{z}) + t(\mathbf{z} + \mathbf{x}) = \mathbf{0}$, then $(r+t)\mathbf{x} + (r+s)\mathbf{y} + (s+t)\mathbf{z} = \mathbf{0}$. Since $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is independent, this implies that $r + t = 0$, $r + s = 0$, and $s + t = 0$. The only solution is $r = s = t = 0$.

d. No. In fact, $(\mathbf{x} + \mathbf{y}) - (\mathbf{y} + \mathbf{z}) + (\mathbf{z} + \mathbf{w}) - (\mathbf{w} + \mathbf{x}) = \mathbf{0}$.

Exercise 5.2.3 Find a basis and calculate the dimension of the following subspaces of $\mathbb{R}^4$.

a. span $\{(1, -1, 2, 0), (2, 3, 0, 3), (1, 9, -6, 6)\}$

b. span $\{(2, 1, 0, -1), (-1, 1, 1, 1), (2, 7, 4, 1)\}$

c. span $\{(-1, 2, 1, 0), (2, 0, 3, -1), (4, 4, 11, -3), (3, -2, 2, -1)\}$

d. span $\{(-2, 0, 3, 1), (1, 2, -1, 0), (-2, 8, 5, 3), (-1, 2, 2, 1)\}$

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b. $\left\{ \begin{bmatrix}2\\1\\0\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\\1\end{bmatrix} \right\}$; dimension 2.

d. $\left\{ \begin{bmatrix}-2\\0\\3\\1\end{bmatrix}, \begin{bmatrix}1\\2\\-1\\0\end{bmatrix} \right\}$; dimension 2.

Exercise 5.2.4 Find a basis and calculate the dimension of the following subspaces of $\mathbb{R}^4$.

a. $U = \left\{ \begin{bmatrix} a \\ a+b \\ a-b \\ b \end{bmatrix} \middle| a \text{ and } b \text{ in } \mathbb{R} \right\}$

b. $U = \left\{ \begin{bmatrix} a+b \\ a-b \\ b \\ a \end{bmatrix} \middle| a \text{ and } b \text{ in } \mathbb{R} \right\}$

c. $U = \left\{ \begin{bmatrix} a \\ b \\ c+a \\ c \end{bmatrix} \middle| a,\ b,\ \text{and } c \text{ in } \mathbb{R} \right\}$

d. $U = \left\{ \begin{bmatrix} a-b \\ b+c \\ a \\ b+c \end{bmatrix} \middle| a,\ b,\ \text{and } c \text{ in } \mathbb{R} \right\}$

e. $U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \middle| a+b-c+d=0 \text{ in } \mathbb{R} \right\}$

f. $U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \middle| a+b=c+d \text{ in } \mathbb{R} \right\}$

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b. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$; dimension 2.

d. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$; dimension 3.

f. $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$; dimension 3.

Exercise 5.2.5   Suppose that $\{\mathbf{x},\ \mathbf{y},\ \mathbf{z},\ \mathbf{w}\}$ is a basis of $\mathbb{R}^4$. Show that:

a. $\{\mathbf{x}+a\mathbf{w},\ \mathbf{y},\ \mathbf{z},\ \mathbf{w}\}$ is also a basis of $\mathbb{R}^4$ for any choice of the scalar a.

b. $\{\mathbf{x}+\mathbf{w},\ \mathbf{y}+\mathbf{w},\ \mathbf{z}+\mathbf{w},\ \mathbf{w}\}$ is also a basis of $\mathbb{R}^4$.

c. $\{\mathbf{x},\ \mathbf{x}+\mathbf{y},\ \mathbf{x}+\mathbf{y}+\mathbf{z},\ \mathbf{x}+\mathbf{y}+\mathbf{z}+\mathbf{w}\}$ is also a basis of $\mathbb{R}^4$.

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b. If $r(\mathbf{x}+\mathbf{w})+s(\mathbf{y}+\mathbf{w})+t(\mathbf{z}+\mathbf{w})+u(\mathbf{w})=\mathbf{0}$, then $r\mathbf{x}+s\mathbf{y}+t\mathbf{z}+(r+s+t+u)\mathbf{w}=\mathbf{0}$, so $r=0$, $s=0$, $t=0$, and $r+s+t+u=0$. The only solution is $r=s=t=u=0$, so the set is independent. Since $\dim \mathbb{R}^4 = 4$, the set is a basis by Theorem 5.2.7.

Exercise 5.2.6   Use Theorem 5.2.3 to determine if the following sets of vectors are a basis of the indicated space.

a. $\{(3,\ -1),\ (2,\ 2)\}$ in $\mathbb{R}^2$

b. $\{(1,\ 1,\ -1),\ (1,\ -1,\ 1),\ (0,\ 0,\ 1)\}$ in $\mathbb{R}^3$

c. $\{(-1,\ 1,\ -1),\ (1,\ -1,\ 2),\ (0,\ 0,\ 1)\}$ in $\mathbb{R}^3$

d. $\{(5,\ 2,\ -1),\ (1,\ 0,\ 1),\ (3,\ -1,\ 0)\}$ in $\mathbb{R}^3$

e. $\{(2,\ 1,\ -1,\ 3),\ (1,\ 1,\ 0,\ 2),\ (0,\ 1,\ 0,\ -3),\ (-1,\ 2,\ 3,\ 1)\}$ in $\mathbb{R}^4$

f. $\{(1,\ 0,\ -2,\ 5),\ (4,\ 4,\ -3,\ 2),\ (0,\ 1,\ 0,\ -3),\ (1,\ 3,\ 3,\ -10)\}$ in $\mathbb{R}^4$

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b. Yes

d. Yes

f. No.

Exercise 5.2.7   In each case show that the statement is true or give an example showing that it is false.

a. If $\{\mathbf{x},\ \mathbf{y}\}$ is independent, then $\{\mathbf{x},\ \mathbf{y},\ \mathbf{x}+\mathbf{y}\}$ is independent.

b. If $\{\mathbf{x},\ \mathbf{y},\ \mathbf{z}\}$ is independent, then $\{\mathbf{y},\ \mathbf{z}\}$ is independent.

c. If $\{\mathbf{y},\ \mathbf{z}\}$ is dependent, then $\{\mathbf{x},\ \mathbf{y},\ \mathbf{z}\}$ is dependent for any $\mathbf{x}$.

d. If all of $\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k$ are nonzero, then $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ is independent.

e. If one of $\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k$ is zero, then $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ is dependent.

f. If $a\mathbf{x}+b\mathbf{y}+c\mathbf{z}=\mathbf{0}$, then $\{\mathbf{x},\ \mathbf{y},\ \mathbf{z}\}$ is independent.

g. If $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is independent, then $a\mathbf{x} + b\mathbf{y} + c\mathbf{z} = \mathbf{0}$ for some a, b, and c in $\mathbb{R}$.

h. If $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is dependent, then $t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k = \mathbf{0}$ for some numbers t_i in $\mathbb{R}$ not all zero.

i. If $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is independent, then $t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k = \mathbf{0}$ for some t_i in $\mathbb{R}$.

j. Every non-empty subset of a linearly independent set is again linearly independent.

k. Every set containing a spanning set is again a spanning set.

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b. T. If $r\mathbf{y} + s\mathbf{z} = \mathbf{0}$, then $0\mathbf{x} + r\mathbf{y} + s\mathbf{z} = \mathbf{0}$ so $r = s = 0$ because $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is independent.

d. F. If $\mathbf{x} \neq \mathbf{0}$, take $k = 2$, $\mathbf{x}_1 = \mathbf{x}$ and $\mathbf{x}_2 = -\mathbf{x}$.

f. F. If $\mathbf{y} = -\mathbf{x}$ and $\mathbf{z} = \mathbf{0}$, then $1\mathbf{x} + 1\mathbf{y} + 1\mathbf{z} = \mathbf{0}$.

h. T. This is a nontrivial, vanishing linear combination, so the $\mathbf{x}_i$ cannot be independent.

Exercise 5.2.8 If A is an $n \times n$ matrix, show that $\det A = 0$ if and only if some column of A is a linear combination of the other columns.

Exercise 5.2.9 Let $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ be a linearly independent set in $\mathbb{R}^4$. Show that $\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{e}_k\}$ is a basis of $\mathbb{R}^4$ for some $\mathbf{e}_k$ in the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$.

Exercise 5.2.10 If $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6\}$ is an independent set of vectors, show that the subset $\{\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_5\}$ is also independent.

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If $r\mathbf{x}_2 + s\mathbf{x}_3 + t\mathbf{x}_5 = \mathbf{0}$ then $0\mathbf{x}_1 + r\mathbf{x}_2 + s\mathbf{x}_3 + 0\mathbf{x}_4 + t\mathbf{x}_5 + 0\mathbf{x}_6 = \mathbf{0}$ so $r = s = t = 0$.

Exercise 5.2.11 Let A be any $m \times n$ matrix, and let $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \ldots, \mathbf{b}_k$ be columns in $\mathbb{R}^m$ such that the system $A\mathbf{x} = \mathbf{b}_i$ has a solution $\mathbf{x}_i$ for each i. If $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \ldots, \mathbf{b}_k\}$ is independent in $\mathbb{R}^m$, show that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_k\}$ is independent in $\mathbb{R}^n$.

Exercise 5.2.12 If $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_k\}$ is independent, show $\{\mathbf{x}_1, \mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3, \ldots, \mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k\}$ is also independent.

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If $t_1\mathbf{x}_1 + t_2(\mathbf{x}_1 + \mathbf{x}_2) + \cdots + t_k(\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k) = \mathbf{0}$, then $(t_1 + t_2 + \cdots + t_k)\mathbf{x}_1 + (t_2 + \cdots + t_k)\mathbf{x}_2 + \cdots + (t_{k-1} + t_k)\mathbf{x}_{k-1} + (t_k)\mathbf{x}_k = \mathbf{0}$. Hence all these coefficients are zero, so we obtain successively $t_k = 0$, $t_{k-1} = 0$, $\ldots$, $t_2 = 0$, $t_1 = 0$.

Exercise 5.2.13 If $\{\mathbf{y}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_k\}$ is independent, show that $\{\mathbf{y} + \mathbf{x}_1, \mathbf{y} + \mathbf{x}_2, \mathbf{y} + \mathbf{x}_3, \ldots, \mathbf{y} + \mathbf{x}_k\}$ is also independent.

Exercise 5.2.14 If $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is independent in $\mathbb{R}^n$, and if $\mathbf{y}$ is not in $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$, show that $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k, \mathbf{y}\}$ is independent.

Exercise 5.2.15 If A and B are matrices and the columns of AB are independent, show that the columns of B are independent.

Exercise 5.2.16 Suppose that $\{\mathbf{x}, \mathbf{y}\}$ is a basis of $\mathbb{R}^2$, and let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

a. If A is invertible, show that $\{a\mathbf{x} + b\mathbf{y}, c\mathbf{x} + d\mathbf{y}\}$ is a basis of $\mathbb{R}^2$.

b. If $\{a\mathbf{x} + b\mathbf{y}, c\mathbf{x} + d\mathbf{y}\}$ is a basis of $\mathbb{R}^2$, show that A is invertible.

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b. We show A^T is invertible (then A is invertible). Let $A^T\mathbf{x} = \mathbf{0}$ where $\mathbf{x} = [s \ t]^T$. This means $as + ct = 0$ and $bs + dt = 0$, so $s(a\mathbf{x} + b\mathbf{y}) + t(c\mathbf{x} + d\mathbf{y}) = (sa + tc)\mathbf{x} + (sb + td)\mathbf{y} = \mathbf{0}$. Hence $s = t = 0$ by hypothesis.

Exercise 5.2.17 Let A denote an $m \times n$ matrix.

a. Show that $\text{null } A = \text{null}(UA)$ for every invertible $m \times m$ matrix U.

b. Show that $\dim(\text{null } A) = \dim(\text{null}(AV))$ for every invertible $n \times n$ matrix V. [*Hint*: If $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is a basis of $\text{null } A$, show that $\{V^{-1}\mathbf{x}_1, V^{-1}\mathbf{x}_2, \ldots, V^{-1}\mathbf{x}_k\}$ is a basis of $\text{null}(AV)$.]

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b. Each $V^{-1}\mathbf{x}_i$ is in $\text{null}(AV)$ because $AV(V^{-1}\mathbf{x}_i) = A\mathbf{x}_i = \mathbf{0}$. The set

$\{V^{-1}\mathbf{x}_1, \ \ldots, \ V^{-1}\mathbf{x}_k\}$ is independent as V^{-1} is invertible. If $\mathbf{y}$ is in $\text{null}(AV)$, then $V\mathbf{y}$ is in $\text{null}(A)$ so let $V\mathbf{y} = t_1\mathbf{x}_1 + \cdots + t_k\mathbf{x}_k$ where each t_k is in $\mathbb{R}$. Thus $\mathbf{y} = t_1 V^{-1}\mathbf{x}_1 + \cdots + t_k V^{-1}\mathbf{x}_k$ is in $\text{span}\{V^{-1}\mathbf{x}_1, \ \ldots, \ V^{-1}\mathbf{x}_k\}$.

Exercise 5.2.18 Let A denote an $m \times n$ matrix.

a. Show that $\text{im}\,A = \text{im}\,(AV)$ for every invertible $n \times n$ matrix V.

b. Show that $\dim(\text{im}\,A) = \dim(\text{im}\,(UA))$ for every invertible $m \times m$ matrix U. [*Hint*: If $\{\mathbf{y}_1, \ \mathbf{y}_2, \ \ldots, \ \mathbf{y}_k\}$ is a basis of $\text{im}\,(UA)$, show that $\{U^{-1}\mathbf{y}_1, U^{-1}\mathbf{y}_2, \ \ldots, \ U^{-1}\mathbf{y}_k\}$ is a basis of $\text{im}\,A$.]

Exercise 5.2.19 Let U and W denote subspaces of $\mathbb{R}^n$, and assume that $U \subseteq W$. If $\dim U = n - 1$, show that either $W = U$ or $W = \mathbb{R}^n$.

Exercise 5.2.20 Let U and W denote subspaces of $\mathbb{R}^n$, and assume that $U \subseteq W$. If $\dim W = 1$, show that either $U = \{\mathbf{0}\}$ or $U = W$. _____
We have $\{\mathbf{0}\} \subseteq U \subseteq W$ where $\dim\{\mathbf{0}\} = 0$ and $\dim W = 1$. Hence $\dim U = 0$ or $\dim U = 1$ by Theorem 5.2.8, that is $U = 0$ or $U = W$, again by Theorem 5.2.8.