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LINEAR ALGEBRA with Applications

Open Edition

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Adapted for

Emory University

Math 221

Linear Algebra

Sections **1** & **2**

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April 15, 2021

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Course page

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Supplementary Exercises for Chapter 6

Exercise 6.1 (Requires calculus) Let V denote the space of all functions $f : \mathbb{R} \to \mathbb{R}$ for which the derivatives f' and f'' exist. Show that f_1, f_2, and f_3 in V are linearly independent provided that their **wronskian** $w(x)$ is nonzero for some x, where

$$w(x) = \det \begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{bmatrix}$$

Exercise 6.2 Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis of $\mathbb{R}^n$ (written as columns), and let A be an $n \times n$ matrix.

a. If A is invertible, show that $\{A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_n\}$ is a basis of $\mathbb{R}^n$.

b. If $\{A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_n\}$ is a basis of $\mathbb{R}^n$, show that A is invertible.

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b. If $YA = 0$, Y a row, we show that $Y = 0$; thus A^T (and hence A) is invertible. Given a column $\mathbf{c}$ in $\mathbb{R}^n$ write $\mathbf{c} = \sum_i r_i(A\mathbf{v}_i)$ where each r_i is in $\mathbb{R}$. Then $Y\mathbf{c} = \sum_i r_i Y A \mathbf{v}_i$, so $Y = YI_n = Y \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} Y\mathbf{e}_1 & Y\mathbf{e}_2 & \cdots & Y\mathbf{e}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix} = 0$, as required.

Exercise 6.3 If A is an $m \times n$ matrix, show that A has rank m if and only if col A contains every column of I_m.

Exercise 6.4 Show that $\text{null}\, A = \text{null}\,(A^TA)$ for any real matrix A.

We have $\text{null}\, A \subseteq \text{null}\,(A^TA)$ because $A\mathbf{x} = \mathbf{0}$ implies $(A^TA)\mathbf{x} = \mathbf{0}$. Conversely, if $(A^TA)\mathbf{x} = \mathbf{0}$, then $\|A\mathbf{x}\|^2 = (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^TA^TA\mathbf{x} = 0$. Thus $A\mathbf{x} = \mathbf{0}$.

Exercise 6.5 Let A be an $m \times n$ matrix of rank r. Show that $\dim(\text{null}\, A) = n - r$ (Theorem 5.4.3) as follows. Choose a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ of $\text{null}\, A$ and extend it to a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{z}_1, \dots, \mathbf{z}_m\}$ of $\mathbb{R}^n$. Show that $\{A\mathbf{z}_1, \dots, A\mathbf{z}_m\}$ is a basis of $\text{col}\, A$.