

# lyryx with Open Texts

# LINEAR ALGEBRA with Applications

Open Edition



ADAPTABLE | ACCESSIBLE | AFFORDABLE

Adapted for

Emory University

Math 221

Linear Algebra

Sections 1 & 2

Lectured and adapted by

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Course page

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## Supplementary Exercises for Chapter 6

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**Exercise 6.1** (Requires calculus) Let  $V$  denote the space of all functions  $f: \mathbb{R} \rightarrow \mathbb{R}$  for which the derivatives  $f'$  and  $f''$  exist. Show that  $f_1, f_2$ , and  $f_3$  in  $V$  are linearly independent provided that their **wronskian**  $w(x)$  is nonzero for some  $x$ , where

$$w(x) = \det \begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{bmatrix}$$

**Exercise 6.2** Let  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  be a basis of  $\mathbb{R}^n$  (written as columns), and let  $A$  be an  $n \times n$  matrix.

- If  $A$  is invertible, show that  $\{A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_n\}$  is a basis of  $\mathbb{R}^n$ .
- If  $\{A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_n\}$  is a basis of  $\mathbb{R}^n$ , show that  $A$  is invertible.

- 
- If  $YA = 0$ ,  $Y$  a row, we show that  $Y = 0$ ; thus  $A^T$  (and hence  $A$ ) is invertible.

Given a column  $\mathbf{c}$  in  $\mathbb{R}^n$  write  $\mathbf{c} = \sum_i r_i(A\mathbf{v}_i)$  where each  $r_i$  is in  $\mathbb{R}$ . Then  $Y\mathbf{c} = \sum_i r_i Y A \mathbf{v}_i$ , so  $Y = Y I_n = Y \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} Y\mathbf{e}_1 & Y\mathbf{e}_2 & \dots & Y\mathbf{e}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix} = 0$ , as required.

**Exercise 6.3** If  $A$  is an  $m \times n$  matrix, show that  $A$  has rank  $m$  if and only if  $\text{col } A$  contains every column of  $I_m$ .

**Exercise 6.4** Show that  $\text{null } A = \text{null } (A^T A)$  for any real matrix  $A$ .

We have  $\text{null } A \subseteq \text{null } (A^T A)$  because  $A\mathbf{x} = \mathbf{0}$  implies  $(A^T A)\mathbf{x} = \mathbf{0}$ . Conversely, if  $(A^T A)\mathbf{x} = \mathbf{0}$ , then  $\|A\mathbf{x}\|^2 = (A\mathbf{x})^T (A\mathbf{x}) = \mathbf{x}^T A^T A \mathbf{x} = 0$ . Thus  $A\mathbf{x} = \mathbf{0}$ .

**Exercise 6.5** Let  $A$  be an  $m \times n$  matrix of rank  $r$ . Show that  $\dim(\text{null } A) = n - r$  (Theorem 5.4.3) as follows. Choose a basis  $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$  of  $\text{null } A$  and extend it to a basis  $\{\mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{z}_1, \dots, \mathbf{z}_m\}$  of  $\mathbb{R}^n$ . Show that  $\{A\mathbf{z}_1, \dots, A\mathbf{z}_m\}$  is a basis of  $\text{col } A$ .

