2. Matrix Algebra

In the study of systems of linear equations in Chapter 1, we found it convenient to manipulate the augmented matrix of the system. Our aim was to reduce it to row-echelon form (using elementary row operations) and hence to write down all solutions to the system. In the present chapter we consider matrices for their own sake. While some of the motivation comes from linear equations, it turns out that matrices can be multiplied and added and so form an algebraic system somewhat analogous to the real numbers. This "matrix algebra" is useful in ways that are quite different from the study of linear equations. For example, the geometrical transformations obtained by rotating the euclidean plane about the origin can be viewed as multiplications by certain 2×2 matrices. These "matrix transformations" are an important tool in geometry and, in turn, the geometry provides a "picture" of the matrices. Furthermore, matrix algebra has many other applications, some of which will be explored in this chapter. This subject is quite old and was first studied systematically in 1858 by Arthur Cayley.[1]

2.1 Matrix Addition, Scalar Multiplication, and Transposition

A rectangular array of numbers is called a **matrix** (the plural is **matrices**), and the numbers are called the **entries** of the matrix. Matrices are usually denoted by uppercase letters: A, B, C, and so on. Hence,

$$A = \left[\begin{array}{rrr} 1 & 2 & -1 \\ 0 & 5 & 6 \end{array} \right] \quad B = \left[\begin{array}{rr} 1 & -1 \\ 0 & 2 \end{array} \right] \quad C = \left[\begin{array}{r} 1 \\ 3 \\ 2 \end{array} \right]$$

are matrices. Clearly matrices come in various shapes depending on the number of **rows** and **columns**. For example, the matrix A shown has 2 rows and 3 columns. In general, a matrix with m rows and n columns is referred to as an $\boldsymbol{m \times n}$ ***matrix*** or as having **size** $\boldsymbol{m \times n}$. Thus matrices A, B, and C above have sizes 2×3, 2×2, and 3×1, respectively. A matrix of size $1 \times n$ is called a **row matrix**, whereas one of size $m \times 1$ is called a **column matrix**. Matrices of size $n \times n$ for some n are called **square** matrices.

Each entry of a matrix is identified by the row and column in which it lies. The rows are numbered from the top down, and the columns are numbered from left to right. Then the $\boldsymbol{(i,\ j)}$**-entry** of a matrix is

[1] Arthur Cayley (1821-1895) showed his mathematical talent early and graduated from Cambridge in 1842 as senior wrangler. With no employment in mathematics in view, he took legal training and worked as a lawyer while continuing to do mathematics, publishing nearly 300 papers in fourteen years. Finally, in 1863, he accepted the Sadlerian professorship in Cambridge and remained there for the rest of his life, valued for his administrative and teaching skills as well as for his scholarship. His mathematical achievements were of the first rank. In addition to originating matrix theory and the theory of determinants, he did fundamental work in group theory, in higher-dimensional geometry, and in the theory of invariants. He was one of the most prolific mathematicians of all time and produced 966 papers.

the number lying simultaneously in row i and column j. For example,

$$\text{The } (1,\ 2)\text{-entry of } \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \text{ is } -1.$$

$$\text{The } (2,\ 3)\text{-entry of } \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 6 \end{bmatrix} \text{ is } 6.$$

A special notation is commonly used for the entries of a matrix. If A is an $m \times n$ matrix, and if the $(i,\ j)$-entry of A is denoted as a_{ij}, then A is displayed as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

This is usually denoted simply as $A = \begin{bmatrix} a_{ij} \end{bmatrix}$. Thus a_{ij} is the entry in row i and column j of A. For example, a 3×4 matrix in this notation is written

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

It is worth pointing out a convention regarding rows and columns: *Rows are mentioned before columns.* For example:

- *If a matrix has size $m \times n$, it has m rows and n columns.*
- *If we speak of the $(i,\ j)$-entry of a matrix, it lies in row i and column j.*
- *If an entry is denoted a_{ij}, the first subscript i refers to the row and the second subscript j to the column in which a_{ij} lies.*

Two points $(x_1,\ y_1)$ and $(x_2,\ y_2)$ in the plane are equal if and only if[2] they have the same coordinates, that is $x_1 = x_2$ and $y_1 = y_2$. Similarly, two matrices A and B are called **equal** (written $A = B$) if and only if:

1. *They have the same size.*
2. *Corresponding entries are equal.*

If the entries of A and B are written in the form $A = \begin{bmatrix} a_{ij} \end{bmatrix}$, $B = \begin{bmatrix} b_{ij} \end{bmatrix}$, described earlier, then the second condition takes the following form:

$$A = \begin{bmatrix} a_{ij} \end{bmatrix} = \begin{bmatrix} b_{ij} \end{bmatrix} \text{ means } a_{ij} = b_{ij} \text{ for all } i \text{ and } j$$

[2]If p and q are statements, we say that p implies q if q is true whenever p is true. Then "p if and only if q" means that both p implies q and q implies p. See Appendix B for more on this.

Example 2.1.1

Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ discuss the possibility that $A = B$, $B = C$, $A = C$.

Solution. $A = B$ is impossible because A and B are of different sizes: A is 2×2 whereas B is 2×3. Similarly, $B = C$ is impossible. But $A = C$ is possible provided that corresponding entries are equal: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ means $a = 1$, $b = 0$, $c = -1$, and $d = 2$.

Matrix Addition

Definition 2.1 Matrix Addition

If A and B are matrices of the same size, their ***sum*** *$A + B$ is the matrix formed by adding corresponding entries.*

If $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$, this takes the form

$$A + B = \left[a_{ij} + b_{ij}\right]$$

Note that addition is *not* defined for matrices of different sizes.

Example 2.1.2

If $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 6 \end{bmatrix}$, compute $A + B$.

Solution.

$$A + B = \begin{bmatrix} 2+1 & 1+1 & 3-1 \\ -1+2 & 2+0 & 0+6 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 6 \end{bmatrix}$$

Example 2.1.3

Find a, b, and c if $\begin{bmatrix} a & b & c \end{bmatrix} + \begin{bmatrix} c & a & b \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}$.

Solution. Add the matrices on the left side to obtain

$$\begin{bmatrix} a+c & b+a & c+b \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}$$

Because corresponding entries must be equal, this gives three equations: $a + c = 3$, $b + a = 2$, and $c + b = -1$. Solving these yields $a = 3$, $b = -1$, $c = 0$.

If A, B, and C are any matrices *of the same size*, then

$$A+B=B+A \qquad \text{(commutative law)}$$
$$A+(B+C)=(A+B)+C \qquad \text{(associative law)}$$

In fact, if $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$, then the $(i,\ j)$-entries of $A+B$ and $B+A$ are, respectively, $a_{ij}+b_{ij}$ and $b_{ij}+a_{ij}$. Since these are equal for all i and j, we get

$$A+B=\left[\begin{array}{c} a_{ij}+b_{ij} \end{array}\right]=\left[\begin{array}{c} b_{ij}+a_{ij} \end{array}\right]=B+A$$

The associative law is verified similarly.

The $m\times n$ matrix in which every entry is zero is called the $m\times n$ **zero matrix** and is denoted as 0 (or 0_{mn} if it is important to emphasize the size). Hence,

$$0+X=X$$

holds for all $m\times n$ matrices X. The **negative** of an $m\times n$ matrix A (written $-A$) is defined to be the $m\times n$ matrix obtained by multiplying each entry of A by -1. If $A=\left[a_{ij}\right]$, this becomes $-A=\left[-a_{ij}\right]$. Hence,

$$A+(-A)=0$$

holds for all matrices A where, of course, 0 is the zero matrix of the same size as A.

A closely related notion is that of subtracting matrices. If A and B are two $m\times n$ matrices, their **difference** $A-B$ is defined by

$$A-B=A+(-B)$$

Note that if $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$, then

$$A-B=\left[a_{ij}\right]+\left[-b_{ij}\right]=\left[a_{ij}-b_{ij}\right]$$

is the $m\times n$ matrix formed by *subtracting* corresponding entries.

Example 2.1.4

Let $A=\left[\begin{array}{rrr} 3 & -1 & 0 \\ 1 & 2 & -4 \end{array}\right]$, $B=\left[\begin{array}{rrr} 1 & -1 & 1 \\ -2 & 0 & 6 \end{array}\right]$, $C=\left[\begin{array}{rrr} 1 & 0 & -2 \\ 3 & 1 & 1 \end{array}\right]$. Compute $-A$, $A-B$, and $A+B-C$.

Solution.

$$\begin{aligned}
-A &= \left[\begin{array}{rrr} -3 & 1 & 0 \\ -1 & -2 & 4 \end{array}\right] \\
A-B &= \left[\begin{array}{ccc} 3-1 & -1-(-1) & 0-1 \\ 1-(-2) & 2-0 & -4-6 \end{array}\right]=\left[\begin{array}{rrr} 2 & 0 & -1 \\ 3 & 2 & -10 \end{array}\right] \\
A+B-C &= \left[\begin{array}{ccc} 3+1-1 & -1-1-0 & 0+1-(-2) \\ 1-2-3 & 2+0-1 & -4+6-1 \end{array}\right]=\left[\begin{array}{rrr} 3 & -2 & 3 \\ -4 & 1 & 1 \end{array}\right]
\end{aligned}$$

Example 2.1.5

Solve $\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} + X = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ where X is a matrix.

Solution. We solve a numerical equation $a + x = b$ by subtracting the number a from both sides to obtain $x = b - a$. This also works for matrices. To solve $\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} + X = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ simply subtract the matrix $\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ from both sides to get

$$X = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1-3 & 0-2 \\ -1-(-1) & 2-1 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix}$$

The reader should verify that this matrix X does indeed satisfy the original equation.

The solution in Example 2.1.5 solves the single matrix equation $A + X = B$ directly via matrix subtraction: $X = B - A$. This ability to work with matrices as entities lies at the heart of matrix algebra.

It is important to note that the sizes of matrices involved in some calculations are often determined by the context. For example, if

$$A + C = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$

then A and C must be the same size (so that $A + C$ makes sense), and that size must be 2×3 (so that the sum is 2×3). For simplicity we shall often omit reference to such facts when they are clear from the context.

Scalar Multiplication

In gaussian elimination, multiplying a row of a matrix by a number k means multiplying *every* entry of that row by k.

Definition 2.2 Matrix Scalar Multiplication

*More generally, if A is any matrix and k is any number, the **scalar multiple** kA is the matrix obtained from A by multiplying each entry of A by k.*

If $A = \left[a_{ij}\right]$, this is

$$kA = \left[ka_{ij}\right]$$

Thus $1A = A$ and $(-1)A = -A$ for any matrix A.

The term *scalar* arises here because the set of numbers from which the entries are drawn is usually referred to as the set of scalars. We have been using real numbers as scalars, but we could equally well have been using complex numbers.

Example 2.1.6

If $A = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \end{bmatrix}$ compute $5A$, $\frac{1}{2}B$, and $3A - 2B$.

Solution.

$$5A = \begin{bmatrix} 15 & -5 & 20 \\ 10 & 0 & 30 \end{bmatrix}, \quad \tfrac{1}{2}B = \begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & \frac{3}{2} & 1 \end{bmatrix}$$

$$3A - 2B = \begin{bmatrix} 9 & -3 & 12 \\ 6 & 0 & 18 \end{bmatrix} - \begin{bmatrix} 2 & 4 & -2 \\ 0 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 7 & -7 & 14 \\ 6 & -6 & 14 \end{bmatrix}$$

If A is any matrix, note that kA is the same size as A for all scalars k. We also have

$$0A = 0 \quad \text{and} \quad k0 = 0$$

because the zero matrix has every entry zero. In other words, $kA = 0$ if either $k = 0$ or $A = 0$. The converse of this statement is also true, as Example 2.1.7 shows.

Example 2.1.7

If $kA = 0$, show that either $k = 0$ or $A = 0$.

Solution. Write $A = \left[a_{ij}\right]$ so that $kA = 0$ means $ka_{ij} = 0$ for all i and j. If $k = 0$, there is nothing to do. If $k \neq 0$, then $ka_{ij} = 0$ implies that $a_{ij} = 0$ for all i and j; that is, $A = 0$.

For future reference, the basic properties of matrix addition and scalar multiplication are listed in Theorem 2.1.1.

Theorem 2.1.1

Let A, B, and C denote arbitrary $m \times n$ matrices where m and n are fixed. Let k and p denote arbitrary real numbers. Then

1. $A + B = B + A$.
2. $A + (B + C) = (A + B) + C$.
3. *There is an $m \times n$ matrix 0, such that $0 + A = A$ for each A.*
4. *For each A there is an $m \times n$ matrix, $-A$, such that $A + (-A) = 0$.*
5. $k(A + B) = kA + kB$.
6. $(k + p)A = kA + pA$.
7. $(kp)A = k(pA)$.
8. $1A = A$.

Proof. Properties 1–4 were given previously. To check Property 5, let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ denote matrices of the same size. Then $A+B = \left[a_{ij}+b_{ij}\right]$, as before, so the $(i,\ j)$-entry of $k(A+B)$ is

$$k(a_{ij}+b_{ij}) = ka_{ij}+kb_{ij}$$

But this is just the $(i,\ j)$-entry of $kA+kB$, and it follows that $k(A+B) = kA+kB$. The other Properties can be similarly verified; the details are left to the reader. □

The Properties in Theorem 2.1.1 enable us to do calculations with matrices in much the same way that numerical calculations are carried out. To begin, Property 2 implies that the sum

$$(A+B)+C = A+(B+C)$$

is the same no matter how it is formed and so is written as $A+B+C$. Similarly, the sum

$$A+B+C+D$$

is independent of how it is formed; for example, it equals both $(A+B)+(C+D)$ and $A+[B+(C+D)]$. Furthermore, property 1 ensures that, for example,

$$B+D+A+C = A+B+C+D$$

In other words, the *order* in which the matrices are added does not matter. A similar remark applies to sums of five (or more) matrices.

Properties 5 and 6 in Theorem 2.1.1 are called **distributive laws** for scalar multiplication, and they extend to sums of more than two terms. For example,

$$k(A+B-C) = kA+kB-kC$$

$$(k+p-m)A = kA+pA-mA$$

Similar observations hold for more than three summands. These facts, together with properties 7 and 8, enable us to simplify expressions by collecting like terms, expanding, and taking common factors in exactly the same way that algebraic expressions involving variables and real numbers are manipulated. The following example illustrates these techniques.

Example 2.1.8

Simplify $2(A+3C)-3(2C-B)-3\left[2(2A+B-4C)-4(A-2C)\right]$ where A, B, and C are all matrices of the same size.

Solution. The reduction proceeds as though A, B, and C were variables.

$$\begin{aligned}
&2(A+3C)-3(2C-B)-3\left[2(2A+B-4C)-4(A-2C)\right]\\
&\quad = 2A+6C-6C+3B-3\left[4A+2B-8C-4A+8C\right]\\
&\quad = 2A+3B-3\left[2B\right]\\
&\quad = 2A-3B
\end{aligned}$$

Transpose of a Matrix

Many results about a matrix A involve the *rows* of A, and the corresponding result for columns is derived in an analogous way, essentially by replacing the word *row* by the word *column* throughout. The following definition is made with such applications in mind.

Definition 2.3 Transpose of a Matrix

If A is an $m \times n$ matrix, the **transpose** *of A, written A^T, is the $n \times m$ matrix whose rows are just the columns of A in the same order.*

In other words, the first row of A^T is the first column of A (that is it consists of the entries of column 1 in order). Similarly the second row of A^T is the second column of A, and so on.

Example 2.1.9

Write down the transpose of each of the following matrices.

$$A = \left[\begin{array}{c} 1 \\ 3 \\ 2 \end{array}\right] \quad B = \left[\begin{array}{ccc} 5 & 2 & 6 \end{array}\right] \quad C = \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{array}\right] \quad D = \left[\begin{array}{rrr} 3 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 1 \end{array}\right]$$

Solution.

$$A^T = \left[\begin{array}{ccc} 1 & 3 & 2 \end{array}\right], \; B^T = \left[\begin{array}{c} 5 \\ 2 \\ 6 \end{array}\right], \; C^T = \left[\begin{array}{ccc} 1 & 3 & 5 \\ 2 & 4 & 6 \end{array}\right], \text{ and } D^T = D.$$

If $A = \left[a_{ij}\right]$ is a matrix, write $A^T = \left[b_{ij}\right]$. Then b_{ij} is the jth element of the ith row of A^T and so is the jth element of the ith *column* of A. This means $b_{ij} = a_{ji}$, so the definition of A^T can be stated as follows:

$$\text{If } A = \left[a_{ij}\right], \text{ then } A^T = \left[a_{ji}\right]. \tag{2.1}$$

This is useful in verifying the following properties of transposition.

Theorem 2.1.2

Let A and B denote matrices of the same size, and let k denote a scalar.

1. *If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix.*
2. $(A^T)^T = A$.
3. $(kA)^T = kA^T$.
4. $(A+B)^T = A^T + B^T$.

Proof. Property 1 is part of the definition of A^T, and Property 2 follows from (2.1). As to Property 3: If $A = [a_{ij}]$, then $kA = [ka_{ij}]$, so (2.1) gives

$$(kA)^T = [ka_{ji}] = k[a_{ji}] = kA^T$$

Finally, if $B = [b_{ij}]$, then $A+B = [c_{ij}]$ where $c_{ij} = a_{ij} + b_{ij}$ Then (2.1) gives Property 4:

$$(A+B)^T = [c_{ij}]^T = [c_{ji}] = [a_{ji} + b_{ji}] = [a_{ji}] + [b_{ji}] = A^T + B^T$$

□

There is another useful way to think of transposition. If $A = [a_{ij}]$ is an $m \times n$ matrix, the elements $a_{11}, a_{22}, a_{33}, \dots$ are called the **main diagonal** of A. Hence the main diagonal extends down and to the right from the upper left corner of the matrix A; it is shaded in the following examples:

$$\left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{array}\right] \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{array}\right] \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right] \left[\begin{array}{c} a_{11} \\ a_{21} \end{array}\right]$$

Thus forming the transpose of a matrix A can be viewed as "flipping" A about its main diagonal, or as "rotating" A through $180°$ about the line containing the main diagonal. This makes Property 2 in Theorem 2.1.2 transparent.

Example 2.1.10

Solve for A if $\left(2A^T - 3\left[\begin{array}{rr} 1 & 2 \\ -1 & 1 \end{array}\right]\right)^T = \left[\begin{array}{rr} 2 & 3 \\ -1 & 2 \end{array}\right]$.

Solution. Using Theorem 2.1.2, the left side of the equation is

$$\left(2A^T - 3\left[\begin{array}{rr} 1 & 2 \\ -1 & 1 \end{array}\right]\right)^T = 2\left(A^T\right)^T - 3\left[\begin{array}{rr} 1 & 2 \\ -1 & 1 \end{array}\right]^T = 2A - 3\left[\begin{array}{rr} 1 & -1 \\ 2 & 1 \end{array}\right]$$

Hence the equation becomes

$$2A - 3\left[\begin{array}{rr} 1 & -1 \\ 2 & 1 \end{array}\right] = \left[\begin{array}{rr} 2 & 3 \\ -1 & 2 \end{array}\right]$$

Thus $2A = \left[\begin{array}{rr} 2 & 3 \\ -1 & 2 \end{array}\right] + 3\left[\begin{array}{rr} 1 & -1 \\ 2 & 1 \end{array}\right] = \left[\begin{array}{rr} 5 & 0 \\ 5 & 5 \end{array}\right]$, so finally $A = \frac{1}{2}\left[\begin{array}{rr} 5 & 0 \\ 5 & 5 \end{array}\right] = \frac{5}{2}\left[\begin{array}{rr} 1 & 0 \\ 1 & 1 \end{array}\right]$.

Note that Example 2.1.10 can also be solved by first transposing both sides, then solving for A^T, and so obtaining $A = (A^T)^T$. The reader should do this.

The matrix $D = \left[\begin{array}{rr} 1 & 2 \\ 2 & 5 \end{array}\right]$ in Example 2.1.9 has the property that $D = D^T$. Such matrices are important; a matrix A is called **symmetric** if $A = A^T$. A symmetric matrix A is necessarily square (if A is $m \times n$, then A^T is $n \times m$, so $A = A^T$ forces $n = m$). The name comes from the fact that these matrices exhibit a symmetry about the main diagonal. That is, entries that are directly across the main diagonal from each other are equal.

For example, $\left[\begin{array}{ccc} a & b & c \\ b' & d & e \\ c' & e' & f \end{array}\right]$ is symmetric when $b = b'$, $c = c'$, and $e = e'$.

Example 2.1.11

If A and B are symmetric $n \times n$ matrices, show that $A + B$ is symmetric.

Solution. We have $A^T = A$ and $B^T = B$, so, by Theorem 2.1.2, we have $(A + B)^T = A^T + B^T = A + B$. Hence $A + B$ is symmetric.

Example 2.1.12

Suppose a square matrix A satisfies $A = 2A^T$. Show that necessarily $A = 0$.

Solution. If we iterate the given equation, Theorem 2.1.2 gives

$$A = 2A^T = 2\left[2A^T\right]^T = 2\left[2(A^T)^T\right] = 4A$$

Subtracting A from both sides gives $3A = 0$, so $A = \frac{1}{3}(0) = 0$.

Exercises for 2.1

Exercise 2.1.1 Find a, b, c, and d if

a. $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right] = \left[\begin{array}{cc} c - 3d & -d \\ 2a + d & a + b \end{array}\right]$

b. $\left[\begin{array}{cc} a - b & b - c \\ c - d & d - a \end{array}\right] = 2\left[\begin{array}{rr} 1 & 1 \\ -3 & 1 \end{array}\right]$

c. $3\left[\begin{array}{c} a \\ b \end{array}\right] + 2\left[\begin{array}{c} b \\ a \end{array}\right] = \left[\begin{array}{c} 1 \\ 2 \end{array}\right]$

d. $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right] = \left[\begin{array}{cc} b & c \\ d & a \end{array}\right]$

Exercise 2.1.2 Compute the following:

a. $\left[\begin{array}{ccc} 3 & 2 & 1 \\ 5 & 1 & 0 \end{array}\right] - 5\left[\begin{array}{rrr} 3 & 0 & -2 \\ 1 & -1 & 2 \end{array}\right]$

b. $3\left[\begin{array}{r} 3 \\ -1 \end{array}\right] - 5\left[\begin{array}{c} 6 \\ 2 \end{array}\right] + 7\left[\begin{array}{r} 1 \\ -1 \end{array}\right]$

c. $\left[\begin{array}{rr} -2 & 1 \\ 3 & 2 \end{array}\right] - 4\left[\begin{array}{rr} 1 & -2 \\ 0 & -1 \end{array}\right] + 3\left[\begin{array}{rr} 2 & -3 \\ -1 & -2 \end{array}\right]$

d. $\left[\begin{array}{ccc} 3 & -1 & 2 \end{array}\right] - 2\left[\begin{array}{ccc} 9 & 3 & 4 \end{array}\right] + \left[\begin{array}{ccc} 3 & 11 & -6 \end{array}\right]$

e. $\left[\begin{array}{rrrr} 1 & -5 & 4 & 0 \\ 2 & 1 & 0 & 6 \end{array}\right]^T$

f. $\left[\begin{array}{rrr} 0 & -1 & 2 \\ 1 & 0 & -4 \\ -2 & 4 & 0 \end{array}\right]^T$

g. $\left[\begin{array}{rr} 3 & -1 \\ 2 & 1 \end{array}\right] - 2\left[\begin{array}{rr} 1 & -2 \\ 1 & 1 \end{array}\right]^T$

h. $3\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^T - 2\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

Exercise 2.1.3 Let $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 1 & 4 \end{bmatrix}$, and $E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Compute the following (where possible).

a. $3A - 2B$
b. $5C$
c. $3E^T$
d. $B + D$
e. $4A^T - 3C$
f. $(A + C)^T$
g. $2B - 3E$
h. $A - D$
i. $(B - 2E)^T$

Exercise 2.1.4 Find A if:

a. $5A - \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = 3A - \begin{bmatrix} 5 & 2 \\ 6 & 1 \end{bmatrix}$

b. $3A - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5A - 2\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

Exercise 2.1.5 Find A in terms of B if:

a. $A + B = 3A + 2B$
b. $2A - B = 5(A + 2B)$

Exercise 2.1.6 If X, Y, A, and B are matrices of the same size, solve the following systems of equations to obtain X and Y in terms of A and B.

a. $\begin{aligned} 5X + 3Y &= A \\ 2X + Y &= B \end{aligned}$

b. $\begin{aligned} 4X + 3Y &= A \\ 5X + 4Y &= B \end{aligned}$

Exercise 2.1.7 Find all matrices X and Y such that:

a. $3X - 2Y = \begin{bmatrix} 3 & -1 \end{bmatrix}$
b. $2X - 5Y = \begin{bmatrix} 1 & 2 \end{bmatrix}$

Exercise 2.1.8 Simplify the following expressions where A, B, and C are matrices.

a. $2[9(A - B) + 7(2B - A)]$
$-2[3(2B + A) - 2(A + 3B) - 5(A + B)]$

b. $5[3(A - B + 2C) - 2(3C - B) - A]$
$+2[3(3A - B + C) + 2(B - 2A) - 2C]$

Exercise 2.1.9 If A is any 2×2 matrix, show that:

a. $A = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ for some numbers a, b, c, and d.

b. $A = p\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + q\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + r\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + s\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for some numbers p, q, r, and s.

Exercise 2.1.10 Let $A = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 0 & 1 \end{bmatrix}$. If $rA + sB + tC = 0$ for some scalars r, s, and t, show that necessarily $r = s = t = 0$.

Exercise 2.1.11

a. If $Q + A = A$ holds for every $m \times n$ matrix A, show that $Q = 0_{mn}$.

b. If A is an $m \times n$ matrix and $A + A' = 0_{mn}$, show that $A' = -A$.

Exercise 2.1.12 If A denotes an $m \times n$ matrix, show that $A = -A$ if and only if $A = 0$.

Exercise 2.1.13 A square matrix is called a **diagonal** matrix if all the entries off the main diagonal are zero. If A and B are diagonal matrices, show that the following matrices are also diagonal.

a. $A + B$
b. $A - B$
c. kA for any number k

Exercise 2.1.14 In each case determine all s and t such that the given matrix is symmetric:

a. $\begin{bmatrix} 1 & s \\ -2 & t \end{bmatrix}$
b. $\begin{bmatrix} s & t \\ st & 1 \end{bmatrix}$
c. $\begin{bmatrix} s & 2s & st \\ t & -1 & s \\ t & s^2 & s \end{bmatrix}$
d. $\begin{bmatrix} 2 & s & t \\ 2s & 0 & s + t \\ 3 & 3 & t \end{bmatrix}$

Exercise 2.1.15 In each case find the matrix A.

a. $\left(A + 3\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix}\right)^T = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 3 & 8 \end{bmatrix}$

b. $\left(3A^T + 2\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)^T = \begin{bmatrix} 8 & 0 \\ 3 & 1 \end{bmatrix}$

c. $(2A - 3\begin{bmatrix} 1 & 2 & 0 \end{bmatrix})^T = 3A^T + \begin{bmatrix} 2 & 1 & -1 \end{bmatrix}^T$

d. $\left(2A^T - 5\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\right)^T = 4A - 9\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$

Exercise 2.1.16 Let A and B be symmetric (of the same size). Show that each of the following is symmetric.

a. $(A - B)$

b. kA for any scalar k

Exercise 2.1.17 Show that $A + A^T$ and AA^T are symmetric for *any* square matrix A.

Exercise 2.1.18 If A is a square matrix and $A = kA^T$ where $k \neq \pm 1$, show that $A = 0$.

Exercise 2.1.19 In each case either show that the statement is true or give an example showing it is false.

a. If $A + B = A + C$, then B and C have the same size.

b. If $A + B = 0$, then $B = 0$.

c. If the $(3, 1)$-entry of A is 5, then the $(1, 3)$-entry of A^T is -5.

d. A and A^T have the same main diagonal for every matrix A.

e. If B is symmetric and $A^T = 3B$, then $A = 3B$.

f. If A and B are symmetric, then $kA + mB$ is symmetric for any scalars k and m.

Exercise 2.1.20 A square matrix W is called **skew-symmetric** if $W^T = -W$. Let A be any square matrix.

a. Show that $A - A^T$ is skew-symmetric.

b. Find a symmetric matrix S and a skew-symmetric matrix W such that $A = S + W$.

c. Show that S and W in part (b) are uniquely determined by A.

Exercise 2.1.21 If W is skew-symmetric (Exercise 2.1.20), show that the entries on the main diagonal are zero.

Exercise 2.1.22 Prove the following parts of Theorem 2.1.1.

a. $(k + p)A = kA + pA$

b. $(kp)A = k(pA)$

Exercise 2.1.23 Let A, A_1, A_2, $\dots$, A_n denote matrices of the same size. Use induction on n to verify the following extensions of properties 5 and 6 of Theorem 2.1.1.

a. $k(A_1 + A_2 + \cdots + A_n) = kA_1 + kA_2 + \cdots + kA_n$ for any number k

b. $(k_1 + k_2 + \cdots + k_n)A = k_1A + k_2A + \cdots + k_nA$ for any numbers k_1, k_2, $\dots$, k_n

Exercise 2.1.24 Let A be a square matrix. If $A = pB^T$ and $B = qA^T$ for some matrix B and numbers p and q, show that either $A = 0 = B$ or $pq = 1$.
[*Hint*: Example 2.1.7.]

2.2 Matrix-Vector Multiplication

Up to now we have used matrices to solve systems of linear equations by manipulating the rows of the augmented matrix. In this section we introduce a different way of describing linear systems that makes more use of the coefficient matrix of the system and leads to a useful way of "multiplying" matrices.

Vectors

It is a well-known fact in analytic geometry that two points in the plane with coordinates (a_1, a_2) and (b_1, b_2) are equal if and only if $a_1 = b_1$ and $a_2 = b_2$. Moreover, a similar condition applies to points (a_1, a_2, a_3) in space. We extend this idea as follows.

An ordered sequence $(a_1, a_2, \dots, a_n)$ of real numbers is called an **ordered n-tuple**. The word "ordered" here reflects our insistence that two ordered n-tuples are equal if and only if corresponding entries are the same. In other words,

$$(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n) \quad \text{if and only if} \quad a_1 = b_1,\ a_2 = b_2,\ \dots, \text{ and } a_n = b_n.$$

Thus the ordered 2-tuples and 3-tuples are just the ordered pairs and triples familiar from geometry.

Definition 2.4 The set $\mathbb{R}^n$ of ordered n-tuples of real numbers

Let $\mathbb{R}$ denote the set of all real numbers. The set of all *ordered n-tuples from $\mathbb{R}$ has a special notation:*

$$\mathbb{R}^n \textit{ denotes the set of all ordered } n\textit{-tuples of real numbers.}$$

There are two commonly used ways to denote the n-tuples in $\mathbb{R}^n$: As rows $(r_1, r_2, \dots, r_n)$ or columns $\left[\begin{array}{c} r_1 \\ r_2 \\ \vdots \\ r_n \end{array}\right]$; the notation we use depends on the context. In any event they are called **vectors** or **n-vectors** and will be denoted using bold type such as $\mathbf{x}$ or $\mathbf{v}$. For example, an $m \times n$ matrix A will be written as a row of columns:

$$A = \left[\begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{array}\right] \text{ where } \mathbf{a}_j \text{ denotes column } j \text{ of } A \text{ for each } j.$$

If $\mathbf{x}$ and $\mathbf{y}$ are two n-vectors in $\mathbb{R}^n$, it is clear that their matrix sum $\mathbf{x} + \mathbf{y}$ is also in $\mathbb{R}^n$ as is the scalar multiple $k\mathbf{x}$ for any real number k. We express this observation by saying that $\mathbb{R}^n$ is **closed** under addition and scalar multiplication. In particular, all the basic properties in Theorem 2.1.1 are true of these n-vectors. These properties are fundamental and will be used frequently below without comment. As for matrices in general, the $n \times 1$ zero matrix is called the **zero n-vector** in $\mathbb{R}^n$ and, if $\mathbf{x}$ is an n-vector, the n-vector $-\mathbf{x}$ is called the **negative $\mathbf{x}$**.

Of course, we have already encountered these n-vectors in Section 1.3 as the solutions to systems of linear equations with n variables. In particular we defined the notion of a linear combination of vectors and showed that a linear combination of solutions to a homogeneous system is again a solution. Clearly, a linear combination of n-vectors in $\mathbb{R}^n$ is again in $\mathbb{R}^n$, a fact that we will be using.

Matrix-Vector Multiplication

Given a system of linear equations, the left sides of the equations depend only on the coefficient matrix A and the column $\mathbf{x}$ of variables, and not on the constants. This observation leads to a fundamental idea in linear algebra: We view the left sides of the equations as the "product" $A\mathbf{x}$ of the matrix A and the vector $\mathbf{x}$. This simple change of perspective leads to a completely new way of viewing linear systems—one that is very useful and will occupy our attention throughout this book.

To motivate the definition of the "product" $A\mathbf{x}$, consider first the following system of two equations in three variables:

$$\begin{aligned} ax_1 + \ bx_2 + \ cx_3 &= b_1 \\ a'x_1 + b'x_2 + c'x_3 &= b_1 \end{aligned} \tag{2.2}$$

and let $A = \left[\begin{array}{ccc} a & b & c \\ a' & b' & c' \end{array}\right]$, $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right]$, $\mathbf{b} = \left[\begin{array}{c} b_1 \\ b_2 \end{array}\right]$ denote the coefficient matrix, the variable matrix, and the constant matrix, respectively. The system (2.2) can be expressed as a single vector equation

$$\left[\begin{array}{c} ax_1 + \ bx_2 + \ cx_3 \\ a'x_1 + b'x_2 + c'x_3 \end{array}\right] = \left[\begin{array}{c} b_1 \\ b_2 \end{array}\right]$$

which in turn can be written as follows:

$$x_1 \left[\begin{array}{c} a \\ a' \end{array}\right] + x_2 \left[\begin{array}{c} b \\ b' \end{array}\right] + x_3 \left[\begin{array}{c} c \\ c' \end{array}\right] = \left[\begin{array}{c} b_1 \\ b_2 \end{array}\right]$$

Now observe that the vectors appearing on the left side are just the columns

$$\mathbf{a}_1 = \left[\begin{array}{c} a \\ a' \end{array}\right], \ \mathbf{a}_2 = \left[\begin{array}{c} b \\ b' \end{array}\right], \ \text{and } \mathbf{a}_3 = \left[\begin{array}{c} c \\ c' \end{array}\right]$$

of the coefficient matrix A. Hence the system (2.2) takes the form

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b} \tag{2.3}$$

This shows that the system (2.2) has a solution if and only if the constant matrix $\mathbf{b}$ is a linear combination[3] of the columns of A, and that in this case the entries of the solution are the coefficients x_1, x_2, and x_3 in this linear combination.

Moreover, this holds in general. If A is any $m \times n$ matrix, it is often convenient to view A as a row of columns. That is, if $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ are the columns of A, we write

$$A = \left[\begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{array}\right]$$

and say that $A = \left[\begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{array}\right]$ is *given in terms of its columns*.

Now consider any system of linear equations with $m \times n$ coefficient matrix A. If $\mathbf{b}$ is the constant matrix of the system, and if $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right]$ is the matrix of variables then, exactly as above, the system can

[3] Linear combinations were introduced in Section 1.3 to describe the solutions of homogeneous systems of linear equations. They will be used extensively in what follows.

be written as a single vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \tag{2.4}$$

Example 2.2.1

Write the system $\begin{cases} 3x_1 + 2x_2 - 4x_3 = 0 \\ x_1 - 3x_2 + x_3 = 3 \\ x_2 - 5x_3 = -1 \end{cases}$ in the form given in (2.4).

Solution.

$$x_1\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$$

As mentioned above, we view the left side of (2.4) as the *product* of the matrix A and the vector $\mathbf{x}$. This basic idea is formalized in the following definition:

Definition 2.5 Matrix-Vector Multiplication

*Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ be an $m \times n$ matrix, written in terms of its columns $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is any n-vector, the **product** $A\mathbf{x}$ is defined to be the m-vector given by:*

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

In other words, if A is $m \times n$ and $\mathbf{x}$ is an n-vector, the product $A\mathbf{x}$ is the linear combination of the columns of A where the coefficients are the entries of $\mathbf{x}$ (in order).

Note that if A is an $m \times n$ matrix, the product $A\mathbf{x}$ is only defined if $\mathbf{x}$ is an n-vector and then the vector $A\mathbf{x}$ is an m-vector because this is true of each column $\mathbf{a}_j$ of A. But in this case the *system* of linear equations with coefficient matrix A and constant vector $\mathbf{b}$ takes the form of a *single* matrix equation

$$A\mathbf{x} = \mathbf{b}$$

The following theorem combines Definition 2.5 and equation (2.4) and summarizes the above discussion. Recall that a system of linear equations is said to be *consistent* if it has at least one solution.

Theorem 2.2.1

1. *Every system of linear equations has the form $A\mathbf{x} = \mathbf{b}$ where A is the coefficient matrix, $\mathbf{b}$ is the constant matrix, and $\mathbf{x}$ is the matrix of variables.*

2. *The system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is a linear combination of the columns of A.*

3. If $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ are the columns of A and if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then $\mathbf{x}$ is a solution to the linear system $A\mathbf{x} = \mathbf{b}$ if and only if $x_1, x_2, \dots, x_n$ are a solution of the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

A system of linear equations in the form $A\mathbf{x} = \mathbf{b}$ as in (1) of Theorem 2.2.1 is said to be written in **matrix form**. This is a useful way to view linear systems as we shall see.

Theorem 2.2.1 transforms the problem of solving the linear system $A\mathbf{x} = \mathbf{b}$ into the problem of expressing the constant matrix B as a linear combination of the columns of the coefficient matrix A. Such a change in perspective is very useful because one approach or the other may be better in a particular situation; the importance of the theorem is that there is a choice.

Example 2.2.2

If $A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 0 & 2 & -3 & 1 \\ -3 & 4 & 1 & 2 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \end{bmatrix}$, compute $A\mathbf{x}$.

Solution. By Definition 2.5: $A\mathbf{x} = 2\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} + 0\begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \\ -6 \end{bmatrix}$.

Example 2.2.3

Given columns $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$ in $\mathbb{R}^3$, write $2\mathbf{a}_1 - 3\mathbf{a}_2 + 5\mathbf{a}_3 + \mathbf{a}_4$ in the form $A\mathbf{x}$ where A is a matrix and $\mathbf{x}$ is a vector.

Solution. Here the column of coefficients is $\mathbf{x} = \begin{bmatrix} 2 \\ -3 \\ 5 \\ 1 \end{bmatrix}$. Hence Definition 2.5 gives

$$A\mathbf{x} = 2\mathbf{a}_1 - 3\mathbf{a}_2 + 5\mathbf{a}_3 + \mathbf{a}_4$$

where $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix}$ is the matrix with $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$ as its columns.

Example 2.2.4

Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix}$ be the 3×4 matrix given in terms of its columns $\mathbf{a}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix}$, and $\mathbf{a}_4 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$. In each case below, either express $\mathbf{b}$ as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$, or show that it is not such a linear combination. Explain what your answer means for the corresponding system $A\mathbf{x} = \mathbf{b}$ of linear equations.

a. $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

b. $\mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$

Solution. By Theorem 2.2.1, $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$ if and only if the system $A\mathbf{x} = \mathbf{b}$ is consistent (that is, it has a solution). So in each case we carry the augmented matrix $[A|\mathbf{b}]$ of the system $A\mathbf{x} = \mathbf{b}$ to reduced form.

a. Here $\left[\begin{array}{rrrr|r} 2 & 1 & 3 & 3 & 1 \\ 0 & 1 & -1 & 1 & 2 \\ -1 & 1 & -3 & 0 & 3 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$, so the system $A\mathbf{x} = \mathbf{b}$ has no solution in this case. Hence $\mathbf{b}$ is *not* a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$.

b. Now $\left[\begin{array}{rrrr|r} 2 & 1 & 3 & 3 & 4 \\ 0 & 1 & -1 & 1 & 2 \\ -1 & 1 & -3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$, so the system $A\mathbf{x} = \mathbf{b}$ is consistent.

Thus $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$ in this case. In fact the general solution is $x_1 = 1 - 2s - t$, $x_2 = 2 + s - t$, $x_3 = s$, and $x_4 = t$ where s and t are arbitrary parameters. Hence $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4 = \mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ for *any* choice of s and t. If we take $s = 0$ and $t = 0$, this becomes $\mathbf{a}_1 + 2\mathbf{a}_2 = \mathbf{b}$, whereas taking $s = 1 = t$ gives $-2\mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 = \mathbf{b}$.

Example 2.2.5

Taking A to be the zero matrix, we have $0\mathbf{x} = \mathbf{0}$ for all vectors $\mathbf{x}$ by Definition 2.5 because every column of the zero matrix is zero. Similarly, $A\mathbf{0} = \mathbf{0}$ for all matrices A because every entry of the zero vector is zero.

Example 2.2.6

If $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that $I\mathbf{x} = \mathbf{x}$ for any vector $\mathbf{x}$ in $\mathbb{R}^3$.

Solution. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then Definition 2.5 gives

$$I\mathbf{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{x}$$

The matrix I in Example 2.2.6 is called the 3×3 **identity matrix**, and we will encounter such matrices again in Example 2.2.11 below. Before proceeding, we develop some algebraic properties of matrix-vector multiplication that are used extensively throughout linear algebra.

Theorem 2.2.2

Let A and B be $m \times n$ matrices, and let $\mathbf{x}$ and $\mathbf{y}$ be n-vectors in $\mathbb{R}^n$. Then:

1. $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$.
2. $A(a\mathbf{x}) = a(A\mathbf{x}) = (aA)\mathbf{x}$ *for all scalars a.*
3. $(A+B)\mathbf{x} = A\mathbf{x} + B\mathbf{x}$.

Proof. We prove (3); the other verifications are similar and are left as exercises. Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}$ be given in terms of their columns. Since adding two matrices is the same as adding their columns, we have

$$A + B = \begin{bmatrix} \mathbf{a}_1 + \mathbf{b}_1 & \mathbf{a}_2 + \mathbf{b}_2 & \cdots & \mathbf{a}_n + \mathbf{b}_n \end{bmatrix}$$

If we write $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ Definition 2.5 gives

$$\begin{aligned} (A+B)\mathbf{x} &= x_1(\mathbf{a}_1 + \mathbf{b}_1) + x_2(\mathbf{a}_2 + \mathbf{b}_2) + \cdots + x_n(\mathbf{a}_n + \mathbf{b}_n) \\ &= (x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n) + (x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n) \\ &= A\mathbf{x} + B\mathbf{x} \end{aligned}$$

□

Theorem 2.2.2 allows matrix-vector computations to be carried out much as in ordinary arithmetic. For example, for any $m \times n$ matrices A and B and any n-vectors $\mathbf{x}$ and $\mathbf{y}$, we have:

$$A(2\mathbf{x} - 5\mathbf{y}) = 2A\mathbf{x} - 5A\mathbf{y} \quad \text{and} \quad (3A - 7B)\mathbf{x} = 3A\mathbf{x} - 7B\mathbf{x}$$

We will use such manipulations throughout the book, often without mention.

Linear Equations

Theorem 2.2.2 also gives a useful way to describe the solutions to a system

$$A\mathbf{x} = \mathbf{b}$$

of linear equations. There is a related system

$$A\mathbf{x} = \mathbf{0}$$

called the **associated homogeneous system**, obtained from the original system $A\mathbf{x} = \mathbf{b}$ by replacing all the constants by zeros. Suppose $\mathbf{x}_1$ is a solution to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x}_0$ is a solution to $A\mathbf{x} = \mathbf{0}$ (that is $A\mathbf{x}_1 = \mathbf{b}$ and $A\mathbf{x}_0 = \mathbf{0}$). Then $\mathbf{x}_1 + \mathbf{x}_0$ is another solution to $A\mathbf{x} = \mathbf{b}$. Indeed, Theorem 2.2.2 gives

$$A(\mathbf{x}_1 + \mathbf{x}_0) = A\mathbf{x}_1 + A\mathbf{x}_0 = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

This observation has a useful converse.

Theorem 2.2.3

Suppose $\mathbf{x}_1$ is any particular solution to the system $A\mathbf{x} = \mathbf{b}$ of linear equations. Then every solution $\mathbf{x}_2$ to $A\mathbf{x} = \mathbf{b}$ has the form

$$\mathbf{x}_2 = \mathbf{x}_0 + \mathbf{x}_1$$

for some solution $\mathbf{x}_0$ of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.

Proof. Suppose $\mathbf{x}_2$ is also a solution to $A\mathbf{x} = \mathbf{b}$, so that $A\mathbf{x}_2 = \mathbf{b}$. Write $\mathbf{x}_0 = \mathbf{x}_2 - \mathbf{x}_1$. Then $\mathbf{x}_2 = \mathbf{x}_0 + \mathbf{x}_1$ and, using Theorem 2.2.2, we compute

$$A\mathbf{x}_0 = A(\mathbf{x}_2 - \mathbf{x}_1) = A\mathbf{x}_2 - A\mathbf{x}_1 = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

Hence $\mathbf{x}_0$ is a solution to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$. □

Note that gaussian elimination provides one such representation.

Example 2.2.7

Express every solution to the following system as the sum of a specific solution plus a solution to the associated homogeneous system.

$$\begin{aligned} x_1 - x_2 - x_3 + 3x_4 &= 2 \\ 2x_1 - x_2 - 3x_3 + 4x_4 &= 6 \\ x_1 - 2x_3 + x_4 &= 4 \end{aligned}$$

Solution. Gaussian elimination gives $x_1 = 4+2s-t$, $x_2 = 2+s+2t$, $x_3 = s$, and $x_4 = t$ where s and t are arbitrary parameters. Hence the general solution can be written

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4+2s-t \\ 2+s+2t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \end{bmatrix} + \left(s \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right)$$

Thus $\mathbf{x}_1 = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution (where $s = 0 = t$), and $\mathbf{x}_0 = s \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ gives *all* solutions to the associated homogeneous system. (To see why this is so, carry out the gaussian elimination again but with all the constants set equal to zero.)

The following useful result is included with no proof.

Theorem 2.2.4

Let $A\mathbf{x} = \mathbf{b}$ be a system of equations with augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$. Write $\operatorname{rank} A = r$.

1. $\operatorname{rank} \left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ *is either r or $r+1$.*
2. *The system is consistent if and only if* $\operatorname{rank} \left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] = r$.
3. *The system is inconsistent if and only if* $\operatorname{rank} \left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] = r+1$.

The Dot Product

Definition 2.5 is not always the easiest way to compute a matrix-vector product $A\mathbf{x}$ because it requires that the columns of A be explicitly identified. There is another way to find such a product which uses the matrix A as a whole with no reference to its columns, and hence is useful in practice. The method depends on the following notion.

Definition 2.6 Dot Product in $\mathbb{R}^n$

*If $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ are two ordered n-tuples, their **dot product** is defined to be the number*

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

obtained by multiplying corresponding entries and adding the results.

To see how this relates to matrix products, let A denote a 3×4 matrix and let $\mathbf{x}$ be a 4-vector. Writing

$$\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\right] \quad \text{and} \quad A = \left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]$$

in the notation of Section 2.1, we compute

$$\begin{aligned} A\mathbf{x} = \left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\right] &= x_1 \left[\begin{array}{c} a_{11} \\ a_{21} \\ a_{31} \end{array}\right] + x_2 \left[\begin{array}{c} a_{12} \\ a_{22} \\ a_{32} \end{array}\right] + x_3 \left[\begin{array}{c} a_{13} \\ a_{23} \\ a_{33} \end{array}\right] + x_4 \left[\begin{array}{c} a_{14} \\ a_{24} \\ a_{34} \end{array}\right] \\ &= \left[\begin{array}{c} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 \end{array}\right] \end{aligned}$$

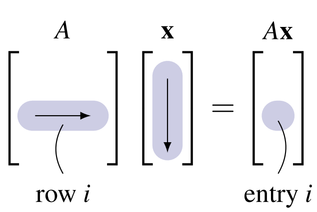
From this we see that each entry of $A\mathbf{x}$ is the dot product of the corresponding row of A with $\mathbf{x}$. This computation goes through in general, and we record the result in Theorem 2.2.5.

Theorem 2.2.5: Dot Product Rule

Let A be an $m \times n$ matrix and let $\mathbf{x}$ be an n-vector. Then each entry of the vector $A\mathbf{x}$ is the dot product of the corresponding row of A with $\mathbf{x}$.

This result is used extensively throughout linear algebra.

If A is $m \times n$ and $\mathbf{x}$ is an n-vector, the computation of $A\mathbf{x}$ by the dot product rule is simpler than using Definition 2.5 because the computation can be carried out directly with no explicit reference to the columns of A (as in Definition 2.5). The first entry of $A\mathbf{x}$ is the dot product of row 1 of A with $\mathbf{x}$. In hand calculations this is computed by going *across* row one of A, going *down* the column $\mathbf{x}$, multiplying corresponding entries, and adding the results. The other entries of $A\mathbf{x}$ are computed in the same way using the other rows of A with the column $\mathbf{x}$.



In general, compute entry i of $A\mathbf{x}$ as follows (see the diagram):

Go *across* row i of A and *down* column $\mathbf{x}$, multiply corresponding entries, and add the results.

As an illustration, we rework Example 2.2.2 using the dot product rule instead of Definition 2.5.

Example 2.2.8

If $A = \left[\begin{array}{rrrr} 2 & -1 & 3 & 5 \\ 0 & 2 & -3 & 1 \\ -3 & 4 & 1 & 2 \end{array}\right]$ and $\mathbf{x} = \left[\begin{array}{r} 2 \\ 1 \\ 0 \\ -2 \end{array}\right]$, compute $A\mathbf{x}$.

Solution. The entries of $A\mathbf{x}$ are the dot products of the rows of A with $\mathbf{x}$:

$$A\mathbf{x} = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 0 & 2 & -3 & 1 \\ -3 & 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2\cdot 2 & + & (-1)1 & + & 3\cdot 0 & + & 5(-2) \\ 0\cdot 2 & + & 2\cdot 1 & + & (-3)0 & + & 1(-2) \\ (-3)2 & + & 4\cdot 1 & + & 1\cdot 0 & + & 2(-2) \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \\ -6 \end{bmatrix}$$

Of course, this agrees with the outcome in Example 2.2.2.

Example 2.2.9

Write the following system of linear equations in the form $A\mathbf{x} = \mathbf{b}$.

$$\begin{aligned} 5x_1 - x_2 + 2x_3 + x_4 - 3x_5 &= 8 \\ x_1 + x_2 + 3x_3 - 5x_4 + 2x_5 &= -2 \\ -x_1 + x_2 - 2x_3 + \quad - 3x_5 &= 0 \end{aligned}$$

Solution. Write $A = \begin{bmatrix} 5 & -1 & 2 & 1 & -3 \\ 1 & 1 & 3 & -5 & 2 \\ -1 & 1 & -2 & 0 & -3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 8 \\ -2 \\ 0 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$. Then the dot product rule gives $A\mathbf{x} = \begin{bmatrix} 5x_1 - x_2 + 2x_3 + x_4 - 3x_5 \\ x_1 + x_2 + 3x_3 - 5x_4 + 2x_5 \\ -x_1 + x_2 - 2x_3 \quad - 3x_5 \end{bmatrix}$, so the entries of $A\mathbf{x}$ are the left sides of the equations in the linear system. Hence the system becomes $A\mathbf{x} = \mathbf{b}$ because matrices are equal if and only corresponding entries are equal.

Example 2.2.10

If A is the zero $m \times n$ matrix, then $A\mathbf{x} = \mathbf{0}$ for each n-vector $\mathbf{x}$.

Solution. For each k, entry k of $A\mathbf{x}$ is the dot product of row k of A with $\mathbf{x}$, and this is zero because row k of A consists of zeros.

Definition 2.7 The Identity Matrix

*For each $n > 2$, the **identity matrix** I_n is the $n \times n$ matrix with 1s on the main diagonal (upper left to lower right), and zeros elsewhere.*

The first few identity matrices are

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \dots$$

In Example 2.2.6 we showed that $I_3\mathbf{x} = \mathbf{x}$ for each 3-vector $\mathbf{x}$ using Definition 2.5. The following result shows that this holds in general, and is the reason for the name.

Example 2.2.11

For each $n \geq 2$ we have $I_n\mathbf{x} = \mathbf{x}$ for each n-vector $\mathbf{x}$ in $\mathbb{R}^n$.

Solution. We verify the case $n = 4$. Given the 4-vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ the dot product rule gives

$$I_4\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1+0+0+0 \\ 0+x_2+0+0 \\ 0+0+x_3+0 \\ 0+0+0+x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{x}$$

In general, $I_n\mathbf{x} = \mathbf{x}$ because entry k of $I_n\mathbf{x}$ is the dot product of row k of I_n with $\mathbf{x}$, and row k of I_n has 1 in position k and zeros elsewhere.

Example 2.2.12

Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ be any $m \times n$ matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$. If $\mathbf{e}_j$ denotes column j of the $n \times n$ identity matrix I_n, then $A\mathbf{e}_j = \mathbf{a}_j$ for each $j = 1, 2, \dots, n$.

Solution. Write $\mathbf{e}_j = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix}$ where $t_j = 1$, but $t_i = 0$ for all $i \neq j$. Then Theorem 2.2.5 gives

$$A\mathbf{e}_j = t_1\mathbf{a}_1 + \cdots + t_j\mathbf{a}_j + \cdots + t_n\mathbf{a}_n = 0 + \cdots + \mathbf{a}_j + \cdots + 0 = \mathbf{a}_j$$

Example 2.2.12 will be referred to later; for now we use it to prove:

Theorem 2.2.6

Let A and B be $m \times n$ matrices. If $A\mathbf{x} = B\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$, then $A = B$.

Proof. Write $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}$ and in terms of their columns. It is enough to show that $\mathbf{a}_k = \mathbf{b}_k$ holds for all k. But we are assuming that $A\mathbf{e}_k = B\mathbf{e}_k$, which gives $\mathbf{a}_k = \mathbf{b}_k$ by Example 2.2.12. □

We have introduced matrix-vector multiplication as a new way to think about systems of linear equations. But it has several other uses as well. It turns out that many geometric operations can be described using matrix multiplication, and we now investigate how this happens. As a bonus, this description provides a geometric "picture" of a matrix by revealing the effect on a vector when it is multiplied by A. This "geometric view" of matrices is a fundamental tool in understanding them.

Transformations

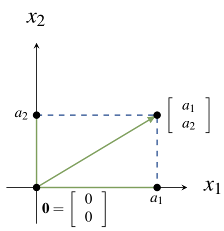


Figure 2.2.1

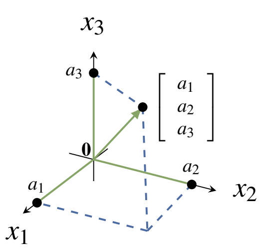


Figure 2.2.2

The set $\mathbb{R}^2$ has a geometrical interpretation as the euclidean plane where a vector $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ in $\mathbb{R}^2$ represents the point (a_1, a_2) in the plane (see Figure 2.2.1). In this way we regard $\mathbb{R}^2$ as the set of all points in the plane. Accordingly, we will refer to vectors in $\mathbb{R}^2$ as points, and denote their coordinates as a column rather than a row. To enhance this geometrical interpretation of the vector $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, it is denoted graphically by an arrow from the origin $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to the vector as in Figure 2.2.1.

Similarly we identify $\mathbb{R}^3$ with 3-dimensional space by writing a point (a_1, a_2, a_3) as the vector $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ in $\mathbb{R}^3$, again represented by an arrow[4] from the origin to the point as in Figure 2.2.2. In this way the terms "point" and "vector" mean the same thing in the plane or in space.

We begin by describing a particular geometrical transformation of the plane $\mathbb{R}^2$.

Example 2.2.13

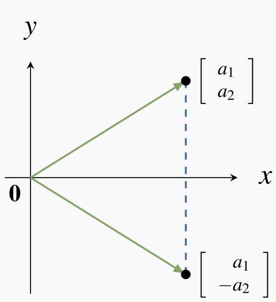


Figure 2.2.3

Consider the transformation of $\mathbb{R}^2$ given by *reflection* in the x axis. This operation carries the vector $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ to its reflection $\begin{bmatrix} a_1 \\ -a_2 \end{bmatrix}$ as in Figure 2.2.3. Now observe that

$$\begin{bmatrix} a_1 \\ -a_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

so reflecting $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ in the x axis can be achieved by multiplying by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

[4]This "arrow" representation of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ will be used extensively in Chapter 4.

If we write $A = \left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array} \right]$, Example 2.2.13 shows that reflection in the x axis carries each vector $\mathbf{x}$ in $\mathbb{R}^2$ to the vector $A\mathbf{x}$ in $\mathbb{R}^2$. It is thus an example of a function

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{where} \quad T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^2$$

As such it is a generalization of the familiar functions $f : \mathbb{R} \to \mathbb{R}$ that carry a *number* x to another real *number* $f(x)$.

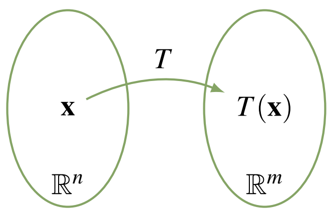


Figure 2.2.4

More generally, functions $T : \mathbb{R}^n \to \mathbb{R}^m$ are called **transformations** from $\mathbb{R}^n$ to $\mathbb{R}^m$. Such a transformation T is a rule that assigns to every vector $\mathbf{x}$ in $\mathbb{R}^n$ a uniquely determined vector $T(\mathbf{x})$ in $\mathbb{R}^m$ called the **image** of $\mathbf{x}$ under T. We denote this state of affairs by writing

$$T : \mathbb{R}^n \to \mathbb{R}^m \quad \text{or} \quad \mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$$

The transformation T can be visualized as in Figure 2.2.4.

To describe a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ we must specify the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ for every $\mathbf{x}$ in $\mathbb{R}^n$. This is referred to as **defining** T, or as specifying the **action** of T. Saying that the action *defines* the transformation means that we regard two transformations $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ as **equal** if they have the **same action**; more formally

$$S = T \quad \text{if and only if} \quad S(\mathbf{x}) = T(\mathbf{x}) \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n.$$

Again, this what we mean by $f = g$ where $f,\ g : \mathbb{R} \to \mathbb{R}$ are ordinary functions.

Functions $f : \mathbb{R} \to \mathbb{R}$ are often described by a formula, examples being $f(x) = x^2 + 1$ and $f(x) = \sin x$. The same is true of transformations; here is an example.

Example 2.2.14

The formula $T \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \right] = \left[\begin{array}{c} x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{array} \right]$ defines a transformation $\mathbb{R}^4 \to \mathbb{R}^3$.

Example 2.2.13 suggests that matrix multiplication is an important way of defining transformations $\mathbb{R}^n \to \mathbb{R}^m$. If A is any $m \times n$ matrix, multiplication by A gives a transformation

$$T_A : \mathbb{R}^n \to \mathbb{R}^m \quad \text{defined by} \quad T_A(\mathbf{x}) = A\mathbf{x} \text{ for every } \mathbf{x} \text{ in } \mathbb{R}^n$$

Definition 2.8 Matrix Transformation T_A

*T_A is called the **matrix transformation induced** by A.*

Thus Example 2.2.13 shows that reflection in the x axis is the matrix transformation $\mathbb{R}^2 \to \mathbb{R}^2$ induced by the matrix $\left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array} \right]$. Also, the transformation $R : \mathbb{R}^4 \to \mathbb{R}^3$ in Example 2.2.13 is the matrix transformation induced by the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ because } \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{bmatrix}$$

Example 2.2.15

Let $R_{\frac{\pi}{2}} : \mathbb{R}^2 \to \mathbb{R}^2$ denote counterclockwise rotation about the origin through $\frac{\pi}{2}$ radians (that is, $90°$)[5]. Show that $R_{\frac{\pi}{2}}$ is induced by the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Solution.

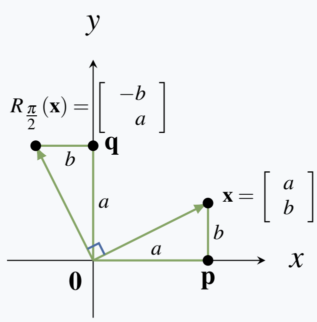


Figure 2.2.5

The effect of $R_{\frac{\pi}{2}}$ is to rotate the vector $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ counterclockwise through $\frac{\pi}{2}$ to produce the vector $R_{\frac{\pi}{2}}(\mathbf{x})$ shown in Figure 2.2.5. Since triangles $\mathbf{0px}$ and $\mathbf{0q}R_{\frac{\pi}{2}}(\mathbf{x})$ are identical, we obtain $R_{\frac{\pi}{2}}(\mathbf{x}) = \begin{bmatrix} -b \\ a \end{bmatrix}$. But $\begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$, so we obtain $R_{\frac{\pi}{2}}(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^2$ where $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. In other words, $R_{\frac{\pi}{2}}$ is the matrix transformation induced by A.

If A is the $m \times n$ zero matrix, then A induces the transformation

$$T : \mathbb{R}^n \to \mathbb{R}^m \quad \text{given by} \quad T(\mathbf{x}) = A\mathbf{x} = \mathbf{0} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

This is called the **zero transformation**, and is denoted $T = 0$.

Another important example is the **identity transformation**

$$1_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n \quad \text{given by} \quad 1_{\mathbb{R}^n}(\mathbf{x}) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

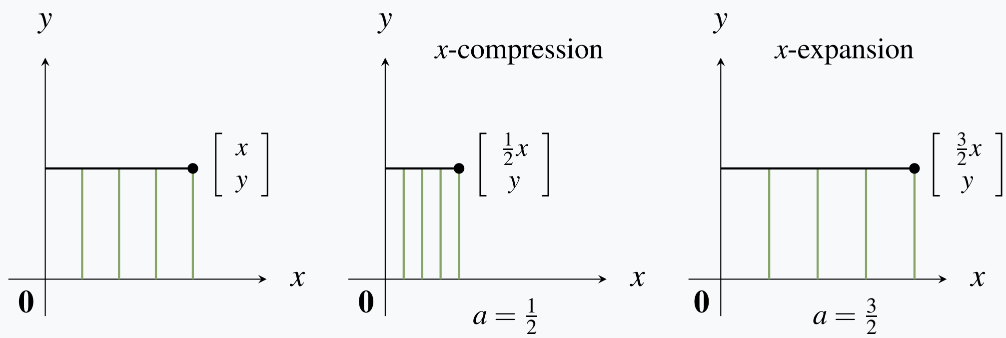
That is, the action of $1_{\mathbb{R}^n}$ on $\mathbf{x}$ is to do nothing to it. If I_n denotes the $n \times n$ identity matrix, we showed in Example 2.2.11 that $I_n\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Hence $1_{\mathbb{R}^n}(\mathbf{x}) = I_n\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$; that is, the identity matrix I_n induces the identity transformation.

Here are two more examples of matrix transformations with a clear geometric description.

[5] *Radian measure* for angles is based on the fact that $360°$ equals 2π radians. Hence π radians $= 180°$ and $\frac{\pi}{2}$ radians $= 90°$.

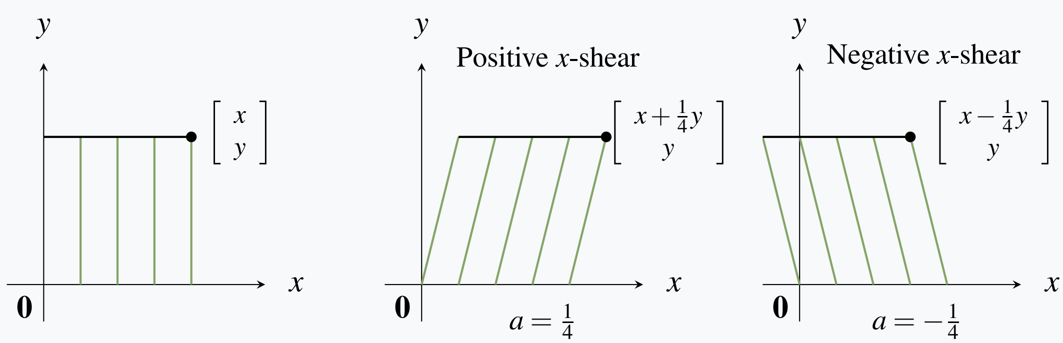
Example 2.2.16

If $a > 0$, the matrix transformation $T\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} ax \\ y \end{array}\right]$ induced by the matrix $A = \left[\begin{array}{cc} a & 0 \\ 0 & 1 \end{array}\right]$ is called an ***x*-expansion** of $\mathbb{R}^2$ if $a > 1$, and an ***x*-compression** if $0 < a < 1$. The reason for the names is clear in the diagram below. Similarly, if $b > 0$ the matrix $A = \left[\begin{array}{cc} 1 & 0 \\ 0 & b \end{array}\right]$ gives rise to ***y*-expansions** and ***y*-compressions**.



Example 2.2.17

If a is a number, the matrix transformation $T\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} x + ay \\ y \end{array}\right]$ induced by the matrix $A = \left[\begin{array}{cc} 1 & a \\ 0 & 1 \end{array}\right]$ is called an ***x*-shear** of $\mathbb{R}^2$ (**positive** if $a > 0$ and **negative** if $a < 0$). Its effect is illustrated below when $a = \frac{1}{4}$ and $a = -\frac{1}{4}$.



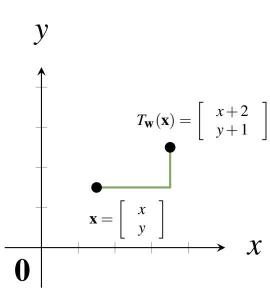


Figure 2.2.6

We hasten to note that there are important geometric transformations that are *not* matrix transformations. For example, if $\mathbf{w}$ is a fixed column in $\mathbb{R}^n$, define the transformation $T_{\mathbf{w}} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$T_{\mathbf{w}}(\mathbf{x}) = \mathbf{x} + \mathbf{w} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

Then $T_{\mathbf{w}}$ is called **translation** by $\mathbf{w}$. In particular, if $\mathbf{w} = \left[\begin{array}{c} 2 \\ 1 \end{array}\right]$ in $\mathbb{R}^2$, the effect of $T_{\mathbf{w}}$ on $\begin{bmatrix} x \\ y \end{bmatrix}$ is to translate it two units to the right and one unit up (see Figure 2.2.6).

The translation $T_{\mathbf{w}}$ is not a matrix transformation unless $\mathbf{w} = \mathbf{0}$. Indeed, if $T_{\mathbf{w}}$ were induced by a matrix A, then $A\mathbf{x} = T_{\mathbf{w}}(\mathbf{x}) = \mathbf{x} + \mathbf{w}$ would hold for every $\mathbf{x}$ in $\mathbb{R}^n$. In particular, taking $\mathbf{x} = \mathbf{0}$ gives $\mathbf{w} = A\mathbf{0} = \mathbf{0}$.

Exercises for 2.2

Exercise 2.2.1 In each case find a system of equations that is equivalent to the given vector equation. (Do not solve the system.)

a. $x_1 \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -3 \end{bmatrix}$

b. $x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 2 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 2 \\ 0 \end{bmatrix}$

Exercise 2.2.2 In each case find a vector equation that is equivalent to the given system of equations. (Do not solve the equation.)

a. $$\begin{aligned} x_1 - x_2 + 3x_3 &= 5 \\ -3x_1 + x_2 + x_3 &= -6 \\ 5x_1 - 8x_2 &= 9 \end{aligned}$$

b. $$\begin{aligned} x_1 - 2x_2 - x_3 + x_4 &= 5 \\ -x_1 + x_3 - 2x_4 &= -3 \\ 2x_1 - 2x_2 + 7x_3 &= 8 \\ 3x_1 - 4x_2 + 9x_3 - 2x_4 &= 12 \end{aligned}$$

Exercise 2.2.3 In each case compute $A\mathbf{x}$ using: (i) Definition 2.5. (ii) Theorem 2.2.5.

a. $A = \begin{bmatrix} 3 & -2 & 0 \\ 5 & -4 & 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

b. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 5 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

c. $A = \begin{bmatrix} -2 & 0 & 5 & 4 \\ 1 & 2 & 0 & 3 \\ -5 & 6 & -7 & 8 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

d. $A = \begin{bmatrix} 3 & -4 & 1 & 6 \\ 0 & 2 & 1 & 5 \\ -8 & 7 & -3 & 0 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

Exercise 2.2.4 Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix}$ be the 3×4 matrix given in terms of its columns $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, and $\mathbf{a}_4 = \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix}$. In each case either express $\mathbf{b}$ as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$, or show that it is not such a linear combination. Explain what your answer means for the corresponding system $A\mathbf{x} = \mathbf{b}$ of linear equations.

a. $\mathbf{b} = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$

b. $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$

Exercise 2.2.5 In each case, express every solution of the system as a sum of a specific solution plus a solution of the associated homogeneous system.

a. $$\begin{aligned} x + y + z &= 2 \\ 2x + y &= 3 \\ x - y - 3z &= 0 \end{aligned}$$

b. $$\begin{aligned} x - y - 4z &= -4 \\ x + 2y + 5z &= 2 \\ x + y + 2z &= 0 \end{aligned}$$

c. $$\begin{aligned} x_1 + x_2 - x_3 - 5x_5 &= 2 \\ x_2 + x_3 - 4x_5 &= -1 \\ x_2 + x_3 + x_4 - x_5 &= -1 \\ 2x_1 - 4x_3 + x_4 + x_5 &= 6 \end{aligned}$$

d. $$\begin{aligned} 2x_1 + x_2 - x_3 - x_4 &= -1 \\ 3x_1 + x_2 + x_3 - 2x_4 &= -2 \\ -x_1 - x_2 + 2x_3 + x_4 &= 2 \\ -2x_1 - x_2 + 2x_4 &= 3 \end{aligned}$$

Exercise 2.2.6 If $\mathbf{x}_0$ and $\mathbf{x}_1$ are solutions to the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$, use Theorem 2.2.2 to show that $s\mathbf{x}_0 + t\mathbf{x}_1$ is also a solution for any scalars s and t (called a **linear combination** of $\mathbf{x}_0$ and $\mathbf{x}_1$).

Exercise 2.2.7 Assume that $A\left[\begin{array}{r} 1 \\ -1 \\ 2 \end{array}\right] = \mathbf{0} = A\left[\begin{array}{r} 2 \\ 0 \\ 3 \end{array}\right]$. Show that $\mathbf{x}_0 = \left[\begin{array}{r} 2 \\ -1 \\ 3 \end{array}\right]$ is a solution to $A\mathbf{x} = \mathbf{b}$. Find a two-parameter family of solutions to $A\mathbf{x} = \mathbf{b}$.

Exercise 2.2.8 In each case write the system in the form $A\mathbf{x} = \mathbf{b}$, use the gaussian algorithm to solve the system, and express the solution as a particular solution plus a linear combination of basic solutions to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.

a. $$\begin{array}{rrrrrr} x_1 - & 2x_2 + & x_3 + & 4x_4 - & x_5 = & 8 \\ -2x_1 + & 4x_2 + & x_3 - & 2x_4 - & 4x_5 = & -1 \\ 3x_1 - & 6x_2 + & 8x_3 + & 4x_4 - & 13x_5 = & 1 \\ 8x_1 - & 16x_2 + & 7x_3 + & 12x_4 - & 6x_5 = & 11 \end{array}$$

b. $$\begin{array}{rrrrrr} x_1 - & 2x_2 + & x_3 + & 2x_4 + & 3x_5 = & -4 \\ -3x_1 + & 6x_2 - & 2x_3 - & 3x_4 - & 11x_5 = & 11 \\ -2x_1 + & 4x_2 - & x_3 + & x_4 - & 8x_5 = & 7 \\ -x_1 + & 2x_2 & & +3x_4 - & 5x_5 = & 3 \end{array}$$

Exercise 2.2.9 Given vectors $\mathbf{a}_1 = \left[\begin{array}{r} 1 \\ 0 \\ 1 \end{array}\right]$, $\mathbf{a}_2 = \left[\begin{array}{r} 1 \\ 1 \\ 0 \end{array}\right]$, and $\mathbf{a}_3 = \left[\begin{array}{r} 0 \\ -1 \\ 1 \end{array}\right]$, find a vector $\mathbf{b}$ that is *not* a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. Justify your answer. [*Hint*: Part (2) of Theorem 2.2.1.]

Exercise 2.2.10 In each case either show that the statement is true, or give an example showing that it is false.

a. $\left[\begin{array}{r} 3 \\ 2 \end{array}\right]$ is a linear combination of $\left[\begin{array}{r} 1 \\ 0 \end{array}\right]$ and $\left[\begin{array}{r} 0 \\ 1 \end{array}\right]$.

b. If $A\mathbf{x}$ has a zero entry, then A has a row of zeros.

c. If $A\mathbf{x} = \mathbf{0}$ where $\mathbf{x} \neq \mathbf{0}$, then $A = 0$.

d. Every linear combination of vectors in $\mathbb{R}^n$ can be written in the form $A\mathbf{x}$.

e. If $A = \left[\begin{array}{ccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{array}\right]$ in terms of its columns, and if $\mathbf{b} = 3\mathbf{a}_1 - 2\mathbf{a}_2$, then the system $A\mathbf{x} = \mathbf{b}$ has a solution.

f. If $A = \left[\begin{array}{ccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{array}\right]$ in terms of its columns, and if the system $A\mathbf{x} = \mathbf{b}$ has a solution, then $\mathbf{b} = s\mathbf{a}_1 + t\mathbf{a}_2$ for some s, t.

g. If A is $m \times n$ and $m < n$, then $A\mathbf{x} = \mathbf{b}$ has a solution for every column $\mathbf{b}$.

h. If $A\mathbf{x} = \mathbf{b}$ has a solution for some column $\mathbf{b}$, then it has a solution for every column $\mathbf{b}$.

i. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x}_1 - \mathbf{x}_2$ is a solution to $A\mathbf{x} = \mathbf{0}$.

j. Let $A = \left[\begin{array}{ccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{array}\right]$ in terms of its columns. If $\mathbf{a}_3 = s\mathbf{a}_1 + t\mathbf{a}_2$, then $A\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = \left[\begin{array}{r} s \\ t \\ -1 \end{array}\right]$.

Exercise 2.2.11 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation. In each case show that T is induced by a matrix and find the matrix.

a. T is a reflection in the y axis.

b. T is a reflection in the line $y = x$.

c. T is a reflection in the line $y = -x$.

d. T is a clockwise rotation through $\frac{\pi}{2}$.

Exercise 2.2.12 The **projection** $P : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $P\left[\begin{array}{c} x \\ y \\ z \end{array}\right] = \left[\begin{array}{c} x \\ y \end{array}\right]$ for all $\left[\begin{array}{c} x \\ y \\ z \end{array}\right]$ in $\mathbb{R}^3$. Show that P is induced by a matrix and find the matrix.

Exercise 2.2.13 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a transformation. In each case show that T is induced by a matrix and find the matrix.

a. T is a reflection in the $x - y$ plane.

b. T is a reflection in the $y - z$ plane.

Exercise 2.2.14 Fix $a > 0$ in $\mathbb{R}$, and define $T_a : \mathbb{R}^4 \to \mathbb{R}^4$ by $T_a(\mathbf{x}) = a\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^4$. Show that T is induced by a matrix and find the matrix. [T is called a **dilation** if $a > 1$ and a **contraction** if $a < 1$.]

Exercise 2.2.15 Let A be $m \times n$ and let $\mathbf{x}$ be in $\mathbb{R}^n$. If A has a row of zeros, show that $A\mathbf{x}$ has a zero entry.

Exercise 2.2.16 If a vector $\mathbf{b}$ is a linear combination of the columns of A, show that the system $A\mathbf{x} = \mathbf{b}$ is consistent (that is, it has at least one solution.)

Exercise 2.2.17 If a system $A\mathbf{x} = \mathbf{b}$ is inconsistent (no solution), show that $\mathbf{b}$ is not a linear combination of the columns of A.

Exercise 2.2.18 Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$.

a. Show that $\mathbf{x}_1 + \mathbf{x}_2$ is a solution to $A\mathbf{x} = \mathbf{0}$.

b. Show that $t\mathbf{x}_1$ is a solution to $A\mathbf{x} = \mathbf{0}$ for any scalar t.

Exercise 2.2.19 Suppose $\mathbf{x}_1$ is a solution to the system $A\mathbf{x} = \mathbf{b}$. If $\mathbf{x}_0$ is any nontrivial solution to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$, show that $\mathbf{x}_1 + t\mathbf{x}_0$, t a scalar, is an infinite one parameter family of solutions to $A\mathbf{x} = \mathbf{b}$. [*Hint*: Example 2.1.7 Section 2.1.]

Exercise 2.2.20 Let A and B be matrices of the same size. If $\mathbf{x}$ is a solution to both the system $A\mathbf{x} = \mathbf{0}$ and the system $B\mathbf{x} = \mathbf{0}$, show that $\mathbf{x}$ is a solution to the system $(A + B)\mathbf{x} = \mathbf{0}$.

Exercise 2.2.21 If A is $m \times n$ and $A\mathbf{x} = \mathbf{0}$ for every $\mathbf{x}$ in $\mathbb{R}^n$, show that $A = 0$ is the zero matrix. [*Hint*: Consider $A\mathbf{e}_j$ where $\mathbf{e}_j$ is the jth column of I_n; that is, $\mathbf{e}_j$ is the vector in $\mathbb{R}^n$ with 1 as entry j and every other entry 0.]

Exercise 2.2.22 Prove part (1) of Theorem 2.2.2.

Exercise 2.2.23 Prove part (2) of Theorem 2.2.2.

2.3 Matrix Multiplication

In Section 2.2 matrix-vector products were introduced. If A is an $m \times n$ matrix, the product $A\mathbf{x}$ was defined for any n-column $\mathbf{x}$ in $\mathbb{R}^n$ as follows: If $A = \left[\begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{array} \right]$ where the $\mathbf{a}_j$ are the columns of A, and if $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right]$, Definition 2.5 reads

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n \tag{2.5}$$

This was motivated as a way of describing systems of linear equations with coefficient matrix A. Indeed every such system has the form $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b}$ is the column of constants.

In this section we extend this matrix-vector multiplication to a way of multiplying matrices in general, and then investigate matrix algebra for its own sake. While it shares several properties of ordinary arithmetic, it will soon become clear that matrix arithmetic is different in a number of ways.

Matrix multiplication is closely related to composition of transformations.

Composition and Matrix Multiplication

Sometimes two transformations "link" together as follows:

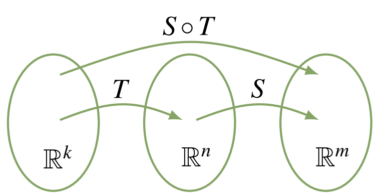
$$\mathbb{R}^k \xrightarrow{T} \mathbb{R}^n \xrightarrow{S} \mathbb{R}^m$$

In this case we can apply T first and then apply S, and the result is a new transformation

$$S \circ T : \mathbb{R}^k \to \mathbb{R}^m$$

called the **composite** of S and T, defined by

$$(S \circ T)(\mathbf{x}) = S[T(\mathbf{x})] \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^k$$



The action of $S \circ T$ can be described as "first T then S" (note the order!)[6]. This new transformation is described in the diagram. The reader will have encountered composition of ordinary functions: For example, consider $\mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R}$ where $f(x) = x^2$ and $g(x) = x + 1$ for all x in $\mathbb{R}$. Then

$$\begin{aligned}
(f \circ g)(x) &= f[g(x)] = f(x+1) = (x+1)^2 \\
(g \circ f)(x) &= g[f(x)] = g(x^2) = x^2 + 1
\end{aligned}$$

for all x in $\mathbb{R}$.

Our concern here is with matrix transformations. Suppose that A is an $m \times n$ matrix and B is an $n \times k$ matrix, and let $\mathbb{R}^k \xrightarrow{T_B} \mathbb{R}^n \xrightarrow{T_A} \mathbb{R}^m$ be the matrix transformations induced by B and A respectively, that is:

$$T_B(\mathbf{x}) = B\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^k \quad \text{and} \quad T_A(\mathbf{y}) = A\mathbf{y} \text{ for all } \mathbf{y} \text{ in } \mathbb{R}^n$$

Write $B = \left[\begin{array}{cccc} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{array} \right]$ where $\mathbf{b}_j$ denotes column j of B for each j. Hence each $\mathbf{b}_j$ is an n-vector (B is $n \times k$) so we can form the matrix-vector product $A\mathbf{b}_j$. In particular, we obtain an $m \times k$ matrix

$$\left[\begin{array}{cccc} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_k \end{array} \right]$$

with columns $A\mathbf{b}_1, A\mathbf{b}_2, \cdots, A\mathbf{b}_k$. Now compute $(T_A \circ T_B)(\mathbf{x})$ for any $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_k \end{array} \right]$ in $\mathbb{R}^k$:

$$\begin{array}{rcll}
(T_A \circ T_B)(\mathbf{x}) & = & T_A[T_B(\mathbf{x})] & \text{Definition of } T_A \circ T_B \\
& = & A(B\mathbf{x}) & A \text{ and } B \text{ induce } T_A \text{ and } T_B \\
& = & A(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_k\mathbf{b}_k) & \text{Equation 2.5 above} \\
& = & A(x_1\mathbf{b}_1) + A(x_2\mathbf{b}_2) + \cdots + A(x_k\mathbf{b}_k) & \text{Theorem 2.2.2} \\
& = & x_1(A\mathbf{b}_1) + x_2(A\mathbf{b}_2) + \cdots + x_k(A\mathbf{b}_k) & \text{Theorem 2.2.2} \\
& = & \left[\begin{array}{cccc} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_k \end{array} \right] \mathbf{x} & \text{Equation 2.5 above}
\end{array}$$

Because $\mathbf{x}$ was an arbitrary vector in $\mathbb{R}^n$, this shows that $T_A \circ T_B$ is the matrix transformation induced by the matrix $\left[\begin{array}{cccc} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_n \end{array} \right]$. This motivates the following definition.

---

[6] When reading the notation $S \circ T$, we read S first and then T even though the action is "first T then S". This annoying state of affairs results because we write $T(\mathbf{x})$ for the effect of the transformation T on $\mathbf{x}$, with T on the left. If we wrote this instead as $(\mathbf{x})T$, the confusion would not occur. However the notation $T(\mathbf{x})$ is well established.

Definition 2.9 Matrix Multiplication

Let A be an $m \times n$ matrix, let B be an $n \times k$ matrix, and write $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{bmatrix}$ where $\mathbf{b}_j$ is column j of B for each j. The product matrix AB is the $m \times k$ matrix defined as follows:

$$AB = A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_k \end{bmatrix}$$

Thus the product matrix AB is given in terms of its columns $A\mathbf{b}_1, A\mathbf{b}_2, \ldots, A\mathbf{b}_n$: Column j of AB is the matrix-vector product $A\mathbf{b}_j$ of A and the corresponding column $\mathbf{b}_j$ of B. Note that each such product $A\mathbf{b}_j$ makes sense by Definition 2.5 because A is $m \times n$ and each $\mathbf{b}_j$ is in $\mathbb{R}^n$ (since B has n rows). Note also that if B is a column matrix, this definition reduces to Definition 2.5 for matrix-vector multiplication.

Given matrices A and B, Definition 2.9 and the above computation give

$$A(B\mathbf{x}) = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_n \end{bmatrix} \mathbf{x} = (AB)\mathbf{x}$$

for all $\mathbf{x}$ in $\mathbb{R}^k$. We record this for reference.

Theorem 2.3.1

Let A be an $m \times n$ matrix and let B be an $n \times k$ matrix. Then the product matrix AB is $m \times k$ and satisfies

$$A(B\mathbf{x}) = (AB)\mathbf{x} \quad \textit{for all } \mathbf{x} \textit{ in } \mathbb{R}^k$$

Here is an example of how to compute the product AB of two matrices using Definition 2.9.

Example 2.3.1

Compute AB if $A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 1 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 9 \\ 7 & 2 \\ 6 & 1 \end{bmatrix}$.

Solution. The columns of B are $\mathbf{b}_1 = \begin{bmatrix} 8 \\ 7 \\ 6 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 9 \\ 2 \\ 1 \end{bmatrix}$, so Definition 2.5 gives

$$A\mathbf{b}_1 = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} 8 \\ 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 67 \\ 78 \\ 55 \end{bmatrix} \text{ and } A\mathbf{b}_2 = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} 9 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 29 \\ 24 \\ 10 \end{bmatrix}$$

Hence Definition 2.9 above gives $AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 67 & 29 \\ 78 & 24 \\ 55 & 10 \end{bmatrix}$.

Example 2.3.2

If A is $m \times n$ and B is $n \times k$, Theorem 2.3.1 gives a simple formula for the composite of the matrix transformations T_A and T_B:

$$T_A \circ T_B = T_{AB}$$

Solution. Given any $\mathbf{x}$ in $\mathbb{R}^k$,

$$\begin{aligned}(T_A \circ T_B)(\mathbf{x}) &= T_A[T_B(\mathbf{x})] \\ &= A[B\mathbf{x}] \\ &= (AB)\mathbf{x} \\ &= T_{AB}(\mathbf{x})\end{aligned}$$

While Definition 2.9 is important, there is another way to compute the matrix product AB that gives a way to calculate each individual entry. In Section 2.2 we defined the dot product of two n-tuples to be the sum of the products of corresponding entries. We went on to show (Theorem 2.2.5) that if A is an $m \times n$ matrix and $\mathbf{x}$ is an n-vector, then entry j of the product $A\mathbf{x}$ is the dot product of row j of A with $\mathbf{x}$. This observation was called the "dot product rule" for matrix-vector multiplication, and the next theorem shows that it extends to matrix multiplication in general.

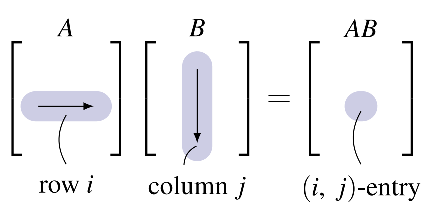
Theorem 2.3.2: Dot Product Rule

Let A and B be matrices of sizes $m \times n$ and $n \times k$, respectively. Then the (i, j)-entry of AB is the dot product of row i of A with column j of B.

Proof. Write $B = \left[\begin{array}{cccc} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{array}\right]$ in terms of its columns. Then $A\mathbf{b}_j$ is column j of AB for each j. Hence the (i, j)-entry of AB is entry i of $A\mathbf{b}_j$, which is the dot product of row i of A with $\mathbf{b}_j$. This proves the theorem. □

Thus to compute the (i, j)-entry of AB, proceed as follows (see the diagram):

Go *across* row i of A, and *down* column j of B, multiply corresponding entries, and add the results.



Note that this requires that the rows of A must be the same length as the columns of B. The following rule is useful for remembering this and for deciding the size of the product matrix AB.

Compatibility Rule

$$\begin{array}{cc} A & B \\ m \times n & n' \times k \end{array}$$

Let A and B denote matrices. If A is $m \times n$ and B is $n' \times k$, the product AB can be formed if and only if $n = n'$. In this case the size of the product matrix AB is $m \times k$, and we say that AB is **defined**, or that A and B are **compatible** for multiplication.

The diagram provides a useful mnemonic for remembering this. We adopt the following convention:

Convention

Whenever a product of matrices is written, it is tacitly assumed that the sizes of the factors are such that the product is defined.

To illustrate the dot product rule, we recompute the matrix product in Example 2.3.1.

Example 2.3.3

Compute AB if $A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 1 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 9 \\ 7 & 2 \\ 6 & 1 \end{bmatrix}$.

Solution. Here A is 3×3 and B is 3×2, so the product matrix AB is defined and will be of size 3×2. Theorem 2.3.2 gives each entry of AB as the dot product of the corresponding row of A with the corresponding column of B_j that is,

$$AB = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} 8 & 9 \\ 7 & 2 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 8 + 3\cdot 7 + 5\cdot 6 & 2\cdot 9 + 3\cdot 2 + 5\cdot 1 \\ 1\cdot 8 + 4\cdot 7 + 7\cdot 6 & 1\cdot 9 + 4\cdot 2 + 7\cdot 1 \\ 0\cdot 8 + 1\cdot 7 + 8\cdot 6 & 0\cdot 9 + 1\cdot 2 + 8\cdot 1 \end{bmatrix} = \begin{bmatrix} 67 & 29 \\ 78 & 24 \\ 55 & 10 \end{bmatrix}$$

Of course, this agrees with Example 2.3.1.

Example 2.3.4

Compute the $(1,\ 3)$- and $(2,\ 4)$-entries of AB where

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix}.$$

Then compute AB.

Solution. The $(1,\ 3)$-entry of AB is the dot product of row 1 of A and column 3 of B (highlighted in the following display), computed by multiplying corresponding entries and adding the results.

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix} \quad (1,\ 3)\text{-entry} = 3\cdot 6 + (-1)\cdot 3 + 2\cdot 5 = 25$$

Similarly, the $(2,\ 4)$-entry of AB involves row 2 of A and column 4 of B.

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix} \quad (2,\ 4)\text{-entry} = 0 \cdot 0 + 1 \cdot 4 + 4 \cdot 8 = 36$$

Since A is 2×3 and B is 3×4, the product is 2×4.

$$AB = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 25 & 12 \\ -4 & 2 & 23 & 36 \end{bmatrix}$$

Example 2.3.5

If $A = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix}$, compute A^2, AB, BA, and B^2 when they are defined.[7]

Solution. Here, A is a 1×3 matrix and B is a 3×1 matrix, so A^2 and B^2 are not defined. However, the compatibility rule reads

$$\begin{array}{cc} A & B \\ 1 \times 3 & 3 \times 1 \end{array} \quad \text{and} \quad \begin{array}{cc} B & A \\ 3 \times 1 & 1 \times 3 \end{array}$$

so both AB and BA can be formed and these are 1×1 and 3×3 matrices, respectively.

$$AB = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 3 \cdot 6 + 2 \cdot 4 \end{bmatrix} = \begin{bmatrix} 31 \end{bmatrix}$$

$$BA = \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 \cdot 1 & 5 \cdot 3 & 5 \cdot 2 \\ 6 \cdot 1 & 6 \cdot 3 & 6 \cdot 2 \\ 4 \cdot 1 & 4 \cdot 3 & 4 \cdot 2 \end{bmatrix} = \begin{bmatrix} 5 & 15 & 10 \\ 6 & 18 & 12 \\ 4 & 12 & 8 \end{bmatrix}$$

Unlike numerical multiplication, matrix products AB and BA *need not be equal*. In fact they need not even be the same size, as Example 2.3.5 shows. It turns out to be rare that $AB = BA$ (although it is by no means impossible), and A and B are said to **commute** when this happens.

Example 2.3.6

Let $A = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$. Compute A^2, AB, BA.

[7] As for numbers, we write $A^2 = A \cdot A$, $A^3 = A \cdot A \cdot A$, etc. Note that A^2 is defined if and only if A is of size $n \times n$ for some n.

Solution. $A^2 = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix} \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $A^2 = 0$ can occur even if $A \neq 0$. Next,

$$AB = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 12 \\ 2 & -8 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ -6 & -9 \end{bmatrix}$$

Hence $AB \neq BA$, even though AB and BA are the same size.

Example 2.3.7

If A is any matrix, then $IA = A$ and $AI = A$, and where I denotes an identity matrix of a size so that the multiplications are defined.

Solution. These both follow from the dot product rule as the reader should verify. For a more formal proof, write $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ where $\mathbf{a}_j$ is column j of A. Then Definition 2.9 and Example 2.2.11 give

$$IA = \begin{bmatrix} I\mathbf{a}_1 & I\mathbf{a}_2 & \cdots & I\mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} = A$$

If $\mathbf{e}_j$ denotes column j of I, then $A\mathbf{e}_j = \mathbf{a}_j$ for each j by Example 2.2.12. Hence Definition 2.9 gives:

$$AI = A\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{e}_1 & A\mathbf{e}_2 & \cdots & A\mathbf{e}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} = A$$

The following theorem collects several results about matrix multiplication that are used everywhere in linear algebra.

Theorem 2.3.3

Assume that a is any scalar, and that A, B, and C are matrices of sizes such that the indicated matrix products are defined. Then:

1. *$IA = A$ and $AI = A$ where I denotes an identity matrix.*
2. *$A(BC) = (AB)C$.*
3. *$A(B+C) = AB + AC$.*
4. *$(B+C)A = BA + CA$.*
5. *$a(AB) = (aA)B = A(aB)$.*
6. *$(AB)^T = B^T A^T$.*

Proof. Condition (1) is Example 2.3.7; we prove (2), (4), and (6) and leave (3) and (5) as exercises.

1. If $C = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_k \end{bmatrix}$ in terms of its columns, then $BC = \begin{bmatrix} B\mathbf{c}_1 & B\mathbf{c}_2 & \cdots & B\mathbf{c}_k \end{bmatrix}$ by Defini-

tion 2.9, so

$$\begin{aligned} A(BC) &= \left[\begin{array}{cccc} A(B\mathbf{c}_1) & A(B\mathbf{c}_2) & \cdots & A(B\mathbf{c}_k) \end{array}\right] && \text{Definition 2.9} \\ &= \left[\begin{array}{cccc} (AB)\mathbf{c}_1 & (AB)\mathbf{c}_2 & \cdots & (AB)\mathbf{c}_k) \end{array}\right] && \text{Theorem 2.3.1} \\ &= (AB)C && \text{Definition 2.9} \end{aligned}$$

4. We know (Theorem 2.2.2) that $(B+C)\mathbf{x} = B\mathbf{x} + C\mathbf{x}$ holds for every column $\mathbf{x}$. If we write $A = \left[\begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{array}\right]$ in terms of its columns, we get

$$\begin{aligned} (B+C)A &= \left[\begin{array}{cccc} (B+C)\mathbf{a}_1 & (B+C)\mathbf{a}_2 & \cdots & (B+C)\mathbf{a}_n \end{array}\right] && \text{Definition 2.9} \\ &= \left[\begin{array}{cccc} B\mathbf{a}_1 + C\mathbf{a}_1 & B\mathbf{a}_2 + C\mathbf{a}_2 & \cdots & B\mathbf{a}_n + C\mathbf{a}_n \end{array}\right] && \text{Theorem 2.2.2} \\ &= \left[\begin{array}{cccc} B\mathbf{a}_1 & B\mathbf{a}_2 & \cdots & B\mathbf{a}_n \end{array}\right] + \left[\begin{array}{cccc} C\mathbf{a}_1 & C\mathbf{a}_2 & \cdots & C\mathbf{a}_n \end{array}\right] && \text{Adding Columns} \\ &= BA + CA && \text{Definition 2.9} \end{aligned}$$

6. As in Section 2.1, write $A = [a_{ij}]$ and $B = [b_{ij}]$, so that $A^T = [a'_{ij}]$ and $B^T = [b'_{ij}]$ where $a'_{ij} = a_{ji}$ and $b'_{ji} = b_{ij}$ for all i and j. If c_{ij} denotes the $(i,\ j)$-entry of $B^T A^T$, then c_{ij} is the dot product of row i of B^T with column j of A^T. Hence

$$\begin{aligned} c_{ij} = b'_{i1}a'_{1j} + b'_{i2}a'_{2j} + \cdots + b'_{im}a'_{mj} &= b_{1i}a_{j1} + b_{2i}a_{j2} + \cdots + b_{mi}a_{jm} \\ &= a_{j1}b_{1i} + a_{j2}b_{2i} + \cdots + a_{jm}b_{mi} \end{aligned}$$

But this is the dot product of row j of A with column i of B; that is, the $(j,\ i)$-entry of AB; that is, the $(i,\ j)$-entry of $(AB)^T$. This proves (6). □

Property 2 in Theorem 2.3.3 is called the **associative law** of matrix multiplication. It asserts that the equation $A(BC) = (AB)C$ holds for all matrices (if the products are defined). Hence this product is the same no matter how it is formed, and so is written simply as ABC. This extends: The product $ABCD$ of four matrices can be formed several ways—for example, $(AB)(CD)$, $[A(BC)]D$, and $A[B(CD)]$—but the associative law implies that they are all equal and so are written as $ABCD$. A similar remark applies in general: Matrix products can be written unambiguously with no parentheses.

However, a note of caution about matrix multiplication must be taken: The fact that AB and BA need *not* be equal means that the *order* of the factors is important in a product of matrices. For example $ABCD$ and $ADCB$ may *not* be equal.

Warning

If the order of the factors in a product of matrices is changed, the product matrix may change (or may not be defined). Ignoring this warning is a source of many errors by students of linear algebra!

Properties 3 and 4 in Theorem 2.3.3 are called **distributive laws**. They assert that $A(B+C) = AB + AC$ and $(B+C)A = BA + CA$ hold whenever the sums and products are defined. These rules extend to more than two terms and, together with Property 5, ensure that many manipulations familiar from ordinary algebra extend to matrices. For example

$$\begin{aligned} A(2B - 3C + D - 5E) &= 2AB - 3AC + AD - 5AE \\ (A + 3C - 2D)B &= AB + 3CB - 2DB \end{aligned}$$

Note again that the warning is in effect: For example $A(B-C)$ need *not* equal $AB - CA$. These rules make possible a lot of simplification of matrix expressions.

Example 2.3.8

Simplify the expression $A(BC - CD) + A(C - B)D - AB(C - D)$.

Solution.

$$\begin{aligned} A(BC - CD) + A(C - B)D - AB(C - D) &= A(BC) - A(CD) + (AC - AB)D - (AB)C + (AB)D \\ &= ABC - ACD + ACD - ABD - ABC + ABD \\ &= 0 \end{aligned}$$

Example 2.3.9 and Example 2.3.10 below show how we can use the properties in Theorem 2.3.2 to deduce other facts about matrix multiplication. Matrices A and B are said to **commute** if $AB = BA$.

Example 2.3.9

Suppose that A, B, and C are $n \times n$ matrices and that both A and B commute with C; that is, $AC = CA$ and $BC = CB$. Show that AB commutes with C.

Solution. Showing that AB commutes with C means verifying that $(AB)C = C(AB)$. The computation uses the associative law several times, as well as the given facts that $AC = CA$ and $BC = CB$.

$$(AB)C = A(BC) = A(CB) = (AC)B = (CA)B = C(AB)$$

Example 2.3.10

Show that $AB = BA$ if and only if $(A - B)(A + B) = A^2 - B^2$.

Solution. The following *always* holds:

$$(A - B)(A + B) = A(A + B) - B(A + B) = A^2 + AB - BA - B^2 \tag{2.6}$$

Hence if $AB = BA$, then $(A - B)(A + B) = A^2 - B^2$ follows. Conversely, if this last equation holds, then equation (2.6) becomes

$$A^2 - B^2 = A^2 + AB - BA - B^2$$

This gives $0 = AB - BA$, and $AB = BA$ follows.

In Section 2.2 we saw (in Theorem 2.2.1) that every system of linear equations has the form

$$A\mathbf{x} = \mathbf{b}$$

where A is the coefficient matrix, $\mathbf{x}$ is the column of variables, and $\mathbf{b}$ is the constant matrix. Thus the *system* of linear equations becomes a single matrix equation. Matrix multiplication can yield information about such a system.

Example 2.3.11

Consider a system $A\mathbf{x} = \mathbf{b}$ of linear equations where A is an $m \times n$ matrix. Assume that a matrix C exists such that $CA = I_n$. If the system $A\mathbf{x} = \mathbf{b}$ *has* a solution, show that this solution must be $C\mathbf{b}$. Give a condition guaranteeing that $C\mathbf{b}$ *is in fact* a solution.

Solution. Suppose that $\mathbf{x}$ is any solution to the system, so that $A\mathbf{x} = \mathbf{b}$. Multiply both sides of this matrix equation by C to obtain, successively,

$$C(A\mathbf{x}) = C\mathbf{b}, \qquad (CA)\mathbf{x} = C\mathbf{b}, \qquad I_n\mathbf{x} = C\mathbf{b}, \qquad \mathbf{x} = C\mathbf{b}$$

This shows that *if* the system has a solution $\mathbf{x}$, then that solution must be $\mathbf{x} = C\mathbf{b}$, as required. But it does *not* guarantee that the system *has* a solution. However, if we write $\mathbf{x}_1 = C\mathbf{b}$, then

$$A\mathbf{x}_1 = A(C\mathbf{b}) = (AC)\mathbf{b}$$

Thus $\mathbf{x}_1 = C\mathbf{b}$ will be a solution if the condition $AC = I_m$ is satisfied.

The ideas in Example 2.3.11 lead to important information about matrices; this will be pursued in the next section.

Block Multiplication

Definition 2.10 Block Partition of a Matrix

It is often useful to consider matrices whose entries are themselves matrices (called ***blocks****). A matrix viewed in this way is said to be* ***partitioned into blocks****.*

For example, writing a matrix B in the form

$$B = \left[\begin{array}{cccc} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{array}\right] \text{ where the } \mathbf{b}_j \text{ are the columns of } B$$

is such a block partition of B. Here is another example.

Consider the matrices

$$A = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 2 & -1 & 4 & 2 & 1 \\ 3 & 1 & -1 & 7 & 5 \end{array}\right] = \left[\begin{array}{cc} I_2 & 0_{23} \\ P & Q \end{array}\right] \quad \text{and} \quad B = \left[\begin{array}{cc} 4 & -2 \\ 5 & 6 \\ \hline 7 & 3 \\ -1 & 0 \\ 1 & 6 \end{array}\right] = \left[\begin{array}{c} X \\ Y \end{array}\right]$$

where the blocks have been labelled as indicated. This is a natural way to partition A into blocks in view of the blocks I_2 and 0_{23} that occur. This notation is particularly useful when we are multiplying the matrices A and B because the product AB can be computed in block form as follows:

$$AB = \begin{bmatrix} I & 0 \\ P & Q \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} IX + 0Y \\ PX + QY \end{bmatrix} = \begin{bmatrix} X \\ PX + QY \end{bmatrix} = \left[\begin{array}{cc} 4 & -2 \\ 5 & 6 \\ \hline 30 & 8 \\ 8 & 27 \end{array}\right]$$

This is easily checked to be the product AB, computed in the conventional manner.

In other words, *we can compute the product AB by ordinary matrix multiplication, using blocks as entries*. The only requirement is that the blocks be **compatible**. That is, *the sizes of the blocks must be such that all (matrix) products of blocks that occur make sense*. This means that the number of columns in each block of A must equal the number of rows in the corresponding block of B.

Theorem 2.3.4: Block Multiplication

If matrices A and B are partitioned compatibly into blocks, the product AB can be computed by matrix multiplication using blocks as entries.

We omit the proof.

We have been using two cases of block multiplication. If $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{bmatrix}$ is a matrix where the $\mathbf{b}_j$ are the columns of B, and if the matrix product AB is defined, then we have

$$AB = A\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_k \end{bmatrix}$$

This is Definition 2.9 and is a block multiplication where $A = [A]$ has only one block. As another illustration,

$$B\mathbf{x} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_k\mathbf{b}_k$$

where $\mathbf{x}$ is any $k \times 1$ column matrix (this is Definition 2.5).

It is not our intention to pursue block multiplication in detail here. However, we give one more example because it will be used below.

Theorem 2.3.5

Suppose matrices $A = \begin{bmatrix} B & X \\ 0 & C \end{bmatrix}$ *and* $A_1 = \begin{bmatrix} B_1 & X_1 \\ 0 & C_1 \end{bmatrix}$ *are partitioned as shown where B and B_1 are square matrices of the same size, and C and C_1 are also square of the same size. These are compatible partitionings and block multiplication gives*

$$AA_1 = \begin{bmatrix} B & X \\ 0 & C \end{bmatrix} \begin{bmatrix} B_1 & X_1 \\ 0 & C_1 \end{bmatrix} = \begin{bmatrix} BB_1 & BX_1 + XC_1 \\ 0 & CC_1 \end{bmatrix}$$

Example 2.3.12

Obtain a formula for A^k where $A = \begin{bmatrix} I & X \\ 0 & 0 \end{bmatrix}$ is square and I is an identity matrix.

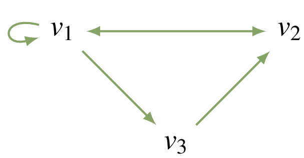
Solution. We have $A^2 = \begin{bmatrix} I & X \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & X \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I^2 & IX + X0 \\ 0 & 0^2 \end{bmatrix} = \begin{bmatrix} I & X \\ 0 & 0 \end{bmatrix} = A$. Hence $A^3 = AA^2 = AA = A^2 = A$. Continuing in this way, we see that $A^k = A$ for every $k \geq 1$.

Block multiplication has theoretical uses as we shall see. However, it is also useful in computing products of matrices in a computer with limited memory capacity. The matrices are partitioned into blocks in such a way that each product of blocks can be handled. Then the blocks are stored in auxiliary memory and their products are computed one by one.

Directed Graphs

The study of directed graphs illustrates how matrix multiplication arises in ways other than the study of linear equations or matrix transformations.

A **directed graph** consists of a set of points (called **vertices**) connected by arrows (called **edges**). For example, the vertices could represent cities and the edges available flights. If the graph has n vertices $v_1, v_2, \dots, v_n$, the **adjacency** matrix $A = \left[a_{ij}\right]$ is the $n \times n$ matrix whose (i, j)-entry a_{ij} is 1 if there is an edge from v_j to v_i (note the order), and zero otherwise. For example, the adjacency matrix of the directed graph shown is $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.



A **path of length** r (or an r-**path**) from vertex j to vertex i is a sequence of r edges leading from v_j to v_i. Thus $v_1 \to v_2 \to v_1 \to v_1 \to v_3$ is a 4-path from v_1 to v_3 in the given graph. The edges are just the paths of length 1, so the (i, j)-entry a_{ij} of the adjacency matrix A is the number of 1-paths from v_j to v_i. This observation has an important extension:

Theorem 2.3.6

If A is the adjacency matrix of a directed graph with n vertices, then the (i, j)-entry of A^r is the number of r-paths $v_j \to v_i$.

As an illustration, consider the adjacency matrix A in the graph shown. Then

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad A^3 = \begin{bmatrix} 4 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Hence, since the $(2, 1)$-entry of A^2 is 2, there are two 2-paths $v_1 \to v_2$ (in fact they are $v_1 \to v_1 \to v_2$ and $v_1 \to v_3 \to v_2$). Similarly, the $(2, 3)$-entry of A^2 is zero, so there are *no* 2-paths $v_3 \to v_2$, as the reader can verify. The fact that no entry of A^3 is zero shows that it is possible to go from any vertex to any other vertex in exactly three steps.

To see why Theorem 2.3.6 is true, observe that it asserts that

$$\text{the } (i,\ j)\text{-entry of } A^r \text{ equals the number of } r\text{-paths } v_j \to v_i \tag{2.7}$$

holds for each $r \geq 1$. We proceed by induction on r (see Appendix C). The case $r = 1$ is the definition of the adjacency matrix. So assume inductively that (2.7) is true for some $r \geq 1$; we must prove that (2.7) also holds for $r+1$. But every $(r+1)$-path $v_j \to v_i$ is the result of an r-path $v_j \to v_k$ for some k, followed by a 1-path $v_k \to v_i$. Writing $A = \left[a_{ij}\right]$ and $A^r = \left[b_{ij}\right]$, there are b_{kj} paths of the former type (by induction) and a_{ik} of the latter type, and so there are $a_{ik}b_{kj}$ such paths in all. Summing over k, this shows that there are

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} \quad (r+1)\text{-paths } v_j \to v_i$$

But this sum is the dot product of the ith row $\left[\begin{array}{cccc} a_{i1} & a_{i2} & \cdots & a_{in} \end{array}\right]$ of A with the jth column $\left[\begin{array}{cccc} b_{1j} & b_{2j} & \cdots & b_{nj} \end{array}\right]^T$ of A^r. As such, it is the $(i,\ j)$-entry of the matrix product $A^rA = A^{r+1}$. This shows that (2.7) holds for $r+1$, as required.

Exercises for 2.3

Exercise 2.3.1 Compute the following matrix products.

a. $\left[\begin{array}{rr} 1 & 3 \\ 0 & -2 \end{array}\right]\left[\begin{array}{rr} 2 & -1 \\ 0 & 1 \end{array}\right]$

b. $\left[\begin{array}{rrr} 1 & -1 & 2 \\ 2 & 0 & 4 \end{array}\right]\left[\begin{array}{rrr} 2 & 3 & 1 \\ 1 & 9 & 7 \\ -1 & 0 & 2 \end{array}\right]$

c. $\left[\begin{array}{rrr} 5 & 0 & -7 \\ 1 & 5 & 9 \end{array}\right]\left[\begin{array}{r} 3 \\ 1 \\ -1 \end{array}\right]$

d. $\left[\begin{array}{rrr} 1 & 3 & -3 \end{array}\right]\left[\begin{array}{rr} 3 & 0 \\ -2 & 1 \\ 0 & 6 \end{array}\right]$

e. $\left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\left[\begin{array}{rr} 3 & -2 \\ 5 & -7 \\ 9 & 7 \end{array}\right]$

f. $\left[\begin{array}{rrr} 1 & -1 & 3 \end{array}\right]\left[\begin{array}{r} 2 \\ 1 \\ -8 \end{array}\right]$

g. $\left[\begin{array}{r} 2 \\ 1 \\ -7 \end{array}\right]\left[\begin{array}{rrr} 1 & -1 & 3 \end{array}\right]$

h. $\left[\begin{array}{rr} 3 & 1 \\ 5 & 2 \end{array}\right]\left[\begin{array}{rr} 2 & -1 \\ -5 & 3 \end{array}\right]$

i. $\left[\begin{array}{rrr} 2 & 3 & 1 \\ 5 & 7 & 4 \end{array}\right]\left[\begin{array}{ccc} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{array}\right]$

j. $\left[\begin{array}{ccc} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{array}\right]\left[\begin{array}{ccc} a' & 0 & 0 \\ 0 & b' & 0 \\ 0 & 0 & c' \end{array}\right]$

Exercise 2.3.2 In each of the following cases, find all possible products A^2, AB, AC, and so on.

a. $A = \left[\begin{array}{rrr} 1 & 2 & 3 \\ -1 & 0 & 0 \end{array}\right]$, $B = \left[\begin{array}{rr} 1 & -2 \\ \frac{1}{2} & 3 \end{array}\right]$, $C = \left[\begin{array}{rr} -1 & 0 \\ 2 & 5 \\ 0 & 5 \end{array}\right]$

b. $A = \left[\begin{array}{rrr} 1 & 2 & 4 \\ 0 & 1 & -1 \end{array}\right]$, $B = \left[\begin{array}{rr} -1 & 6 \\ 1 & 0 \end{array}\right]$, $C = \left[\begin{array}{rr} 2 & 0 \\ -1 & 1 \\ 1 & 2 \end{array}\right]$

Exercise 2.3.3 Find a, b, a_1, and b_1 if:

a. $\begin{bmatrix} a & b \\ a_1 & b_1 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ a_1 & b_1 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ -1 & 4 \end{bmatrix}$

Exercise 2.3.4 Verify that $A^2 - A - 6I = 0$ if:

a. $\begin{bmatrix} 3 & -1 \\ 0 & -2 \end{bmatrix}$ b. $\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$

Exercise 2.3.5

Given $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 5 & 8 \end{bmatrix}$, and $D = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 5 \end{bmatrix}$, verify the following facts from Theorem 2.3.1.

a. $A(B-D) = AB - AD$ b. $A(BC) = (AB)C$

c. $(CD)^T = D^T C^T$

Exercise 2.3.6 Let A be a 2×2 matrix.

a. If A commutes with $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, show that $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ for some a and b.

b. If A commutes with $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, show that $A = \begin{bmatrix} a & 0 \\ c & a \end{bmatrix}$ for some a and c.

c. Show that A commutes with *every* 2×2 matrix if and only if $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ for some a.

Exercise 2.3.7

a. If A^2 can be formed, what can be said about the size of A?

b. If AB and BA can both be formed, describe the sizes of A and B.

c. If ABC can be formed, A is 3×3, and C is 5×5, what size is B?

Exercise 2.3.8

a. Find two 2×2 matrices A such that $A^2 = 0$.

b. Find three 2×2 matrices A such that (i) $A^2 = I$; (ii) $A^2 = A$.

c. Find 2×2 matrices A and B such that $AB = 0$ but $BA \neq 0$.

Exercise 2.3.9 Write $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, and let A be $3 \times n$ and B be $m \times 3$.

a. Describe PA in terms of the rows of A.

b. Describe BP in terms of the columns of B.

Exercise 2.3.10 Let A, B, and C be as in Exercise 2.3.5. Find the $(3,\ 1)$-entry of CAB using exactly six numerical multiplications.

Exercise 2.3.11 Compute AB, using the indicated block partitioning.

$$A = \left[\begin{array}{rr|rr} 2 & -1 & 3 & 1 \\ 1 & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad B = \left[\begin{array}{rr|r} 1 & 2 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 5 & 1 \\ 1 & -1 & 0 \end{array}\right]$$

Exercise 2.3.12 In each case give formulas for all powers A, A^2, A^3, $\dots$ of A using the block decomposition indicated.

a. $A = \left[\begin{array}{r|rr} 1 & 0 & 0 \\ \hline 1 & 1 & -1 \\ 1 & -1 & 1 \end{array}\right]$

b. $A = \left[\begin{array}{rr|rr} 1 & -1 & 2 & -1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$

Exercise 2.3.13 Compute the following using block multiplication (all blocks are $k \times k$).

a. $\begin{bmatrix} I & X \\ -Y & I \end{bmatrix}\begin{bmatrix} I & 0 \\ Y & I \end{bmatrix}$ b. $\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}$

c. $\begin{bmatrix} I & X \end{bmatrix}\begin{bmatrix} I & X \end{bmatrix}^T$ d. $\begin{bmatrix} I & X^T \end{bmatrix}\begin{bmatrix} -X & I \end{bmatrix}^T$

e. $\begin{bmatrix} I & X \\ 0 & -I \end{bmatrix}^n$ any $n \geq 1$

f. $\begin{bmatrix} 0 & X \\ I & 0 \end{bmatrix}^n$ any $n \geq 1$

Exercise 2.3.14 Let A denote an $m \times n$ matrix.

a. If $AX = 0$ for every $n \times 1$ matrix X, show that $A = 0$.

b. If $YA = 0$ for every $1 \times m$ matrix Y, show that $A = 0$.

Exercise 2.3.15

a. If $U = \left[\begin{array}{rr} 1 & 2 \\ 0 & -1 \end{array}\right]$, and $AU = 0$, show that $A = 0$.

b. Let U be such that $AU = 0$ implies that $A = 0$. If $PU = QU$, show that $P = Q$.

Exercise 2.3.16 Simplify the following expressions where A, B, and C represent matrices.

a. $A(3B - C) + (A - 2B)C + 2B(C + 2A)$

b. $A(B + C - D) + B(C - A + D) - (A + B)C$ $+ (A - B)D$

c. $AB(BC - CB) + (CA - AB)BC + CA(A - B)C$

d. $(A - B)(C - A) + (C - B)(A - C) + (C - A)^2$

Exercise 2.3.17 If $A = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ where $a \neq 0$, show that A factors in the form $A = \left[\begin{array}{cc} 1 & 0 \\ x & 1 \end{array}\right] \left[\begin{array}{cc} y & z \\ 0 & w \end{array}\right]$.

Exercise 2.3.18 If A and B commute with C, show that the same is true of:

a. $A + B$

b. kA, k any scalar

Exercise 2.3.19 If A is any matrix, show that both AA^T and A^TA are symmetric.

Exercise 2.3.20 If A and B are symmetric, show that AB is symmetric if and only if $AB = BA$.

Exercise 2.3.21 If A is a 2×2 matrix, show that $A^TA = AA^T$ if and only if A is symmetric or $A = \left[\begin{array}{rr} a & b \\ -b & a \end{array}\right]$ for some a and b.

Exercise 2.3.22

a. Find all symmetric 2×2 matrices A such that $A^2 = 0$.

b. Repeat (a) if A is 3×3.

c. Repeat (a) if A is $n \times n$.

Exercise 2.3.23 Show that there exist no 2×2 matrices A and B such that $AB - BA = I$. [*Hint*: Examine the (1, 1)- and (2, 2)-entries.]

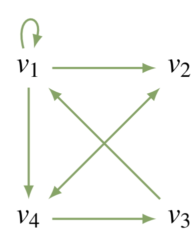
Exercise 2.3.24 Let B be an $n \times n$ matrix. Suppose $AB = 0$ for some nonzero $m \times n$ matrix A. Show that no $n \times n$ matrix C exists such that $BC = I$.

Exercise 2.3.25 An autoparts manufacturer makes fenders, doors, and hoods. Each requires assembly and packaging carried out at factories: Plant 1, Plant 2, and Plant 3. Matrix A below gives the number of hours for assembly and packaging, and matrix B gives the hourly rates at the three plants. Explain the meaning of the (3, 2)-entry in the matrix AB. Which plant is the most economical to operate? Give reasons.

$$\begin{array}{l} \\ \text{Fenders} \\ \text{Doors} \\ \text{Hoods} \end{array} \begin{array}{c} \begin{array}{cc} \text{Assembly} & \text{Packaging} \end{array} \\ \left[\begin{array}{cc} 12 & 2 \\ 21 & 3 \\ 10 & 2 \end{array}\right] \end{array} = A$$

$$\begin{array}{l} \\ \text{Assembly} \\ \text{Packaging} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Plant 1} & \text{Plant 2} & \text{Plant 3} \end{array} \\ \left[\begin{array}{ccc} 21 & 18 & 20 \\ 14 & 10 & 13 \end{array}\right] \end{array} = B$$

Exercise 2.3.26 For the directed graph below, find the adjacency matrix A, compute A^3, and determine the number of paths of length 3 from v_1 to v_4 and from v_2 to v_3.



Exercise 2.3.27 In each case either show the statement is true, or give an example showing that it is false.

a. If $A^2 = I$, then $A = I$.

b. If $AJ = A$, then $J = I$.

c. If A is square, then $(A^T)^3 = (A^3)^T$.

d. If A is symmetric, then $I + A$ is symmetric.

e. If $AB = AC$ and $A \neq 0$, then $B = C$.

f. If $A \neq 0$, then $A^2 \neq 0$.

g. If A has a row of zeros, so also does BA for all B.

h. If A commutes with $A+B$, then A commutes with B.

i. If B has a column of zeros, so also does AB.

j. If AB has a column of zeros, so also does B.

k. If A has a row of zeros, so also does AB.

l. If AB has a row of zeros, so also does A.

Exercise 2.3.28

a. If A and B are 2×2 matrices whose rows sum to 1, show that the rows of AB also sum to 1.

b. Repeat part (a) for the case where A and B are $n \times n$.

Exercise 2.3.29 Let A and B be $n \times n$ matrices for which the systems of equations $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ each have only the trivial solution $\mathbf{x} = \mathbf{0}$. Show that the system $(AB)\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Exercise 2.3.30 The **trace** of a square matrix A, denoted $\text{tr}\, A$, is the sum of the elements on the main diagonal of A. Show that, if A and B are $n \times n$ matrices:

a. $\text{tr}\,(A+B) = \text{tr}\, A + \text{tr}\, B$.

b. $\text{tr}\,(kA) = k\, \text{tr}\,(A)$ for any number k.

c. $\text{tr}\,(A^T) = \text{tr}\,(A)$.

d. $\text{tr}\,(AB) = \text{tr}\,(BA)$.

e. $\text{tr}\,(AA^T)$ is the sum of the squares of all entries of A.

Exercise 2.3.31 Show that $AB - BA = I$ is impossible.

[*Hint*: See the preceding exercise.]

Exercise 2.3.32 A square matrix P is called an **idempotent** if $P^2 = P$. Show that:

a. 0 and I are idempotents.

b. $\left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right]$, $\left[\begin{array}{cc} 1 & 0 \\ 1 & 0 \end{array}\right]$, and $\frac{1}{2}\left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right]$, are idempotents.

c. If P is an idempotent, so is $I - P$. Show further that $P(I-P) = 0$.

d. If P is an idempotent, so is P^T.

e. If P is an idempotent, so is $Q = P + AP - PAP$ for any square matrix A (of the same size as P).

f. If A is $n \times m$ and B is $m \times n$, and if $AB = I_n$, then BA is an idempotent.

Exercise 2.3.33 Let A and B be $n \times n$ **diagonal matrices** (all entries off the main diagonal are zero).

a. Show that AB is diagonal and $AB = BA$.

b. Formulate a rule for calculating XA if X is $m \times n$.

c. Formulate a rule for calculating AY if Y is $n \times k$.

Exercise 2.3.34 If A and B are $n \times n$ matrices, show that:

a. $AB = BA$ if and only if

$$(A+B)^2 = A^2 + 2AB + B^2$$

b. $AB = BA$ if and only if

$$(A+B)(A-B) = (A-B)(A+B)$$

Exercise 2.3.35 In Theorem 2.3.3, prove

a. part 3;

b. part 5.

2.4 Matrix Inverses

Three basic operations on matrices, addition, multiplication, and subtraction, are analogs for matrices of the same operations for numbers. In this section we introduce the matrix analog of numerical division.

To begin, consider how a numerical equation $ax = b$ is solved when a and b are known numbers. If $a = 0$, there is no solution (unless $b = 0$). But if $a \neq 0$, we can multiply both sides by the inverse $a^{-1} = \frac{1}{a}$ to obtain the solution $x = a^{-1}b$. Of course multiplying by a^{-1} is just dividing by a, and the property of a^{-1} that makes this work is that $a^{-1}a = 1$. Moreover, we saw in Section 2.2 that the role that 1 plays in arithmetic is played in matrix algebra by the identity matrix I. This suggests the following definition.

Definition 2.11 Matrix Inverses

If A is a square matrix, a matrix B is called an ***inverse*** *of A if and only if*

$$AB = I \quad \text{and} \quad BA = I$$

A matrix A that has an inverse is called an ***invertible matrix****.*[8]

Example 2.4.1

Show that $B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ is an inverse of $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

Solution. Compute AB and BA.

$$AB = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad BA = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence $AB = I = BA$, so B is indeed an inverse of A.

Example 2.4.2

Show that $A = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}$ has no inverse.

Solution. Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ denote an arbitrary 2×2 matrix. Then

$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a+3c & b+3d \end{bmatrix}$$

so AB has a row of zeros. Hence AB cannot equal I for any B.

[8] Only square matrices have inverses. Even though it is plausible that nonsquare matrices A and B could exist such that $AB = I_m$ and $BA = I_n$, where A is $m \times n$ and B is $n \times m$, we claim that this forces $n = m$. Indeed, if $m < n$ there exists a nonzero column $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$ (by Theorem 1.3.1), so $\mathbf{x} = I_n\mathbf{x} = (BA)\mathbf{x} = B(A\mathbf{x}) = B(\mathbf{0}) = \mathbf{0}$, a contradiction. Hence $m \geq n$. Similarly, the condition $AB = I_m$ implies that $n \geq m$. Hence $m = n$ so A is square.

The argument in Example 2.4.2 shows that no zero matrix has an inverse. But Example 2.4.2 also shows that, unlike arithmetic, *it is possible for a nonzero matrix to have no inverse*. However, if a matrix *does* have an inverse, it has only one.

Theorem 2.4.1

If B and C are both inverses of A, then $B = C$.

Proof. Since B and C are both inverses of A, we have $CA = I = AB$. Hence

$$B = IB = (CA)B = C(AB) = CI = C$$

□

If A is an invertible matrix, the (unique) inverse of A is denoted A^{-1}. Hence A^{-1} (when it exists) is a square matrix of the same size as A with the property that

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I$$

These equations characterize A^{-1} in the following sense:

> **Inverse Criterion:** *If somehow a matrix B can be found such that $AB = I$ and $BA = I$, then A is invertible and B is the inverse of A; in symbols, $B = A^{-1}$.*

This is a way to verify that the inverse of a matrix exists. Example 2.4.3 and Example 2.4.4 offer illustrations.

Example 2.4.3

If $A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$, show that $A^3 = I$ and so find A^{-1}.

Solution. We have $A^2 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, and so

$$A^3 = A^2A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence $A^3 = I$, as asserted. This can be written as $A^2A = I = AA^2$, so it shows that A^2 is the inverse of A. That is, $A^{-1} = A^2 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$.

The next example presents a useful formula for the inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ when it exists. To state it, we define the **determinant** $\det A$ and the **adjugate** $\text{adj}\, A$ of the matrix A as follows:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc, \quad \text{and} \quad \text{adj} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 2.4.4

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, show that A has an inverse if and only if $\det A \neq 0$, and in this case

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

Solution. For convenience, write $e = \det A = ad - bc$ and $B = \operatorname{adj} A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Then $AB = eI = BA$ as the reader can verify. So if $e \neq 0$, scalar multiplication by $\frac{1}{e}$ gives

$$A(\tfrac{1}{e}B) = I = (\tfrac{1}{e}B)A$$

Hence A is invertible and $A^{-1} = \frac{1}{e}B$. Thus it remains only to show that if A^{-1} exists, then $e \neq 0$. We prove this by showing that assuming $e = 0$ leads to a contradiction. In fact, if $e = 0$, then $AB = eI = 0$, so left multiplication by A^{-1} gives $A^{-1}AB = A^{-1}0$; that is, $IB = 0$, so $B = 0$. But this implies that a, b, c, and d are *all* zero, so $A = 0$, contrary to the assumption that A^{-1} exists.

As an illustration, if $A = \begin{bmatrix} 2 & 4 \\ -3 & 8 \end{bmatrix}$ then $\det A = 2 \cdot 8 - 4 \cdot (-3) = 28 \neq 0$. Hence A is invertible and $A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{28} \begin{bmatrix} 8 & -4 \\ 3 & 2 \end{bmatrix}$, as the reader is invited to verify.

The determinant and adjugate will be defined in Chapter 3 for any square matrix, and the conclusions in Example 2.4.4 will be proved in full generality.

Inverses and Linear Systems

Matrix inverses can be used to solve certain systems of linear equations. Recall that a *system* of linear equations can be written as a *single* matrix equation

$$A\mathbf{x} = \mathbf{b}$$

where A and $\mathbf{b}$ are known and $\mathbf{x}$ is to be determined. If A is invertible, we multiply each side of the equation on the left by A^{-1} to get

$$\begin{aligned} A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ I\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

This gives the solution to the system of equations (the reader should verify that $\mathbf{x} = A^{-1}\mathbf{b}$ really does satisfy $A\mathbf{x} = \mathbf{b}$). Furthermore, the argument shows that if $\mathbf{x}$ is *any* solution, then necessarily $\mathbf{x} = A^{-1}\mathbf{b}$, so the solution is unique. Of course the technique works only when the coefficient matrix A has an inverse. This proves Theorem 2.4.2.

Theorem 2.4.2

Suppose a system of n equations in n variables is written in matrix form as

$$A\mathbf{x} = \mathbf{b}$$

If the $n \times n$ coefficient matrix A is invertible, the system has the unique solution

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Example 2.4.5

Use Example 2.4.4 to solve the system $\begin{cases} 5x_1 - 3x_2 = -4 \\ 7x_1 + 4x_2 = 8 \end{cases}$.

Solution. In matrix form this is $A\mathbf{x} = \mathbf{b}$ where $A = \left[\begin{array}{rr} 5 & -3 \\ 7 & 4 \end{array}\right]$, $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]$, and $\mathbf{b} = \left[\begin{array}{r} -4 \\ 8 \end{array}\right]$. Then det $A = 5 \cdot 4 - (-3) \cdot 7 = 41$, so A is invertible and $A^{-1} = \frac{1}{41}\left[\begin{array}{rr} 4 & 3 \\ -7 & 5 \end{array}\right]$ by Example 2.4.4. Thus Theorem 2.4.2 gives

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{41}\left[\begin{array}{rr} 4 & 3 \\ -7 & 5 \end{array}\right]\left[\begin{array}{r} -4 \\ 8 \end{array}\right] = \frac{1}{41}\left[\begin{array}{r} 8 \\ 68 \end{array}\right]$$

so the solution is $x_1 = \frac{8}{41}$ and $x_2 = \frac{68}{41}$.

An Inversion Method

If a matrix A is $n \times n$ and invertible, it is desirable to have an efficient technique for finding the inverse. The following procedure will be justified in Section 2.5.

Matrix Inversion Algorithm

If A is an invertible (square) matrix, there exists a sequence of elementary row operations that carry A to the identity matrix I of the same size, written $A \to I$. This same series of row operations carries I to A^{-1}; that is, $I \to A^{-1}$. The algorithm can be summarized as follows:

$$\left[\begin{array}{cc} A & I \end{array}\right] \to \left[\begin{array}{cc} I & A^{-1} \end{array}\right]$$

where the row operations on A and I are carried out simultaneously.

Example 2.4.6

Use the inversion algorithm to find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix}$$

Solution. Apply elementary row operations to the double matrix

$$\left[\begin{array}{cc} A & I \end{array}\right] = \left[\begin{array}{rrr|rrr} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

so as to carry A to I. First interchange rows 1 and 2.

$$\left[\begin{array}{rrr|rrr} 1 & 4 & -1 & 0 & 1 & 0 \\ 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

Next subtract 2 times row 1 from row 2, and subtract row 1 from row 3.

$$\left[\begin{array}{rrr|rrr} 1 & 4 & -1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{array}\right]$$

Continue to reduced row-echelon form.

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{-3}{2} & \frac{-3}{2} & \frac{11}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{-3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \end{array}\right]$$

Hence $A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -3 & 11 \\ 1 & 1 & -3 \\ 1 & -1 & -1 \end{bmatrix}$, as is readily verified.

Given any $n \times n$ matrix A, Theorem 1.2.1 shows that A can be carried by elementary row operations to a matrix R in reduced row-echelon form. If $R = I$, the matrix A is invertible (this will be proved in the next section), so the algorithm produces A^{-1}. If $R \neq I$, then R has a row of zeros (it is square), so no system of linear equations $A\mathbf{x} = \mathbf{b}$ can have a unique solution. But then A is not invertible by Theorem 2.4.2. Hence, the algorithm is effective in the sense conveyed in Theorem 2.4.3.

Theorem 2.4.3

If A is an $n \times n$ matrix, either A can be reduced to I by elementary row operations or it cannot. In the first case, the algorithm produces A^{-1}; in the second case, A^{-1} does not exist.

Properties of Inverses

The following properties of an invertible matrix are used everywhere.

Example 2.4.7: Cancellation Laws

Let A be an invertible matrix. Show that:

1. If $AB = AC$, then $B = C$.
2. If $BA = CA$, then $B = C$.

Solution. Given the equation $AB = AC$, left multiply both sides by A^{-1} to obtain $A^{-1}AB = A^{-1}AC$. Thus $IB = IC$, that is $B = C$. This proves (1) and the proof of (2) is left to the reader.

Properties (1) and (2) in Example 2.4.7 are described by saying that an invertible matrix can be "left cancelled" and "right cancelled", respectively. Note however that "mixed" cancellation does not hold in general: If A is invertible and $AB = CA$, then B and C may *not* be equal, even if both are 2×2. Here is a specific example:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Sometimes the inverse of a matrix is given by a formula. Example 2.4.4 is one illustration; Example 2.4.8 and Example 2.4.9 provide two more. The idea is the *Inverse Criterion*: If a matrix B can be found such that $AB = I = BA$, then A is invertible and $A^{-1} = B$.

Example 2.4.8

If A is an invertible matrix, show that the transpose A^T is also invertible. Show further that the inverse of A^T is just the transpose of A^{-1}; in symbols, $(A^T)^{-1} = (A^{-1})^T$.

Solution. A^{-1} exists (by assumption). Its transpose $(A^{-1})^T$ is the candidate proposed for the inverse of A^T. Using the inverse criterion, we test it as follows:

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$
$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

Hence $(A^{-1})^T$ is indeed the inverse of A^T; that is, $(A^T)^{-1} = (A^{-1})^T$.

Example 2.4.9

If A and B are invertible $n \times n$ matrices, show that their product AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Solution. We are given a candidate for the inverse of AB, namely $B^{-1}A^{-1}$. We test it as follows:

$$\begin{aligned}(B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I \\ (AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I\end{aligned}$$

Hence $B^{-1}A^{-1}$ is the inverse of AB; in symbols, $(AB)^{-1} = B^{-1}A^{-1}$.

We now collect several basic properties of matrix inverses for reference.

Theorem 2.4.4

All the following matrices are square matrices of the same size.

1. *I is invertible and $I^{-1} = I$.*
2. *If A is invertible, so is A^{-1}, and $(A^{-1})^{-1} = A$.*
3. *If A and B are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.*
4. *If $A_1, A_2, \ldots, A_k$ are all invertible, so is their product $A_1A_2\cdots A_k$, and*
$$(A_1A_2\cdots A_k)^{-1} = A_k^{-1}\cdots A_2^{-1}A_1^{-1}.$$
5. *If A is invertible, so is A^k for any $k \geq 1$, and $(A^k)^{-1} = (A^{-1})^k$.*
6. *If A is invertible and $a \neq 0$ is a number, then aA is invertible and $(aA)^{-1} = \frac{1}{a}A^{-1}$.*
7. *If A is invertible, so is its transpose A^T, and $(A^T)^{-1} = (A^{-1})^T$.*

Proof.

1. This is an immediate consequence of the fact that $I^2 = I$.
2. The equations $AA^{-1} = I = A^{-1}A$ show that A is the inverse of A^{-1}; in symbols, $(A^{-1})^{-1} = A$.
3. This is Example 2.4.9.
4. Use induction on k. If $k = 1$, there is nothing to prove, and if $k = 2$, the result is property 3. If $k > 2$, assume inductively that $(A_1A_2\cdots A_{k-1})^{-1} = A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}$. We apply this fact together with property 3 as follows:
$$\begin{aligned}\left[A_1A_2\cdots A_{k-1}A_k\right]^{-1} &= \left[(A_1A_2\cdots A_{k-1})A_k\right]^{-1} \\ &= A_k^{-1}(A_1A_2\cdots A_{k-1})^{-1} \\ &= A_k^{-1}\left(A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}\right)\end{aligned}$$

So the proof by induction is complete.

5. This is property 4 with $A_1 = A_2 = \cdots = A_k = A$.

6. This is left as Exercise 2.4.29.

7. This is Example 2.4.8. □

The reversal of the order of the inverses in properties 3 and 4 of Theorem 2.4.4 is a consequence of the fact that matrix multiplication is not commutative. Another manifestation of this comes when matrix equations are dealt with. If a matrix equation $B = C$ is given, it can be *left-multiplied* by a matrix A to yield $AB = AC$. Similarly, *right-multiplication* gives $BA = CA$. However, we cannot mix the two: If $B = C$, it need *not* be the case that $AB = CA$ even if A is invertible, for example, $A = \left[\begin{array}{rr} 1 & 1 \\ 0 & 1 \end{array} \right]$, $B = \left[\begin{array}{rr} 0 & 0 \\ 1 & 0 \end{array} \right] = C$.

Part 7 of Theorem 2.4.4 together with the fact that $(A^T)^T = A$ gives

Corollary 2.4.1

A square matrix A is invertible if and only if A^T is invertible.

Example 2.4.10

Find A if $(A^T - 2I)^{-1} = \left[\begin{array}{rr} 2 & 1 \\ -1 & 0 \end{array} \right]$.

Solution. By Theorem 2.4.4(2) and Example 2.4.4, we have

$$(A^T - 2I) = \left[\left(A^T - 2I\right)^{-1} \right]^{-1} = \left[\begin{array}{rr} 2 & 1 \\ -1 & 0 \end{array} \right]^{-1} = \left[\begin{array}{rr} 0 & -1 \\ 1 & 2 \end{array} \right]$$

Hence $A^T = 2I + \left[\begin{array}{rr} 0 & -1 \\ 1 & 2 \end{array} \right] = \left[\begin{array}{rr} 2 & -1 \\ 1 & 4 \end{array} \right]$, so $A = \left[\begin{array}{rr} 2 & 1 \\ -1 & 4 \end{array} \right]$ by Theorem 2.4.4(7).

The following important theorem collects a number of conditions all equivalent[9] to invertibility. It will be referred to frequently below.

Theorem 2.4.5: Inverse Theorem

The following conditions are equivalent for an $n \times n$ matrix A:

1. *A is invertible.*

2. *The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.*

3. *A can be carried to the identity matrix I_n by elementary row operations.*

[9] If p and q are statements, we say that p **implies** q (written $p \Rightarrow q$) if q is true whenever p is true. The statements are called **equivalent** if both $p \Rightarrow q$ and $q \Rightarrow p$ (written $p \Leftrightarrow q$, spoken "p if and only if q"). See Appendix B.

4. *The system $A\mathbf{x} = \mathbf{b}$ has at least one solution $\mathbf{x}$ for every choice of column $\mathbf{b}$.*

5. *There exists an $n \times n$ matrix C such that $AC = I_n$.*

Proof. We show that each of these conditions implies the next, and that (5) implies (1).

(1) $\Rightarrow$ (2). If A^{-1} exists, then $A\mathbf{x} = \mathbf{0}$ gives $\mathbf{x} = I_n\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$.

(2) $\Rightarrow$ (3). Assume that (2) is true. Certainly $A \to R$ by row operations where R is a reduced, row-echelon matrix. It suffices to show that $R = I_n$. Suppose that this is not the case. Then R has a row of zeros (being square). Now consider the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{0} \end{array}\right]$ of the system $A\mathbf{x} = \mathbf{0}$. Then $\left[\begin{array}{c|c} A & \mathbf{0} \end{array}\right] \to \left[\begin{array}{c|c} R & \mathbf{0} \end{array}\right]$ is the reduced form, and $\left[\begin{array}{c|c} R & \mathbf{0} \end{array}\right]$ also has a row of zeros. Since R is square there must be at least one nonleading variable, and hence at least one parameter. Hence the system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, contrary to (2). So $R = I_n$ after all.

(3) $\Rightarrow$ (4). Consider the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ of the system $A\mathbf{x} = \mathbf{b}$. Using (3), let $A \to I_n$ by a sequence of row operations. Then these same operations carry $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] \to \left[\begin{array}{c|c} I_n & \mathbf{c} \end{array}\right]$ for some column $\mathbf{c}$. Hence the system $A\mathbf{x} = \mathbf{b}$ has a solution (in fact unique) by gaussian elimination. This proves (4).

(4) $\Rightarrow$ (5). Write $I_n = \left[\begin{array}{cccc} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{array}\right]$ where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ are the columns of I_n. For each $j = 1, 2, \dots, n$, the system $A\mathbf{x} = \mathbf{e}_j$ has a solution $\mathbf{c}_j$ by (4), so $A\mathbf{c}_j = \mathbf{e}_j$. Now let $C = \left[\begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{array}\right]$ be the $n \times n$ matrix with these matrices $\mathbf{c}_j$ as its columns. Then Definition 2.9 gives (5):

$$AC = A\left[\begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{array}\right] = \left[\begin{array}{cccc} A\mathbf{c}_1 & A\mathbf{c}_2 & \cdots & A\mathbf{c}_n \end{array}\right] = \left[\begin{array}{cccc} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{array}\right] = I_n$$

(5) $\Rightarrow$ (1). Assume that (5) is true so that $AC = I_n$ for some matrix C. Then $C\mathbf{x} = 0$ implies $\mathbf{x} = \mathbf{0}$ (because $\mathbf{x} = I_n\mathbf{x} = AC\mathbf{x} = A\mathbf{0} = \mathbf{0}$). Thus condition (2) holds for the matrix C rather than A. Hence the argument above that (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) (with A replaced by C) shows that a matrix C' exists such that $CC' = I_n$. But then

$$A = AI_n = A(CC') = (AC)C' = I_nC' = C'$$

Thus $CA = CC' = I_n$ which, together with $AC = I_n$, shows that C is the inverse of A. This proves (1). □

The proof of (5) $\Rightarrow$ (1) in Theorem 2.4.5 shows that if $AC = I$ for square matrices, then necessarily $CA = I$, and hence that C and A are inverses of each other. We record this important fact for reference.

Corollary 2.4.1

If A and C are square matrices such that $AC = I$, then also $CA = I$. In particular, both A and C are invertible, $C = A^{-1}$, and $A = C^{-1}$.

Here is a quick way to remember Corollary 2.4.1. If A is a square matrix, then

1. If $AC = I$ then $C = A^{-1}$.
2. If $CA = I$ then $C = A^{-1}$.

Observe that Corollary 2.4.1 is false if A and C are not square matrices. For example, we have

$$\left[\begin{array}{ccc} 1 & 2 & 1 \\ 1 & 1 & 1 \end{array}\right] \left[\begin{array}{rr} -1 & 1 \\ 1 & -1 \\ 0 & 1 \end{array}\right] = I_2 \quad \text{but} \quad \left[\begin{array}{rr} -1 & 1 \\ 1 & -1 \\ 0 & 1 \end{array}\right] \left[\begin{array}{ccc} 1 & 2 & 1 \\ 1 & 1 & 1 \end{array}\right] \neq I_3$$

In fact, it is verified in the footnote on page 80 that if $AB = I_m$ and $BA = I_n$, where A is $m \times n$ and B is $n \times m$, then $m = n$ and A and B are (square) inverses of each other.

An $n \times n$ matrix A has rank n if and only if (3) of Theorem 2.4.5 holds. Hence

Corollary 2.4.2

An $n \times n$ matrix A is invertible if and only if rank $A = n$.

Here is a useful fact about inverses of block matrices.

Example 2.4.11

Let $P = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix}$ and $Q = \begin{bmatrix} A & 0 \\ Y & B \end{bmatrix}$ be block matrices where A is $m \times m$ and B is $n \times n$ (possibly $m \neq n$).

a. Show that P is invertible if and only if A and B are both invertible. In this case, show that

$$P^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}XB^{-1} \\ 0 & B^{-1} \end{bmatrix}$$

b. Show that Q is invertible if and only if A and B are both invertible. In this case, show that

$$Q^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -B^{-1}YA^{-1} & B^{-1} \end{bmatrix}$$

Solution. We do (a.) and leave (b.) for the reader.

a. If A^{-1} and B^{-1} both exist, write $R = \begin{bmatrix} A^{-1} & -A^{-1}XB^{-1} \\ 0 & B^{-1} \end{bmatrix}$. Using block multiplication, one verifies that $PR = I_{m+n} = RP$, so P is invertible, and $P^{-1} = R$. Conversely, suppose that P is invertible, and write $P^{-1} = \begin{bmatrix} C & V \\ W & D \end{bmatrix}$ in block form, where C is $m \times m$ and D is $n \times n$.

Then the equation $PP^{-1} = I_{n+m}$ becomes

$$\begin{bmatrix} A & X \\ 0 & B \end{bmatrix} \begin{bmatrix} C & V \\ W & D \end{bmatrix} = \begin{bmatrix} AC + XW & AV + XD \\ BW & BD \end{bmatrix} = I_{m+n} = \begin{bmatrix} I_m & 0 \\ 0 & I_n \end{bmatrix}$$

using block notation. Equating corresponding blocks, we find

$$AC + XW = I_m, \qquad BW = 0, \qquad \text{and } BD = I_n$$

Hence B is invertible because $BD = I_n$ (by Corollary 2.4.1), then $W = 0$ because $BW = 0$, and finally, $AC = I_m$ (so A is invertible, again by Corollary 2.4.1).

Inverses of Matrix Transformations

Let $T = T_A : \mathbb{R}^n \to \mathbb{R}^n$ denote the matrix transformation induced by the $n \times n$ matrix A. Since A is square, it may very well be invertible, and this leads to the question:

What does it mean geometrically for T that A is invertible?

To answer this, let $T' = T_{A^{-1}} : \mathbb{R}^n \to \mathbb{R}^n$ denote the transformation induced by A^{-1}. Then

$$\left.\begin{array}{l} T'[T(\mathbf{x})] = A^{-1}[A\mathbf{x}] = I\mathbf{x} = \mathbf{x} \\ T[T'(\mathbf{x})] = A\left[A^{-1}\mathbf{x}\right] = I\mathbf{x} = \mathbf{x} \end{array}\right. \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \tag{2.8}$$

The first of these equations asserts that, if T carries $\mathbf{x}$ to a vector $T(\mathbf{x})$, then T' carries $T(\mathbf{x})$ right back to $\mathbf{x}$; that is T' "reverses" the action of T. Similarly T "reverses" the action of T'. Conditions (2.8) can be stated compactly in terms of composition:

$$T' \circ T = 1_{\mathbb{R}^n} \quad \text{and} \quad T \circ T' = 1_{\mathbb{R}^n} \tag{2.9}$$

When these conditions hold, we say that the matrix transformation T' is an **inverse** of T, and we have shown that if the matrix A of T is invertible, then T has an inverse (induced by A^{-1}).

The converse is also true: If T has an inverse, then its matrix A must be invertible. Indeed, suppose $S : \mathbb{R}^n \to \mathbb{R}^n$ is any inverse of T, so that $S \circ T = 1_{\mathbb{R}_n}$ and $T \circ S = 1_{\mathbb{R}_n}$. It can be shown that S is also a matrix transformation. If B is the matrix of S, we have

$$BA\mathbf{x} = S[T(\mathbf{x})] = (S \circ T)(\mathbf{x}) = 1_{\mathbb{R}^n}(\mathbf{x}) = \mathbf{x} = I_n\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

It follows by Theorem 2.2.6 that $BA = I_n$, and a similar argument shows that $AB = I_n$. Hence A is invertible with $A^{-1} = B$. Furthermore, the inverse transformation S has matrix A^{-1}, so $S = T'$ using the earlier notation. This proves the following important theorem.

Theorem 2.4.6

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ denote the matrix transformation induced by an $n \times n$ matrix A. Then

A is invertible if and only if T has an inverse.

In this case, T has exactly one inverse (which we denote as T^{-1}), and $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is the transformation induced by the matrix A^{-1}. In other words

$$(T_A)^{-1} = T_{A^{-1}}$$

The geometrical relationship between T and T^{-1} is embodied in equations (2.8) above:

$$T^{-1}[T(\mathbf{x})] = \mathbf{x} \quad \text{and} \quad T\left[T^{-1}(\mathbf{x})\right] = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

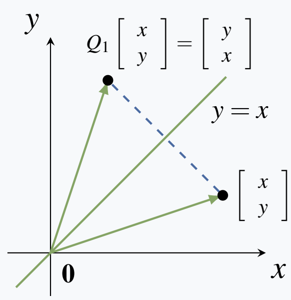
These equations are called the **fundamental identities** relating T and T^{-1}. Loosely speaking, they assert that each of T and T^{-1} "reverses" or "undoes" the action of the other.

This geometric view of the inverse of a linear transformation provides a new way to find the inverse of a matrix A. More precisely, if A is an invertible matrix, we proceed as follows:

1. *Let T be the linear transformation induced by A.*
2. *Obtain the linear transformation T^{-1} which "reverses" the action of T.*
3. *Then A^{-1} is the matrix of T^{-1}.*

Here is an example.

Example 2.4.12



Find the inverse of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ by viewing it as a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$.

Solution. If $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ the vector $A\mathbf{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$ is the result of reflecting $\mathbf{x}$ in the line $y = x$ (see the diagram). Hence, if $Q_1 : \mathbb{R}^2 \to \mathbb{R}^2$ denotes reflection in the line $y = x$, then A is the matrix of Q_1. Now observe that Q_1 reverses itself because reflecting a vector $\mathbf{x}$ twice results in $\mathbf{x}$. Consequently $Q_1^{-1} = Q_1$. Since A^{-1} is the matrix of Q_1^{-1} and A is the matrix of Q, it follows that $A^{-1} = A$. Of course this conclusion is clear by simply observing directly that $A^2 = I$, but the geometric method can often work where these other methods may be less straightforward.

Exercises for 2.4

Exercise 2.4.1 In each case, show that the matrices are inverses of each other.

a. $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 0 \\ 1 & -4 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 4 & 0 \\ 1 & -3 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 7 & 2 & -6 \\ -3 & -1 & 3 \\ 2 & 1 & -2 \end{bmatrix}$

d. $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$

Exercise 2.4.2 Find the inverse of each of the following matrices.

a. $\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 0 \\ -1 & -1 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & -1 & 2 \\ -5 & 7 & -11 \\ -2 & 3 & -5 \end{bmatrix}$

e. $\begin{bmatrix} 3 & 5 & 0 \\ 3 & 7 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

f. $\begin{bmatrix} 3 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 5 & -1 \end{bmatrix}$

g. $\begin{bmatrix} 2 & 4 & 1 \\ 3 & 3 & 2 \\ 4 & 1 & 4 \end{bmatrix}$

h. $\begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$

i. $\begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$

j. $\begin{bmatrix} -1 & 4 & 5 & 2 \\ 0 & 0 & 0 & -1 \\ 1 & -2 & -2 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix}$

k. $\begin{bmatrix} 1 & 0 & 7 & 5 \\ 0 & 1 & 3 & 6 \\ 1 & -1 & 5 & 2 \\ 1 & -1 & 5 & 1 \end{bmatrix}$ l. $\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Exercise 2.4.3 In each case, solve the systems of equations by finding the inverse of the coefficient matrix.

a. $\begin{aligned} 3x - y &= 5 \\ 2x + 2y &= 1 \end{aligned}$

b. $\begin{aligned} 2x - 3y &= 0 \\ x - 4y &= 1 \end{aligned}$

c. $\begin{aligned} x + y + 2z &= 5 \\ x + y + z &= 0 \\ x + 2y + 4z &= -2 \end{aligned}$

d. $\begin{aligned} x + 4y + 2z &= 1 \\ 2x + 3y + 3z &= -1 \\ 4x + y + 4z &= 0 \end{aligned}$

Exercise 2.4.4 Given $A^{-1} = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ -1 & 1 & 0 \end{bmatrix}$:

a. Solve the system of equations $A\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.

b. Find a matrix B such that
$AB = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

c. Find a matrix C such that
$CA = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \end{bmatrix}$.

Exercise 2.4.5 Find A when

a. $(3A)^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

b. $(2A)^T = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}^{-1}$

c. $(I + 3A)^{-1} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$

d. $(I - 2A^T)^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

e. $\left(A \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$

f. $\left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} A\right)^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$

g. $\left(A^T - 2I\right)^{-1} = 2\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

h. $\left(A^{-1} - 2I\right)^T = -2\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Exercise 2.4.6 Find A when:

a. $A^{-1} = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 1 \\ 0 & 2 & -2 \end{bmatrix}$ b. $A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

Exercise 2.4.7 Given $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 4 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 0 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, express the variables x_1, x_2, and x_3 in terms of z_1, z_2, and z_3.

Exercise 2.4.8

a. In the system $\begin{aligned} 3x + 4y &= 7 \\ 4x + 5y &= 1 \end{aligned}$, substitute the new variables x' and y' given by $\begin{aligned} x &= -5x' + 4y' \\ y &= 4x' - 3y' \end{aligned}$. Then find x and y.

b. Explain part (a) by writing the equations as $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix} = B\begin{bmatrix} x' \\ y' \end{bmatrix}$. What is the relationship between A and B?

Exercise 2.4.9 In each case either prove the assertion or give an example showing that it is false.

a. If $A \neq 0$ is a square matrix, then A is invertible.

b. If A and B are both invertible, then $A + B$ is invertible.

c. If A and B are both invertible, then $(A^{-1}B)^T$ is invertible.

d. If $A^4 = 3I$, then A is invertible.

e. If $A^2 = A$ and $A \neq 0$, then A is invertible.

f. If $AB = B$ for some $B \neq 0$, then A is invertible.

g. If A is invertible and skew symmetric ($A^T = -A$), the same is true of A^{-1}.

h. If A^2 is invertible, then A is invertible.

i. If $AB = I$, then A and B commute.

Exercise 2.4.10

a. If A, B, and C are square matrices and $AB = I$, $I = CA$, show that A is invertible and $B = C = A^{-1}$.

b. If $C^{-1} = A$, find the inverse of C^T in terms of A.

Exercise 2.4.11 Suppose $CA = I_m$, where C is $m \times n$ and A is $n \times m$. Consider the system $A\mathbf{x} = \mathbf{b}$ of n equations in m variables.

a. Show that this system has a unique solution CB if it is consistent.

b. If $C = \left[\begin{array}{rrr} 0 & -5 & 1 \\ 3 & 0 & -1 \end{array} \right]$ and $A = \left[\begin{array}{rr} 2 & -3 \\ 1 & -2 \\ 6 & -10 \end{array} \right]$, find $\mathbf{x}$ (if it exists) when (i) $\mathbf{b} = \left[\begin{array}{r} 1 \\ 0 \\ 3 \end{array} \right]$; and (ii) $\mathbf{b} = \left[\begin{array}{r} 7 \\ 4 \\ 22 \end{array} \right]$.

Exercise 2.4.12 Verify that $A = \left[\begin{array}{rr} 1 & -1 \\ 0 & 2 \end{array} \right]$ satisfies $A^2 - 3A + 2I = 0$, and use this fact to show that $A^{-1} = \frac{1}{2}(3I - A)$.

Exercise 2.4.13 Let $Q = \left[\begin{array}{rrrr} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{array} \right]$. Compute QQ^T and so find Q^{-1} if $Q \neq 0$.

Exercise 2.4.14 Let $U = \left[\begin{array}{rr} 0 & 1 \\ 1 & 0 \end{array} \right]$. Show that each of U, $-U$, and $-I_2$ is its own inverse and that the product of any two of these is the third.

Exercise 2.4.15 Consider $A = \left[\begin{array}{rr} 1 & 1 \\ -1 & 0 \end{array} \right]$, $B = \left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array} \right]$, $C = \left[\begin{array}{rrr} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{array} \right]$. Find the inverses by computing (a) A^6; (b) B^4; and (c) C^3.

Exercise 2.4.16 Find the inverse of $\left[\begin{array}{rrr} 1 & 0 & 1 \\ c & 1 & c \\ 3 & c & 2 \end{array} \right]$ in terms of c.

Exercise 2.4.17 If $c \neq 0$, find the inverse of $\left[\begin{array}{rrr} 1 & -1 & 1 \\ 2 & -1 & 2 \\ 0 & 2 & c \end{array} \right]$ in terms of c.

Exercise 2.4.18 Show that A has no inverse when:

a. A has a row of zeros.

b. A has a column of zeros.

c. each row of A sums to 0. [*Hint*: Theorem 2.4.5(2).]

d. each column of A sums to 0.

[*Hint*: Corollary 2.4.1, Theorem 2.4.4.]

Exercise 2.4.19 Let A denote a square matrix.

a. Let $YA = 0$ for some matrix $Y \neq 0$. Show that A has no inverse. [*Hint*: Corollary 2.4.1, Theorem 2.4.4.]

b. Use part (a) to show that (i) $\left[\begin{array}{rrr} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{array} \right]$; and (ii) $\left[\begin{array}{rrr} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{array} \right]$ have no inverse.

[*Hint*: For part (ii) compare row 3 with the difference between row 1 and row 2.]

Exercise 2.4.20 If A is invertible, show that

a. $A^2 \neq 0$.

b. $A^k \neq 0$ for all $k = 1, 2, \dots$.

Exercise 2.4.21 Suppose $AB = 0$, where A and B are square matrices. Show that:

a. If one of A and B has an inverse, the other is zero.

b. It is impossible for both A and B to have inverses.

c. $(BA)^2 = 0$.

Exercise 2.4.22 Find the inverse of the x-expansion in Example 2.2.16 and describe it geometrically.

Exercise 2.4.23 Find the inverse of the shear transformation in Example 2.2.17 and describe it geometrically.

Exercise 2.4.24 In each case assume that A is a square matrix that satisfies the given condition. Show that A is invertible and find a formula for A^{-1} in terms of A.

a. $A^3 - 3A + 2I = 0$.

b. $A^4 + 2A^3 - A - 4I = 0$.

Exercise 2.4.25 Let A and B denote $n \times n$ matrices.

a. If A and AB are invertible, show that B is invertible using only (2) and (3) of Theorem 2.4.4.

b. If AB is invertible, show that both A and B are invertible using Theorem 2.4.5.

Exercise 2.4.26 In each case find the inverse of the matrix A using Example 2.4.11.

a. $A = \left[\begin{array}{rrr} -1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 1 & -1 \end{array} \right]$ b. $A = \left[\begin{array}{rrr} 3 & 1 & 0 \\ 5 & 2 & 0 \\ 1 & 3 & -1 \end{array} \right]$

c. $A = \left[\begin{array}{rrrr} 3 & 4 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & -1 & 1 & 3 \\ 3 & 1 & 1 & 4 \end{array} \right]$

d. $A = \left[\begin{array}{rrrr} 2 & 1 & 5 & 2 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{array} \right]$

Exercise 2.4.27 If A and B are invertible symmetric matrices such that $AB = BA$, show that A^{-1}, AB, AB^{-1}, and $A^{-1}B^{-1}$ are also invertible and symmetric.

Exercise 2.4.28 Let A be an $n \times n$ matrix and let I be the $n \times n$ identity matrix.

a. If $A^2 = 0$, verify that $(I - A)^{-1} = I + A$.

b. If $A^3 = 0$, verify that $(I - A)^{-1} = I + A + A^2$.

c. Find the inverse of $\left[\begin{array}{rrr} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array} \right]$.

d. If $A^n = 0$, find the formula for $(I - A)^{-1}$.

Exercise 2.4.29 Prove property 6 of Theorem 2.4.4: If A is invertible and $a \neq 0$, then aA is invertible and $(aA)^{-1} = \frac{1}{a}A^{-1}$

Exercise 2.4.30 Let A, B, and C denote $n \times n$ matrices. Using only Theorem 2.4.4, show that:

a. If A, C, and ABC are all invertible, B is invertible.

b. If AB and BA are both invertible, A and B are both invertible.

Exercise 2.4.31 Let A and B denote invertible $n \times n$ matrices.

a. If $A^{-1} = B^{-1}$, does it mean that $A = B$? Explain.

b. Show that $A = B$ if and only if $A^{-1}B = I$.

Exercise 2.4.32 Let A, B, and C be $n \times n$ matrices, with A and B invertible. Show that

a. If A commutes with C, then A^{-1} commutes with C.

b. If A commutes with B, then A^{-1} commutes with B^{-1}.

Exercise 2.4.33 Let A and B be square matrices of the same size.

a. Show that $(AB)^2 = A^2B^2$ if $AB = BA$.

b. If A and B are invertible and $(AB)^2 = A^2B^2$, show that $AB = BA$.

c. If $A = \left[\begin{array}{rr} 1 & 0 \\ 0 & 0 \end{array} \right]$ and $B = \left[\begin{array}{rr} 1 & 1 \\ 0 & 0 \end{array} \right]$, show that $(AB)^2 = A^2B^2$ but $AB \neq BA$.

Exercise 2.4.34 Let A and B be $n \times n$ matrices for which AB is invertible. Show that A and B are both invertible.

Exercise 2.4.35 Consider $A = \left[\begin{array}{rrr} 1 & 3 & -1 \\ 2 & 1 & 5 \\ 1 & -7 & 13 \end{array} \right]$,

$B = \left[\begin{array}{rrr} 1 & 1 & 2 \\ 3 & 0 & -3 \\ -2 & 5 & 17 \end{array} \right]$.

a. Show that A is not invertible by finding a nonzero 1×3 matrix Y such that $YA = 0$.

[*Hint*: Row 3 of A equals 2(row 2) $-$ 3(row 1).]

b. Show that B is not invertible.

[*Hint*: Column 3 = 3(column 2) − column 1.]

Exercise 2.4.36 Show that a square matrix A is invertible if and only if it can be left-cancelled: $AB = AC$ implies $B = C$.

Exercise 2.4.37 If $U^2 = I$, show that $I + U$ is not invertible unless $U = I$.

Exercise 2.4.38

a. If J is the 4×4 matrix with every entry 1, show that $I - \frac{1}{2}J$ is self-inverse and symmetric.

b. If X is $n \times m$ and satisfies $X^T X = I_m$, show that $I_n - 2XX^T$ is self-inverse and symmetric.

Exercise 2.4.39 An $n \times n$ matrix P is called an idempotent if $P^2 = P$. Show that:

a. I is the only invertible idempotent.

b. P is an idempotent if and only if $I - 2P$ is self-inverse.

c. U is self-inverse if and only if $U = I - 2P$ for some idempotent P.

d. $I - aP$ is invertible for any $a \neq 1$, and that $(I - aP)^{-1} = I + \left(\frac{a}{1-a}\right)^P$.

Exercise 2.4.40 If $A^2 = kA$, where $k \neq 0$, show that A is invertible if and only if $A = kI$.

Exercise 2.4.41 Let A and B denote $n \times n$ invertible matrices.

a. Show that $A^{-1} + B^{-1} = A^{-1}(A + B)B^{-1}$.

b. If $A + B$ is also invertible, show that $A^{-1} + B^{-1}$ is invertible and find a formula for $(A^{-1} + B^{-1})^{-1}$.

Exercise 2.4.42 Let A and B be $n \times n$ matrices, and let I be the $n \times n$ identity matrix.

a. Verify that $A(I + BA) = (I + AB)A$ and that $(I + BA)B = B(I + AB)$.

b. If $I + AB$ is invertible, verify that $I + BA$ is also invertible and that $(I + BA)^{-1} = I - B(I + AB)^{-1}A$.

2.5 Elementary Matrices

It is now clear that elementary row operations are important in linear algebra: They are essential in solving linear systems (using the gaussian algorithm) and in inverting a matrix (using the matrix inversion algorithm). It turns out that they can be performed by left multiplying by certain invertible matrices. These matrices are the subject of this section.

Definition 2.12 Elementary Matrices

An $n \times n$ matrix E is called an ***elementary matrix*** *if it can be obtained from the identity matrix I_n by a single elementary row operation (called the operation* ***corresponding*** *to E). We say that E is of type I, II, or III if the operation is of that type (see Definition 1.2).*

Hence

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}, \quad \text{and} \quad E_3 = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$$

are elementary of types I, II, and III, respectively, obtained from the 2×2 identity matrix by interchanging rows 1 and 2, multiplying row 2 by 9, and adding 5 times row 2 to row 1.

Suppose now that the matrix $A = \begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix}$ is left multiplied by the above elementary matrices E_1, E_2, and E_3. The results are:

$$E_1A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix} = \begin{bmatrix} p & q & r \\ a & b & c \end{bmatrix}$$
$$E_2A = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix} = \begin{bmatrix} a & b & c \\ 9p & 9q & 9r \end{bmatrix}$$
$$E_3A = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix} = \begin{bmatrix} a+5p & b+5q & c+5r \\ p & q & r \end{bmatrix}$$

In each case, left multiplying A by the elementary matrix has the *same* effect as doing the corresponding row operation to A. This works in general.

Lemma 2.5.1: [10]

If an elementary row operation is performed on an $m \times n$ matrix A, the result is EA where E is the elementary matrix obtained by performing the same operation on the $m \times m$ identity matrix.

Proof. We prove it for operations of type III; the proofs for types I and II are left as exercises. Let E be the elementary matrix corresponding to the operation that adds k times row p to row $q \neq p$. The proof depends on the fact that each row of EA is equal to the corresponding row of E times A. Let $K_1, K_2, \dots, K_m$ denote the rows of I_m. Then row i of E is K_i if $i \neq q$, while row q of E is $K_q + kK_p$. Hence:

$$\begin{aligned} \text{If } i \neq q \text{ then row } i \text{ of } EA = &\; K_iA = (\text{row } i \text{ of } A). \\ \text{Row } q \text{ of } EA = (K_q + kK_p)A = &\; K_qA + k(K_pA) \\ = &\; (\text{row } q \text{ of } A) \text{ plus } k \text{ (row } p \text{ of } A). \end{aligned}$$

Thus EA is the result of adding k times row p of A to row q, as required. □

The effect of an elementary row operation can be reversed by another such operation (called its inverse) which is also elementary of the same type (see the discussion following (Example 1.1.3). It follows that each elementary matrix E is invertible. In fact, if a row operation on I produces E, then the inverse operation carries E back to I. If F is the elementary matrix corresponding to the inverse operation, this means $FE = I$ (by Lemma 2.5.1). Thus $F = E^{-1}$ and we have proved

Lemma 2.5.2

Every elementary matrix E is invertible, and E^{-1} is also a elementary matrix (of the same type). Moreover, E^{-1} corresponds to the inverse of the row operation that produces E.

The following table gives the inverse of each type of elementary row operation:

Type	Operation	Inverse Operation
I	Interchange rows p and q	Interchange rows p and q
II	Multiply row p by $k \neq 0$	Multiply row p by $1/k$, $k \neq 0$
III	Add k times row p to row $q \neq p$	Subtract k times row p from row q, $q \neq p$

[10] A *lemma* is an auxiliary theorem used in the proof of other theorems.

Note that elementary matrices of type I are self-inverse.

Example 2.5.1

Find the inverse of each of the elementary matrices

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}, \quad \text{and} \quad E_3 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution. E_1, E_2, and E_3 are of type I, II, and III respectively, so the table gives

$$E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{9} \end{bmatrix}, \quad \text{and} \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Inverses and Elementary Matrices

Suppose that an $m \times n$ matrix A is carried to a matrix B (written $A \to B$) by a series of k elementary row operations. Let $E_1, E_2, \dots, E_k$ denote the corresponding elementary matrices. By Lemma 2.5.1, the reduction becomes

$$A \to E_1A \to E_2E_1A \to E_3E_2E_1A \to \cdots \to E_kE_{k-1}\cdots E_2E_1A = B$$

In other words,

$$A \to UA = B \quad \text{where } U = E_kE_{k-1}\cdots E_2E_1$$

The matrix $U = E_kE_{k-1}\cdots E_2E_1$ is invertible, being a product of invertible matrices by Lemma 2.5.2. Moreover, U can be computed without finding the E_i as follows: If the above series of operations carrying $A \to B$ is performed on I_m in place of A, the result is $I_m \to UI_m = U$. Hence this series of operations carries the block matrix $\begin{bmatrix} A & I_m \end{bmatrix} \to \begin{bmatrix} B & U \end{bmatrix}$. This, together with the above discussion, proves

Theorem 2.5.1

Suppose A is $m \times n$ and $A \to B$ by elementary row operations.

1. *$B = UA$ where U is an $m \times m$ invertible matrix.*
2. *U can be computed by $\begin{bmatrix} A & I_m \end{bmatrix} \to \begin{bmatrix} B & U \end{bmatrix}$ using the operations carrying $A \to B$.*
3. *$U = E_kE_{k-1}\cdots E_2E_1$ where $E_1, E_2, \dots, E_k$ are the elementary matrices corresponding (in order) to the elementary row operations carrying A to B.*

Example 2.5.2

If $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, express the reduced row-echelon form R of A as $R = UA$ where U is invertible.

Solution. Reduce the double matrix $\begin{bmatrix} A & I \end{bmatrix} \to \begin{bmatrix} R & U \end{bmatrix}$ as follows:

$$\begin{bmatrix} A & I \end{bmatrix} = \left[\begin{array}{ccc|cc} 2 & 3 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 0 & 1 \\ 2 & 3 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 0 & 1 \\ 0 & -1 & -1 & 1 & -2 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|cc} 1 & 0 & -1 & 2 & -3 \\ 0 & 1 & 1 & -1 & 2 \end{array}\right]$$

Hence $R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$.

Now suppose that A is invertible. We know that $A \to I$ by Theorem 2.4.5, so taking $B = I$ in Theorem 2.5.1 gives $\begin{bmatrix} A & I \end{bmatrix} \to \begin{bmatrix} I & U \end{bmatrix}$ where $I = UA$. Thus $U = A^{-1}$, so we have $\begin{bmatrix} A & I \end{bmatrix} \to \begin{bmatrix} I & A^{-1} \end{bmatrix}$. This is the matrix inversion algorithm in Section 2.4. However, more is true: Theorem 2.5.1 gives $A^{-1} = U = E_k E_{k-1} \cdots E_2 E_1$ where $E_1, E_2, \dots, E_k$ are the elementary matrices corresponding (in order) to the row operations carrying $A \to I$. Hence

$$A = \left(A^{-1}\right)^{-1} = (E_k E_{k-1} \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1} \tag{2.10}$$

By Lemma 2.5.2, this shows that every invertible matrix A is a product of elementary matrices. Since elementary matrices are invertible (again by Lemma 2.5.2), this proves the following important characterization of invertible matrices.

Theorem 2.5.2

A square matrix is invertible if and only if it is a product of elementary matrices.

It follows from Theorem 2.5.1 that $A \to B$ by row operations if and only if $B = UA$ for some invertible matrix B. In this case we say that A and B are **row-equivalent**. (See Exercise 2.5.17.)

Example 2.5.3

Express $A = \begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix}$ as a product of elementary matrices.

Solution. Using Lemma 2.5.1, the reduction of $A \to I$ is as follows:

$$A = \begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix} \to E_1 A = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix} \to E_2 E_1 A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \to E_3 E_2 E_1 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where the corresponding elementary matrices are

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

Hence $(E_3\, E_2\, E_1)A = I$, so:

$$A = (E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1} = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]\left[\begin{array}{cc} 1 & 0 \\ -2 & 1 \end{array}\right]\left[\begin{array}{cc} 1 & 0 \\ 0 & 3 \end{array}\right]$$

Smith Normal Form

Let A be an $m \times n$ matrix of rank r, and let R be the reduced row-echelon form of A. Theorem 2.5.1 shows that $R = UA$ where U is invertible, and that U can be found from $\left[\begin{array}{cc} A & I_m \end{array}\right] \to \left[\begin{array}{cc} R & U \end{array}\right]$.

The matrix R has r leading ones (since $\operatorname{rank} A = r$) so, as R is reduced, the $n \times m$ matrix R^T contains each row of I_r in the first r columns. Thus row operations will carry $R^T \to \left[\begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array}\right]_{n \times m}$. Hence Theorem 2.5.1 (again) shows that $\left[\begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array}\right]_{n \times m} = U_1R^T$ where U_1 is an $n \times n$ invertible matrix. Writing $V = U_1^T$, we obtain

$$UAV = RV = RU_1^T = \left(U_1R^T\right)^T = \left(\left[\begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array}\right]_{n \times m}\right)^T = \left[\begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array}\right]_{m \times n}$$

Moreover, the matrix $U_1 = V^T$ can be computed by $\left[\begin{array}{cc} R^T & I_n \end{array}\right] \to \left[\begin{array}{cc} \left[\begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array}\right]_{n \times m} & V^T \end{array}\right]$. This proves

Theorem 2.5.3

Let A be an $m \times n$ matrix of rank r. There exist invertible matrices U and V of size $m \times m$ and $n \times n$, respectively, such that

$$UAV = \left[\begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array}\right]_{m \times n}$$

Moreover, if R is the reduced row-echelon form of A, then:

1. *U can be computed by $\left[\begin{array}{cc} A & I_m \end{array}\right] \to \left[\begin{array}{cc} R & U \end{array}\right]$;*
2. *V can be computed by $\left[\begin{array}{cc} R^T & I_n \end{array}\right] \to \left[\begin{array}{cc} \left[\begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array}\right]_{n \times m} & V^T \end{array}\right]$.*

If A is an $m \times n$ matrix of rank r, the matrix $\left[\begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array}\right]$ is called the **Smith normal form**[11] of A. Whereas the reduced row-echelon form of A is the "nicest" matrix to which A can be carried by row operations, the Smith canonical form is the "nicest" matrix to which A can be carried by *row and column* operations. This is because doing row operations to R^T amounts to doing *column* operations to R and then transposing.

[11]Named after Henry John Stephen Smith (1826–83).

Example 2.5.4

Given $A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 2 & -2 & 1 & -1 \\ -1 & 1 & 0 & 3 \end{bmatrix}$, find invertible matrices U and V such that $UAV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, where $r = \text{rank } A$.

Solution. The matrix U and the reduced row-echelon form R of A are computed by the row reduction $\begin{bmatrix} A & I_3 \end{bmatrix} \to \begin{bmatrix} R & U \end{bmatrix}$:

$$\left[\begin{array}{rrrr|rrr} 1 & -1 & 1 & 2 & 1 & 0 & 0 \\ 2 & -2 & 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrrr|rrr} 1 & -1 & 0 & -3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 5 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

Hence

$$R = \begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

In particular, $r = \text{rank } R = 2$. Now row-reduce $\begin{bmatrix} R^T & I_4 \end{bmatrix} \to \left[\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \; V^T \right]$:

$$\left[\begin{array}{rrr|rrrr} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ -3 & 5 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrrr} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & -5 & 1 \end{array}\right]$$

whence

$$V^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 0 & -5 & -1 \end{bmatrix} \quad \text{so} \quad V = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then $UAV = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ as is easily verified.

Uniqueness of the Reduced Row-echelon Form

In this short subsection, Theorem 2.5.1 is used to prove the following important theorem.

Theorem 2.5.4

If a matrix A is carried to reduced row-echelon matrices R and S by row operations, then $R = S$.

Proof. Observe first that $UR = S$ for some invertible matrix U (by Theorem 2.5.1 there exist invertible matrices P and Q such that $R = PA$ and $S = QA$; take $U = QP^{-1}$). We show that $R = S$ by induction on the number m of rows of R and S. The case $m=1$ is left to the reader. If R_j and S_j denote column j in R and S respectively, the fact that $UR=S$ gives

$$UR_j = S_j \quad \text{for each } j \tag{2.11}$$

Since U is invertible, this shows that R and S have the same zero columns. Hence, by passing to the matrices obtained by deleting the zero columns from R and S, we may assume that R and S have no zero columns.

But then the first column of R and S is the first column of I_m because R and S are row-echelon, so (2.11) shows that the first column of U is column 1 of I_m. Now write U, R, and S in block form as follows.

$$U = \left[\begin{array}{cc} 1 & X \\ 0 & V \end{array} \right], \quad R = \left[\begin{array}{cc} 1 & X \\ 0 & R' \end{array} \right], \quad \text{and} \quad S = \left[\begin{array}{cc} 1 & Z \\ 0 & S' \end{array} \right]$$

Since $UR=S$, block multiplication gives $VR'=S'$ so, since V is invertible (U is invertible) and both R' and S' are reduced row-echelon, we obtain $R'=S'$ by induction. Hence R and S have the same number (say r) of leading 1s, and so both have m–r zero rows.

In fact, R and S have leading ones in the same columns, say r of them. Applying (2.11) to these columns shows that the first r columns of U are the first r columns of I_m. Hence we can write U, R, and S in block form as follows:

$$U = \left[\begin{array}{cc} I_r & M \\ 0 & W \end{array} \right], \quad R = \left[\begin{array}{cc} R_1 & R_2 \\ 0 & 0 \end{array} \right], \quad \text{and} \quad S = \left[\begin{array}{cc} S_1 & S_2 \\ 0 & 0 \end{array} \right]$$

where R_1 and S_1 are $r \times r$. Then block multiplication gives $UR=R$; that is, $S=R$. This completes the proof. □

Exercises for 2.5

Exercise 2.5.1 For each of the following elementary matrices, describe the corresponding elementary row operation and write the inverse.

a. $E = \left[\begin{array}{ccc} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$

b. $E = \left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array} \right]$

c. $E = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{array} \right]$

d. $E = \left[\begin{array}{rcc} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$

e. $E = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right]$

f. $E = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{array} \right]$

Exercise 2.5.2 In each case find an elementary matrix E such that $B=EA$.

a. $A = \left[\begin{array}{cr} 2 & 1 \\ 3 & -1 \end{array} \right], B = \left[\begin{array}{cr} 2 & 1 \\ 1 & -2 \end{array} \right]$

b. $A = \left[\begin{array}{rc} -1 & 2 \\ 0 & 1 \end{array} \right], B = \left[\begin{array}{cr} 1 & -2 \\ 0 & 1 \end{array} \right]$

c. $A = \left[\begin{array}{rc} 1 & 1 \\ -1 & 2 \end{array} \right], B = \left[\begin{array}{rc} -1 & 2 \\ 1 & 1 \end{array} \right]$

d. $A = \left[\begin{array}{cc} 4 & 1 \\ 3 & 2 \end{array} \right], B = \left[\begin{array}{cr} 1 & -1 \\ 3 & 2 \end{array} \right]$

e. $A = \left[\begin{array}{rr} -1 & 1 \\ 1 & -1 \end{array} \right], B = \left[\begin{array}{rc} -1 & 1 \\ -1 & 1 \end{array} \right]$

f. $A=\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}, B=\begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$

Exercise 2.5.3 Let $A=\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ and $C=\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$.

a. Find elementary matrices E_1 and E_2 such that $C=E_2E_1A$.

b. Show that there is *no* elementary matrix E such that $C=EA$.

Exercise 2.5.4 If E is elementary, show that A and EA differ in at most two rows.

Exercise 2.5.5

a. Is I an elementary matrix? Explain.

b. Is 0 an elementary matrix? Explain.

Exercise 2.5.6 In each case find an invertible matrix U such that $UA=R$ is in reduced row-echelon form, and express U as a product of elementary matrices.

a. $A=\begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 0 \end{bmatrix}$

b. $A=\begin{bmatrix} 1 & 2 & 1 \\ 5 & 12 & -1 \end{bmatrix}$

c. $A=\begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & 1 & 1 & 2 \\ 1 & -3 & 3 & 2 \end{bmatrix}$

d. $A=\begin{bmatrix} 2 & 1 & -1 & 0 \\ 3 & -1 & 2 & 1 \\ 1 & -2 & 3 & 1 \end{bmatrix}$

Exercise 2.5.7 In each case find an invertible matrix U such that $UA=B$, and express U as a product of elementary matrices.

a. $A=\begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & 2 \end{bmatrix}, B=\begin{bmatrix} 1 & -1 & -2 \\ 3 & 0 & 1 \end{bmatrix}$

b. $A=\begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, B=\begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$

Exercise 2.5.8 In each case factor A as a product of elementary matrices.

a. $A=\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

b. $A=\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

c. $A=\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 6 \end{bmatrix}$

d. $A=\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ -2 & 2 & 15 \end{bmatrix}$

Exercise 2.5.9 Let E be an elementary matrix.

a. Show that E^T is also elementary of the same type.

b. Show that $E^T=E$ if E is of type I or II.

Exercise 2.5.10 Show that every matrix A can be factored as $A=UR$ where U is invertible and R is in reduced row-echelon form.

Exercise 2.5.11 If $A=\begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}$ and $B=\begin{bmatrix} 5 & 2 \\ -5 & -3 \end{bmatrix}$ find an elementary matrix F such that $AF=B$.

[*Hint*: See Exercise 2.5.9.]

Exercise 2.5.12 In each case find invertible U and V such that $UAV=\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, where $r=\operatorname{rank} A$.

a. $A=\begin{bmatrix} 1 & 1 & -1 \\ -2 & -2 & 4 \end{bmatrix}$

b. $A=\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$

c. $A=\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & 0 & 3 \\ 0 & 1 & -4 & 1 \end{bmatrix}$

d. $A=\begin{bmatrix} 1 & 1 & 0 & -1 \\ 3 & 2 & 1 & 1 \\ 1 & 0 & 1 & 3 \end{bmatrix}$

Exercise 2.5.13 Prove Lemma 2.5.1 for elementary matrices of:

a. type I;

b. type II.

Exercise 2.5.14 While trying to invert A, $\begin{bmatrix} A & I \end{bmatrix}$ is carried to $\begin{bmatrix} P & Q \end{bmatrix}$ by row operations. Show that $P=QA$.

Exercise 2.5.15 If A and B are $n\times n$ matrices and AB is a product of elementary matrices, show that the same is true of A.

Exercise 2.5.16 If U is invertible, show that the reduced row-echelon form of a matrix $\left[\begin{array}{ll} U & A \end{array}\right]$ is $\left[\begin{array}{ll} I & U^{-1}A \end{array}\right]$.

Exercise 2.5.17 Two matrices A and B are called **row-equivalent** (written $A \stackrel{r}{\sim} B$) if there is a sequence of elementary row operations carrying A to B.

a. Show that $A \stackrel{r}{\sim} B$ if and only if $A = UB$ for some invertible matrix U.

b. Show that:

   i. $A \stackrel{r}{\sim} A$ for all matrices A.

   ii. If $A \stackrel{r}{\sim} B$, then $B \stackrel{r}{\sim} A$

   iii. If $A \stackrel{r}{\sim} B$ and $B \stackrel{r}{\sim} C$, then $A \stackrel{r}{\sim} C$.

c. Show that, if A and B are both row-equivalent to some third matrix, then $A \stackrel{r}{\sim} B$.

d. Show that $\left[\begin{array}{rrrr} 1 & -1 & 3 & 2 \\ 0 & 1 & 4 & 1 \\ 1 & 0 & 8 & 6 \end{array}\right]$ and $\left[\begin{array}{rrrr} 1 & -1 & 4 & 5 \\ -2 & 1 & -11 & -8 \\ -1 & 2 & 2 & 2 \end{array}\right]$ are row-equivalent. [*Hint*: Consider (c) and Theorem 1.2.1.]

Exercise 2.5.18 If U and V are invertible $n \times n$ matrices, show that $U \stackrel{r}{\sim} V$. (See Exercise 2.5.17.)

Exercise 2.5.19 (See Exercise 2.5.17.) Find all matrices that are row-equivalent to:

a. $\left[\begin{array}{lll} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

b. $\left[\begin{array}{lll} 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right]$

c. $\left[\begin{array}{lll} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$

d. $\left[\begin{array}{lll} 1 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right]$

Exercise 2.5.20 Let A and B be $m \times n$ and $n \times m$ matrices, respectively. If $m > n$, show that AB is not invertible. [*Hint*: Use Theorem 1.3.1 to find $\mathbf{x} \neq \mathbf{0}$ with $B\mathbf{x} = \mathbf{0}$.]

Exercise 2.5.21 Define an *elementary column operation* on a matrix to be one of the following: (I) Interchange two columns. (II) Multiply a column by a nonzero scalar. (III) Add a multiple of a column to another column. Show that:

a. If an elementary column operation is done to an $m \times n$ matrix A, the result is AF, where F is an $n \times n$ elementary matrix.

b. Given any $m \times n$ matrix A, there exist $m \times m$ elementary matrices $E_1, \dots, E_k$ and $n \times n$ elementary matrices $F_1, \dots, F_p$ such that, in block form,

$$E_k \cdots E_1 A F_1 \cdots F_p = \left[\begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array}\right]$$

Exercise 2.5.22 Suppose B is obtained from A by:

a. interchanging rows i and j;

b. multiplying row i by $k \neq 0$;

c. adding k times row i to row j ($i \neq j$).

In each case describe how to obtain B^{-1} from A^{-1}. [*Hint*: See part (a) of the preceding exercise.]

Exercise 2.5.23 Two $m \times n$ matrices A and B are called **equivalent** (written $A \stackrel{e}{\sim} B$) if there exist invertible matrices U and V (sizes $m \times m$ and $n \times n$) such that $A = UBV$.

a. Prove the following the properties of equivalence.

   i. $A \stackrel{e}{\sim} A$ for all $m \times n$ matrices A.

   ii. If $A \stackrel{e}{\sim} B$, then $B \stackrel{e}{\sim} A$.

   iii. If $A \stackrel{e}{\sim} B$ and $B \stackrel{e}{\sim} C$, then $A \stackrel{e}{\sim} C$.

b. Prove that two $m \times n$ matrices are equivalent if they have the same rank. [*Hint*: Use part (a) and Theorem 2.5.3.]

2.6 Linear Transformations

If A is an $m \times n$ matrix, recall that the transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T_A(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

is called the *matrix transformation induced* by A. In Section 2.2, we saw that many important geometric transformations were in fact matrix transformations. These transformations can be characterized in a different way. The new idea is that of a linear transformation, one of the basic notions in linear algebra. We define these transformations in this section, and show that they are really just the matrix transformations looked at in another way. Having these two ways to view them turns out to be useful because, in a given situation, one perspective or the other may be preferable.

Linear Transformations

Definition 2.13 Linear Transformations $\mathbb{R}^n \to \mathbb{R}^m$

*A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a **linear transformation** if it satisfies the following two conditions for all vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and all scalars a:*

T1 $\quad T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$

T2 $\quad T(a\mathbf{x}) = aT(\mathbf{x})$

Of course, $\mathbf{x}+\mathbf{y}$ and $a\mathbf{x}$ here are computed in $\mathbb{R}^n$, while $T(\mathbf{x}) + T(\mathbf{y})$ and $aT(\mathbf{x})$ are in $\mathbb{R}^m$. We say that T *preserves addition* if T1 holds, and that T *preserves scalar multiplication* if T2 holds. Moreover, taking $a = 0$ and $a = -1$ in T2 gives

$$T(\mathbf{0}) = \mathbf{0} \quad \text{and} \quad T(-\mathbf{x}) = -T(\mathbf{x}) \quad \text{for all } \mathbf{x}$$

Hence T preserves the zero vector and the negative of a vector. Even more is true.

Recall that a vector $\mathbf{y}$ in $\mathbb{R}^n$ is called a **linear combination** of vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ if $\mathbf{y}$ has the form

$$\mathbf{y} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k$$

for some scalars $a_1, a_2, \dots, a_k$. Conditions T1 and T2 combine to show that every linear transformation T *preserves linear combinations* in the sense of the following theorem. This result is used repeatedly in linear algebra.

Theorem 2.6.1: Linearity Theorem

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then for each $k = 1, 2, \dots$

$$T(a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k) = a_1T(\mathbf{x}_1) + a_2T(\mathbf{x}_2) + \cdots + a_kT(\mathbf{x}_k)$$

for all scalars a_i and all vectors $\mathbf{x}_i$ in $\mathbb{R}^n$.

Proof. If $k=1$, it reads $T(a_1\mathbf{x}_1) = a_1T(\mathbf{x}_1)$ which is Condition T1. If $k=2$, we have

$$\begin{array}{rcll} T(a_1\mathbf{x}_1 + a_2\mathbf{x}_2) & = & T(a_1\mathbf{x}_1) + T(a_2\mathbf{x}_2) & \text{by Condition T1} \\ & = & a_1T(\mathbf{x}_1) + a_2T(\mathbf{x}_2) & \text{by Condition T2} \end{array}$$

If $k=3$, we use the case $k=2$ to obtain

$$\begin{array}{rcll} T(a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + a_3\mathbf{x}_3) & = & T\left[(a_1\mathbf{x}_1 + a_2\mathbf{x}_2) + a_3\mathbf{x}_3\right] & \text{collect terms} \\ & = & T(a_1\mathbf{x}_1 + a_2\mathbf{x}_2) + T(a_3\mathbf{x}_3) & \text{by Condition T1} \\ & = & [a_1T(\mathbf{x}_1) + a_2T(\mathbf{x}_2)] + T(a_3\mathbf{x}_3) & \text{by the case } k=2 \\ & = & [a_1T(\mathbf{x}_1) + a_2T(\mathbf{x}_2)] + a_3T(\mathbf{x}_3) & \text{by Condition T2} \end{array}$$

The proof for any k is similar, using the previous case $k-1$ and Conditions T1 and T2. □

The method of proof in Theorem 2.6.1 is called *mathematical induction* (Appendix C).

Theorem 2.6.1 shows that if T is a linear transformation and $T(\mathbf{x}_1)$, $T(\mathbf{x}_2)$, $\dots$, $T(\mathbf{x}_k)$ are all known, then $T(\mathbf{y})$ can be easily computed for any linear combination $\mathbf{y}$ of $\mathbf{x}_1$, $\mathbf{x}_2$, $\dots$, $\mathbf{x}_k$. This is a very useful property of linear transformations, and is illustrated in the next example.

Example 2.6.1

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, $T\left[\begin{array}{c} 1 \\ 1 \end{array}\right] = \left[\begin{array}{r} 2 \\ -3 \end{array}\right]$ and $T\left[\begin{array}{r} 1 \\ -2 \end{array}\right] = \left[\begin{array}{c} 5 \\ 1 \end{array}\right]$, find $T\left[\begin{array}{c} 4 \\ 3 \end{array}\right]$.

Solution. Write $\mathbf{z} = \left[\begin{array}{c} 4 \\ 3 \end{array}\right]$, $\mathbf{x} = \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$, and $\mathbf{y} = \left[\begin{array}{r} 1 \\ -2 \end{array}\right]$ for convenience. Then we know $T(\mathbf{x})$ and $T(\mathbf{y})$ and we want $T(\mathbf{z})$, so it is enough by Theorem 2.6.1 to express $\mathbf{z}$ as a linear combination of $\mathbf{x}$ and $\mathbf{y}$. That is, we want to find numbers a and b such that $\mathbf{z} = a\mathbf{x} + b\mathbf{y}$. Equating entries gives two equations $4 = a+b$ and $3 = a-2b$. The solution is, $a = \frac{11}{3}$ and $b = \frac{1}{3}$, so $\mathbf{z} = \frac{11}{3}\mathbf{x} + \frac{1}{3}\mathbf{y}$. Thus Theorem 2.6.1 gives

$$T(\mathbf{z}) = \tfrac{11}{3}T(\mathbf{x}) + \tfrac{1}{3}T(\mathbf{y}) = \tfrac{11}{3}\left[\begin{array}{r} 2 \\ -3 \end{array}\right] + \tfrac{1}{3}\left[\begin{array}{c} 5 \\ 1 \end{array}\right] = \tfrac{1}{3}\left[\begin{array}{r} 27 \\ -32 \end{array}\right]$$

This is what we wanted.

Example 2.6.2

If A is $m \times n$, the matrix transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$, is a linear transformation.

Solution. We have $T_A(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$, so Theorem 2.2.2 gives

$$T_A(\mathbf{x}+\mathbf{y}) = A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = T_A(\mathbf{x}) + T_A(\mathbf{y})$$

and

$$T_A(a\mathbf{x}) = A(a\mathbf{x}) = a(A\mathbf{x}) = aT_A(\mathbf{x})$$

hold for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and all scalars a. Hence T_A satisfies T1 and T2, and so is linear.

The remarkable thing is that the *converse* of Example 2.6.2 is true: Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is actually a matrix transformation. To see why, we define the **standard basis** of $\mathbb{R}^n$ to be the set of columns

$$\{\mathbf{e}_1,\ \mathbf{e}_2,\ \dots,\ \mathbf{e}_n\}$$

of the identity matrix I_n. Then each $\mathbf{e}_i$ is in $\mathbb{R}^n$ and every vector $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right]$ in $\mathbb{R}^n$ is a linear combination of the $\mathbf{e}_i$. In fact:

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

as the reader can verify. Hence Theorem 2.6.1 shows that

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n) = x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) + \cdots + x_nT(\mathbf{e}_n)$$

Now observe that each $T(\mathbf{e}_i)$ is a column in $\mathbb{R}^m$, so

$$A = \left[\begin{array}{cccc} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{array}\right]$$

is an $m \times n$ matrix. Hence we can apply Definition 2.5 to get

$$T(\mathbf{x}) = x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) + \cdots + x_nT(\mathbf{e}_n) = \left[\begin{array}{cccc} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right] = A\mathbf{x}$$

Since this holds for every $\mathbf{x}$ in $\mathbb{R}^n$, it shows that T is the matrix transformation induced by A, and so proves most of the following theorem.

Theorem 2.6.2

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a transformation.

1. *T is linear if and only if it is a matrix transformation.*
2. *In this case $T = T_A$ is the matrix transformation induced by a unique $m \times n$ matrix A, given in terms of its columns by*

$$A = \left[\begin{array}{cccc} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{array}\right]$$

where $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \dots,\ \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$.

Proof. It remains to verify that the matrix A is unique. Suppose that T is induced by another matrix B. Then $T(\mathbf{x}) = B\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. But $T(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x}$, so $B\mathbf{x} = A\mathbf{x}$ for every $\mathbf{x}$. Hence $A = B$ by Theorem 2.2.6. □

Hence we can speak of *the* matrix of a linear transformation. Because of Theorem 2.6.2 we may (and shall) use the phrases "linear transformation" and "matrix transformation" interchangeably.

Example 2.6.3

Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for all $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$. Show that T is a linear transformation and use Theorem 2.6.2 to find its matrix.

Solution. Write $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, so that $\mathbf{x}+\mathbf{y} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}$. Hence

$$T(\mathbf{x}+\mathbf{y}) = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = T(\mathbf{x}) + T(\mathbf{y})$$

Similarly, the reader can verify that $T(a\mathbf{x}) = aT(\mathbf{x})$ for all a in $\mathbb{R}$, so T is a linear transformation. Now the standard basis of $\mathbb{R}^3$ is

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

so, by Theorem 2.6.2, the matrix of T is

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Of course, the fact that $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ shows directly that T is a matrix transformation (hence linear) and reveals the matrix.

To illustrate how Theorem 2.6.2 is used, we rederive the matrices of the transformations in Examples 2.2.13 and 2.2.15.

Example 2.6.4

Let $Q_0 : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection in the x axis (as in Example 2.2.13) and let $R_{\frac{\pi}{2}} : \mathbb{R}^2 \to \mathbb{R}^2$ denote counterclockwise rotation through $\frac{\pi}{2}$ about the origin (as in Example 2.2.15). Use Theorem 2.6.2 to find the matrices of Q_0 and $R_{\frac{\pi}{2}}$.

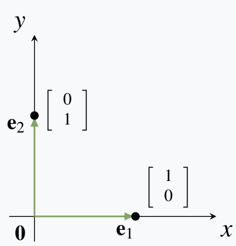


Figure 2.6.1

Solution. Observe that Q_0 and $R_{\frac{\pi}{2}}$ are linear by Example 2.6.2 (they are matrix transformations), so Theorem 2.6.2 applies to them. The standard basis of $\mathbb{R}^2$ is $\{\mathbf{e}_1, \mathbf{e}_2\}$ where $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ points along the positive x axis, and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ points along the positive y axis (see Figure 2.6.1).

The reflection of $\mathbf{e}_1$ in the x axis is $\mathbf{e}_1$ itself because $\mathbf{e}_1$ points along the x axis, and the reflection of $\mathbf{e}_2$ in the x axis is $-\mathbf{e}_2$ because $\mathbf{e}_2$ is perpendicular to the x axis. In other words, $Q_0(\mathbf{e}_1) = \mathbf{e}_1$ and $Q_0(\mathbf{e}_2) = -\mathbf{e}_2$. Hence Theorem 2.6.2 shows that the matrix of Q_0 is

$$\left[\begin{array}{cc} Q_0(\mathbf{e}_1) & Q_0(\mathbf{e}_2) \end{array}\right] = \left[\begin{array}{cc} \mathbf{e}_1 & -\mathbf{e}_2 \end{array}\right] = \left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right]$$

which agrees with Example 2.2.13.

Similarly, rotating $\mathbf{e}_1$ through $\frac{\pi}{2}$ counterclockwise about the origin produces $\mathbf{e}_2$, and rotating $\mathbf{e}_2$ through $\frac{\pi}{2}$ counterclockwise about the origin gives $-\mathbf{e}_1$. That is, $R_{\frac{\pi}{2}}(\mathbf{e}_1) = \mathbf{e}_2$ and $R_{\frac{\pi}{2}}(\mathbf{e}_2) = -\mathbf{e}_2$. Hence, again by Theorem 2.6.2, the matrix of $R_{\frac{\pi}{2}}$ is

$$\left[\begin{array}{cc} R_{\frac{\pi}{2}}(\mathbf{e}_1) & R_{\frac{\pi}{2}}(\mathbf{e}_2) \end{array}\right] = \left[\begin{array}{cc} \mathbf{e}_2 & -\mathbf{e}_1 \end{array}\right] = \left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array}\right]$$

agreeing with Example 2.2.15.

Example 2.6.5

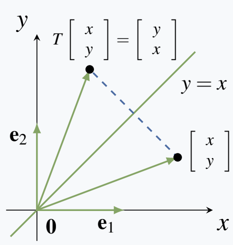


Figure 2.6.2

Let $Q_1 : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection in the line $y = x$. Show that Q_1 is a matrix transformation, find its matrix, and use it to illustrate Theorem 2.6.2.

Solution. Figure 2.6.2 shows that $Q_1 \left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} y \\ x \end{array}\right]$. Hence $Q_1 \left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] \left[\begin{array}{c} y \\ x \end{array}\right]$, so Q_1 is the matrix transformation induced by the matrix $A = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]$. Hence Q_1 is linear (by Example 2.6.2) and so Theorem 2.6.2 applies. If $\mathbf{e}_1 = \left[\begin{array}{c} 1 \\ 0 \end{array}\right]$ and $\mathbf{e}_2 = \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$ are the standard basis of $\mathbb{R}^2$, then it is clear geometrically that $Q_1(\mathbf{e}_1) = \mathbf{e}_2$ and $Q_1(\mathbf{e}_2) = \mathbf{e}_1$. Thus (by Theorem 2.6.2) the matrix of Q_1 is $\left[\begin{array}{cc} Q_1(\mathbf{e}_1) & Q_1(\mathbf{e}_2) \end{array}\right] = \left[\begin{array}{cc} \mathbf{e}_2 & \mathbf{e}_1 \end{array}\right] = A$ as before.

Recall that, given two "linked" transformations

$$\mathbb{R}^k \xrightarrow{T} \mathbb{R}^n \xrightarrow{S} \mathbb{R}^m$$

we can apply T first and then apply S, and so obtain a new transformation

$$S \circ T : \mathbb{R}^k \to \mathbb{R}^m$$

called the **composite** of S and T, defined by

$$(S \circ T)(\mathbf{x}) = S[T(\mathbf{x})] \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^k$$

If S and T are linear, the action of $S \circ T$ can be computed by multiplying their matrices.

Theorem 2.6.3

Let $\mathbb{R}^k \xrightarrow{T} \mathbb{R}^n \xrightarrow{S} \mathbb{R}^m$ be linear transformations, and let A and B be the matrices of S and T respectively. Then $S \circ T$ is linear with matrix AB.

Proof. $(S \circ T)(\mathbf{x}) = S[T(\mathbf{x})] = A[B\mathbf{x}] = (AB)\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^k$. □

Theorem 2.6.3 shows that the action of the composite $S \circ T$ is determined by the matrices of S and T. But it also provides a very useful interpretation of matrix multiplication. If A and B are matrices, the product matrix AB induces the transformation resulting from first applying B and then applying A. Thus the study of matrices can cast light on geometrical transformations and vice-versa. Here is an example.

Example 2.6.6

Show that reflection in the x axis followed by rotation through $\frac{\pi}{2}$ is reflection in the line $y = x$.

Solution. The composite in question is $R_{\frac{\pi}{2}} \circ Q_0$ where Q_0 is reflection in the x axis and $R_{\frac{\pi}{2}}$ is rotation through $\frac{\pi}{2}$. By Example 2.6.4, $R_{\frac{\pi}{2}}$ has matrix $A = \left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array}\right]$ and Q_0 has matrix $B = \left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right]$. Hence Theorem 2.6.3 shows that the matrix of $R_{\frac{\pi}{2}} \circ Q_0$ is $AB = \left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array}\right]\left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right] = \left[\begin{array}{rr} 0 & 1 \\ 1 & 0 \end{array}\right]$, which is the matrix of reflection in the line $y = x$ by Example 2.6.3.

This conclusion can also be seen geometrically. Let $\mathbf{x}$ be a typical point in $\mathbb{R}^2$, and assume that $\mathbf{x}$ makes an angle α with the positive x axis. The effect of first applying Q_0 and then applying $R_{\frac{\pi}{2}}$ is shown in Figure 2.6.3. The fact that $R_{\frac{\pi}{2}}[Q_0(\mathbf{x})]$ makes the angle α with the positive y axis shows that $R_{\frac{\pi}{2}}[Q_0(\mathbf{x})]$ is the reflection of $\mathbf{x}$ in the line $y = x$.

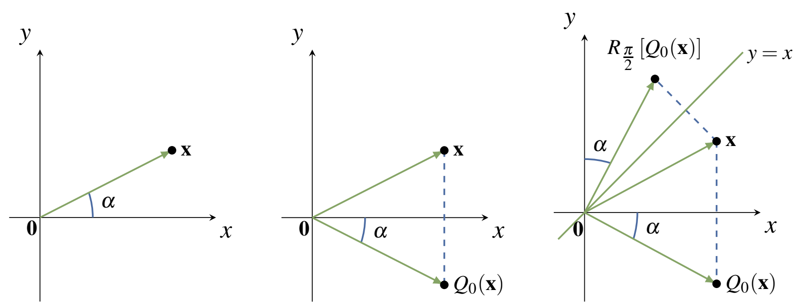


Figure 2.6.3

In Theorem 2.6.3, we saw that the matrix of the composite of two linear transformations is the product of their matrices (in fact, matrix products were defined so that this is the case). We are going to apply this fact to rotations, reflections, and projections in the plane. Before proceeding, we pause to present useful geometrical descriptions of vector addition and scalar multiplication in the plane, and to give a short review of angles and the trigonometric functions.

Some Geometry

As we have seen, it is convenient to view a vector $\mathbf{x}$ in $\mathbb{R}^2$ as an arrow from the origin to the point $\mathbf{x}$ (see Section 2.2). This enables us to visualize what sums and scalar multiples mean geometrically. For example consider $\mathbf{x} = \left[\begin{array}{c} 1 \\ 2 \end{array}\right]$ in $\mathbb{R}^2$. Then $2\mathbf{x} = \left[\begin{array}{c} 2 \\ 4 \end{array}\right]$, $\frac{1}{2}\mathbf{x} = \left[\begin{array}{c} \frac{1}{2} \\ 1 \end{array}\right]$ and $-\frac{1}{2}\mathbf{x} = \left[\begin{array}{c} -\frac{1}{2} \\ -1 \end{array}\right]$, and these are shown as arrows in Figure 2.6.4.

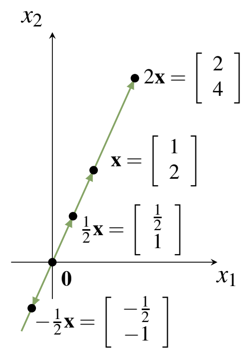


Figure 2.6.4

Observe that the arrow for $2\mathbf{x}$ is twice as long as the arrow for $\mathbf{x}$ and in the same direction, and that the arrows for $\frac{1}{2}\mathbf{x}$ is also in the same direction as the arrow for $\mathbf{x}$, but only half as long. On the other hand, the arrow for $-\frac{1}{2}\mathbf{x}$ is half as long as the arrow for $\mathbf{x}$, but in the *opposite* direction. More generally, we have the following geometrical description of scalar multiplication in $\mathbb{R}^2$:

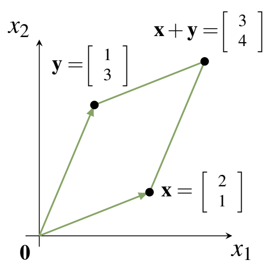


Figure 2.6.5

Scalar Multiple Law

Let $\mathbf{x}$ be a vector in $\mathbb{R}^2$. The arrow for $k\mathbf{x}$ is $|k|$ times[12] *as long as the arrow for $\mathbf{x}$, and is in the same direction as the arrow for $\mathbf{x}$ if $k > 0$, and in the opposite direction if $k < 0$.*

Now consider two vectors $\mathbf{x} = \left[\begin{array}{c} 2 \\ 1 \end{array}\right]$ and $\mathbf{y} = \left[\begin{array}{c} 1 \\ 3 \end{array}\right]$ in $\mathbb{R}^2$. They are plotted in Figure 2.6.5 along with their sum $\mathbf{x}+\mathbf{y} = \left[\begin{array}{c} 3 \\ 4 \end{array}\right]$. It is a routine matter to verify that the four points $\mathbf{0}$, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{x}+\mathbf{y}$ form the vertices of a **parallelogram**–that is opposite sides are parallel and of the same length. (The reader should verify that the side from $\mathbf{0}$ to $\mathbf{x}$ has slope of $\frac{1}{2}$, as does the side from $\mathbf{y}$ to $\mathbf{x}+\mathbf{y}$, so these sides are parallel.) We state this as follows:

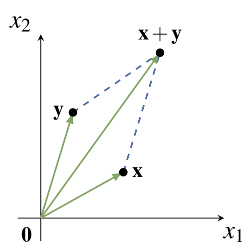


Figure 2.6.6

Parallelogram Law

Consider vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$. If the arrows for $\mathbf{x}$ and $\mathbf{y}$ are drawn (see Figure 2.6.6), the arrow for $\mathbf{x}+\mathbf{y}$ corresponds to the fourth vertex of the parallelogram determined by the points $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{0}$.

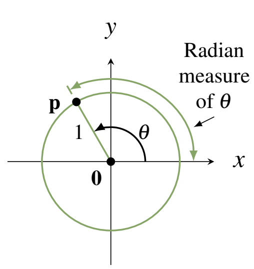


Figure 2.6.7

We will have more to say about this in Chapter 4.

Before proceeding we turn to a brief review of angles and the trigonometric functions. Recall that an angle θ is said to be in **standard position** if it is measured counterclockwise from the positive x axis (as in Figure 2.6.7). Then θ uniquely determines a point $\mathbf{p}$ on the **unit circle**

[12]If k is a real number, $|k|$ denotes the **absolute value** of k; that is, $|k| = k$ if $k \geq 0$ and $|k| = -k$ if $k < 0$.

(radius 1, centre at the origin). The **radian** measure of θ is the length of the arc on the unit circle from the positive x axis to $\mathbf{p}$. Thus $360° = 2\pi$ radians, $180° = \pi$, $90° = \frac{\pi}{2}$, and so on.

The point $\mathbf{p}$ in Figure 2.6.7 is also closely linked to the trigonometric functions **cosine** and **sine**, written $\cos\theta$ and $\sin\theta$ respectively. In fact these functions are *defined* to be the x and y coordinates of $\mathbf{p}$; that is $\mathbf{p} = \left[\begin{array}{c} \cos\theta \\ \sin\theta \end{array}\right]$. This defines $\cos\theta$ and $\sin\theta$ for the arbitrary angle θ (possibly negative), and agrees with the usual values when θ is an acute angle $\left(0 \leq \theta \leq \frac{\pi}{2}\right)$ as the reader should verify. For more discussion of this, see Appendix A.

Rotations

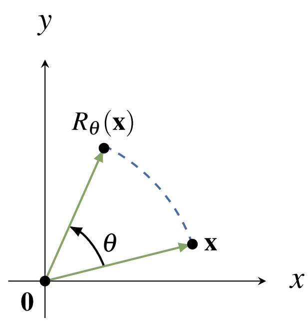


Figure 2.6.8

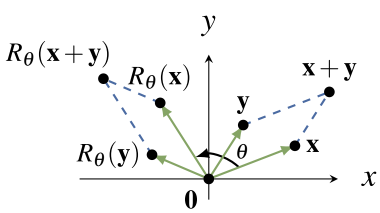


Figure 2.6.9

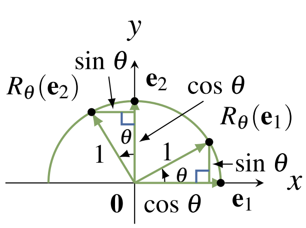


Figure 2.6.10

We can now describe rotations in the plane. Given an angle θ, let

$$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$

denote counterclockwise rotation of $\mathbb{R}^2$ about the origin through the angle θ. The action of R_θ is depicted in Figure 2.6.8. We have already looked at $R_{\frac{\pi}{2}}$ (in Example 2.2.15) and found it to be a matrix transformation. It turns out that R_θ is a matrix transformation for *every* angle θ (with a simple formula for the matrix), but it is not clear how to find the matrix. Our approach is to first establish the (somewhat surprising) fact that R_θ is *linear*, and then obtain the matrix from Theorem 2.6.2.

Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in $\mathbb{R}^2$. Then $\mathbf{x}+\mathbf{y}$ is the diagonal of the parallelogram determined by $\mathbf{x}$ and $\mathbf{y}$ as in Figure 2.6.9.

The effect of R_θ is to rotate the *entire* parallelogram to obtain the new parallelogram determined by $R_\theta(\mathbf{x})$ and $R_\theta(\mathbf{y})$, with diagonal $R_\theta(\mathbf{x}+\mathbf{y})$. But this diagonal is $R_\theta(\mathbf{x})+R_\theta(\mathbf{y})$ by the parallelogram law (applied to the new parallelogram). It follows that

$$R_\theta(\mathbf{x}+\mathbf{y}) = R_\theta(\mathbf{x}) + R_\theta(\mathbf{y})$$

A similar argument shows that $R_\theta(a\mathbf{x}) = aR_\theta(\mathbf{x})$ for any scalar a, so $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is indeed a linear transformation.

With linearity established we can find the matrix of R_θ. Let $\mathbf{e}_1 = \left[\begin{array}{c} 1 \\ 0 \end{array}\right]$ and $\mathbf{e}_2 = \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$ denote the standard basis of $\mathbb{R}^2$. By Figure 2.6.10 we see that

$$R_\theta(\mathbf{e}_1) = \left[\begin{array}{c} \cos\theta \\ \sin\theta \end{array}\right] \quad \text{and} \quad R_\theta(\mathbf{e}_2) = \left[\begin{array}{c} -\sin\theta \\ \cos\theta \end{array}\right]$$

Hence Theorem 2.6.2 shows that R_θ is induced by the matrix

$$\left[\begin{array}{cc} R_\theta(\mathbf{e}_1) & R_\theta(\mathbf{e}_2) \end{array}\right] = \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right]$$

We record this as

Theorem 2.6.4

The rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation with matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

For example, $R_{\frac{\pi}{2}}$ and R_π have matrices $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, respectively, by Theorem 2.6.4. The first of these confirms the result in Example 2.2.15. The second shows that rotating a vector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ through the angle π results in $R_\pi(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix} = -\mathbf{x}$. Thus applying R_π is the same as negating $\mathbf{x}$, a fact that is evident without Theorem 2.6.4.

Example 2.6.7

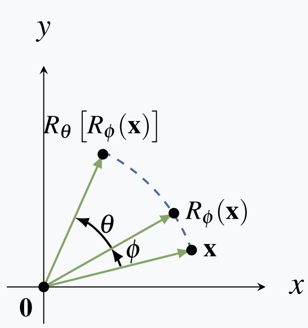


Figure 2.6.11

Let θ and ϕ be angles. By finding the matrix of the composite $R_\theta \circ R_\phi$, obtain expressions for $\cos(\theta+\phi)$ and $\sin(\theta+\phi)$.

Solution. Consider the transformations $\mathbb{R}^2 \xrightarrow{R_\phi} \mathbb{R}^2 \xrightarrow{R_\theta} \mathbb{R}^2$. Their composite $R_\theta \circ R_\phi$ is the transformation that first rotates the plane through ϕ and then rotates it through θ, and so is the rotation through the angle $\theta + \phi$ (see Figure 2.6.11).
In other words

$$R_{\theta+\phi} = R_\theta \circ R_\phi$$

Theorem 2.6.3 shows that the corresponding equation holds for the matrices of these transformations, so Theorem 2.6.4 gives:

$$\begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

If we perform the matrix multiplication on the right, and then compare first column entries, we obtain

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\sin(\theta+\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$$

These are the two basic identities from which most of trigonometry can be derived.

Reflections

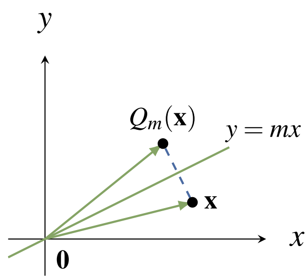


Figure 2.6.12

The line through the origin with slope m has equation $y = mx$, and we let $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection in the line $y = mx$.

This transformation is described geometrically in Figure 2.6.12. In words, $Q_m(\mathbf{x})$ is the "mirror image" of $\mathbf{x}$ in the line $y = mx$. If $m = 0$ then Q_0 is reflection in the x axis, so we already know Q_0 is linear. While we could show directly that Q_m is linear (with an argument like that for R_θ), we prefer to do it another way that is instructive and derives the matrix of Q_m directly without using Theorem 2.6.2.

Let θ denote the angle between the positive x axis and the line $y = mx$. The key observation is that the transformation Q_m can be accomplished in three steps: First rotate through $-\theta$ (so our line coincides with the x axis), then reflect in the x axis, and finally rotate back through θ. In other words:

$$Q_m = R_\theta \circ Q_0 \circ R_{-\theta}$$

Since $R_{-\theta}$, Q_0, and R_θ are all linear, this (with Theorem 2.6.3) shows that Q_m is linear and that its matrix is the product of the matrices of R_θ, Q_0, and $R_{-\theta}$. If we write $c = \cos\theta$ and $s = \sin\theta$ for simplicity, then the matrices of R_θ, $R_{-\theta}$, and Q_0 are

$$\left[\begin{array}{rr} c & -s \\ s & c \end{array}\right], \quad \left[\begin{array}{rr} c & s \\ -s & c \end{array}\right], \quad \text{and} \quad \left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right] \text{respectively.}^{13}$$

Hence, by Theorem 2.6.3, the matrix of $Q_m = R_\theta \circ Q_0 \circ R_{-\theta}$ is

$$\left[\begin{array}{rr} c & -s \\ s & c \end{array}\right]\left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right]\left[\begin{array}{rr} c & s \\ -s & c \end{array}\right] = \left[\begin{array}{cc} c^2 - s^2 & 2sc \\ 2sc & s^2 - c^2 \end{array}\right]$$

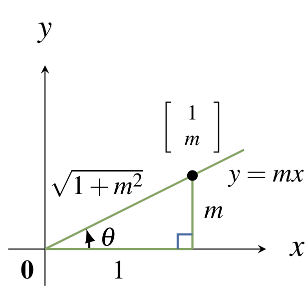


Figure 2.6.13

We can obtain this matrix in terms of m alone. Figure 2.6.13 shows that

$$\cos\theta = \frac{1}{\sqrt{1+m^2}} \text{ and } \sin\theta = \frac{m}{\sqrt{1+m^2}}$$

so the matrix $\left[\begin{array}{cc} c^2 - s^2 & 2sc \\ 2sc & s^2 - c^2 \end{array}\right]$ of Q_m becomes $\frac{1}{1+m^2}\left[\begin{array}{cc} 1 - m^2 & 2m \\ 2m & m^2 - 1 \end{array}\right]$.

Theorem 2.6.5

Let Q_m denote reflection in the line $y = mx$. Then Q_m is a linear transformation with matrix $\frac{1}{1+m^2}\left[\begin{array}{cc} 1 - m^2 & 2m \\ 2m & m^2 - 1 \end{array}\right]$.

[13] The matrix of $R_{-\theta}$ comes from the matrix of R_θ using the fact that, for all angles θ, $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin(\theta)$.

Note that if $m = 0$, the matrix in Theorem 2.6.5 becomes $\left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right]$, as expected. Of course this analysis fails for reflection in the y axis because vertical lines have no slope. However it is an easy exercise to verify directly that reflection in the y axis is indeed linear with matrix $\left[\begin{array}{rr} -1 & 0 \\ 0 & 1 \end{array}\right]$.[14]

Example 2.6.8

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation through $-\frac{\pi}{2}$ followed by reflection in the y axis. Show that T is a reflection in a line through the origin and find the line.

Solution. The matrix of $R_{-\frac{\pi}{2}}$ is $\left[\begin{array}{cc} \cos(-\frac{\pi}{2}) & -\sin(-\frac{\pi}{2}) \\ \sin(-\frac{\pi}{2}) & \cos(-\frac{\pi}{2}) \end{array}\right] = \left[\begin{array}{rr} 0 & 1 \\ -1 & 0 \end{array}\right]$ and the matrix of reflection in the y axis is $\left[\begin{array}{rr} -1 & 0 \\ 0 & 1 \end{array}\right]$. Hence the matrix of T is $\left[\begin{array}{rr} -1 & 0 \\ 0 & 1 \end{array}\right]\left[\begin{array}{rr} 0 & 1 \\ -1 & 0 \end{array}\right] = \left[\begin{array}{rr} 0 & -1 \\ -1 & 0 \end{array}\right]$ and this is reflection in the line $y = -x$ (take $m = -1$ in Theorem 2.6.5).

Projections

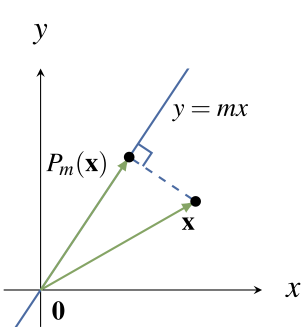


Figure 2.6.14

The method in the proof of Theorem 2.6.5 works more generally. Let $P_m : \mathbb{R}^2 \to \mathbb{R}^2$ denote projection on the line $y = mx$. This transformation is described geometrically in Figure 2.6.14.

If $m = 0$, then $P_0 \left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} x \\ 0 \end{array}\right]$ for all $\left[\begin{array}{c} x \\ y \end{array}\right]$ in $\mathbb{R}^2$, so P_0 is linear with matrix $\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right]$. Hence the argument above for Q_m goes through for P_m. First observe that

$$P_m = R_\theta \circ P_0 \circ R_{-\theta}$$

as before. So, P_m is linear with matrix

$$\left[\begin{array}{rr} c & -s \\ s & c \end{array}\right]\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right]\left[\begin{array}{rr} c & s \\ -s & c \end{array}\right] = \left[\begin{array}{cc} c^2 & sc \\ sc & s^2 \end{array}\right]$$

where $c = \cos\theta = \frac{1}{\sqrt{1+m^2}}$ and $s = \sin\theta = \frac{m}{\sqrt{1+m^2}}$.

[14]Note that $\left[\begin{array}{rr} -1 & 0 \\ 0 & 1 \end{array}\right] = \lim\limits_{m\to\infty} \frac{1}{1+m^2}\left[\begin{array}{cc} 1-m^2 & 2m \\ 2m & m^2-1 \end{array}\right]$.

This gives:

Theorem 2.6.6

Let $P_m : \mathbb{R}^2 \to \mathbb{R}^2$ be projection on the line $y = mx$. Then P_m is a linear transformation with matrix $\frac{1}{1+m^2}\left[\begin{array}{cc} 1 & m \\ m & m^2 \end{array}\right]$.

Again, if $m = 0$, then the matrix in Theorem 2.6.6 reduces to $\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right]$ as expected. As the y axis has no slope, the analysis fails for projection on the y axis, but this transformation is indeed linear with matrix $\left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right]$ as is easily verified directly.

Note that the formula for the matrix of Q_m in Theorem 2.6.5 can be derived from the above formula for the matrix of P_m. Using Figure 2.6.12, observe that $Q_m(\mathbf{x}) = \mathbf{x} + 2[P_m(\mathbf{x}) - \mathbf{x}]$ so $Q_m(x) = 2P_m(\mathbf{x}) - \mathbf{x}$. Substituting the matrices for $P_m(\mathbf{x})$ and $1_{\mathbb{R}^2}(\mathbf{x})$ gives the desired formula.

Example 2.6.9

Given $\mathbf{x}$ in $\mathbb{R}^2$, write $\mathbf{y} = P_m(\mathbf{x})$. The fact that $\mathbf{y}$ lies on the line $y = mx$ means that $P_m(\mathbf{y}) = \mathbf{y}$. But then

$$(P_m \circ P_m)(\mathbf{x}) = P_m(\mathbf{y}) = \mathbf{y} = P_m(\mathbf{x}) \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^2, \quad \text{that is, } P_m \circ P_m = P_m.$$

In particular, if we write the matrix of P_m as $A = \frac{1}{1+m^2}\left[\begin{array}{cc} 1 & m \\ m & m^2 \end{array}\right]$, then $A^2 = A$. The reader should verify this directly.

Exercises for 2.6

Exercise 2.6.1 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation.

a. Find $T\left[\begin{array}{r} 8 \\ 3 \\ 7 \end{array}\right]$ if $T\left[\begin{array}{r} 1 \\ 0 \\ -1 \end{array}\right] = \left[\begin{array}{r} 2 \\ 3 \end{array}\right]$ and $T\left[\begin{array}{r} 2 \\ 1 \\ 3 \end{array}\right] = \left[\begin{array}{r} -1 \\ 0 \end{array}\right]$.

b. Find $T\left[\begin{array}{r} 5 \\ 6 \\ -13 \end{array}\right]$ if $T\left[\begin{array}{r} 3 \\ 2 \\ -1 \end{array}\right] = \left[\begin{array}{r} 3 \\ 5 \end{array}\right]$ and $T\left[\begin{array}{r} 2 \\ 0 \\ 5 \end{array}\right] = \left[\begin{array}{r} -1 \\ 2 \end{array}\right]$.

Exercise 2.6.2 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation.

a. Find $T\left[\begin{array}{r} 1 \\ 3 \\ -2 \\ -3 \end{array}\right]$ if $T\left[\begin{array}{r} 1 \\ 1 \\ 0 \\ -1 \end{array}\right] = \left[\begin{array}{r} 2 \\ 3 \\ -1 \end{array}\right]$ and $T\left[\begin{array}{r} 0 \\ -1 \\ 1 \\ 1 \end{array}\right] = \left[\begin{array}{r} 5 \\ 0 \\ 1 \end{array}\right]$.

b. Find $T\left[\begin{array}{r} 5 \\ -1 \\ 2 \\ -4 \end{array}\right]$ if $T\left[\begin{array}{r} 1 \\ 1 \\ 1 \\ 1 \end{array}\right] = \left[\begin{array}{r} 5 \\ 1 \\ -3 \end{array}\right]$

and $T\begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

Exercise 2.6.3 In each case assume that the transformation T is linear, and use Theorem 2.6.2 to obtain the matrix A of T.

a. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is reflection in the line $y = -x$.

b. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T(\mathbf{x}) = -\mathbf{x}$ for each $\mathbf{x}$ in $\mathbb{R}^2$.

c. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is clockwise rotation through $\frac{\pi}{4}$.

d. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is counterclockwise rotation through $\frac{\pi}{4}$.

Exercise 2.6.4 In each case use Theorem 2.6.2 to obtain the matrix A of the transformation T. You may assume that T is linear in each case.

a. $T : \mathbb{R}^3 \to \mathbb{R}^3$ is reflection in the $x - z$ plane.

b. $T : \mathbb{R}^3 \to \mathbb{R}^3$ is reflection in the $y - z$ plane.

Exercise 2.6.5 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

a. If $\mathbf{x}$ is in $\mathbb{R}^n$, we say that $\mathbf{x}$ is in the *kernel* of T if $T(\mathbf{x}) = \mathbf{0}$. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are both in the kernel of T, show that $a\mathbf{x}_1 + b\mathbf{x}_2$ is also in the kernel of T for all scalars a and b.

b. If $\mathbf{y}$ is in $\mathbb{R}^n$, we say that $\mathbf{y}$ is in the *image* of T if $\mathbf{y} = T(\mathbf{x})$ for some $\mathbf{x}$ in $\mathbb{R}^n$. If $\mathbf{y}_1$ and $\mathbf{y}_2$ are both in the image of T, show that $a\mathbf{y}_1 + b\mathbf{y}_2$ is also in the image of T for all scalars a and b.

Exercise 2.6.6 Use Theorem 2.6.2 to find the matrix of the **identity transformation** $1_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $1_{\mathbb{R}^n}(\mathbf{x}) = \mathbf{x}$ for each $\mathbf{x}$ in $\mathbb{R}^n$.

Exercise 2.6.7 In each case show that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is not a linear transformation.

a. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xy \\ 0 \end{bmatrix}$

b. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y^2 \end{bmatrix}$

Exercise 2.6.8 In each case show that T is either reflection in a line or rotation through an angle, and find the line or angle.

a. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3x+4y \\ 4x+3y \end{bmatrix}$

b. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} x+y \\ -x+y \end{bmatrix}$

c. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} x-\sqrt{3}y \\ \sqrt{3}x+y \end{bmatrix}$

d. $T\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{10}\begin{bmatrix} 8x+6y \\ 6x-8y \end{bmatrix}$

Exercise 2.6.9 Express reflection in the line $y = -x$ as the composition of a rotation followed by reflection in the line $y = x$.

Exercise 2.6.10 Find the matrix of $T : \mathbb{R}^3 \to \mathbb{R}^3$ in each case:

a. T is rotation through θ about the x axis (from the y axis to the z axis).

b. T is rotation through θ about the y axis (from the x axis to the z axis).

Exercise 2.6.11 Let $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection in the line making an angle θ with the positive x axis.

a. Show that the matrix of T_θ is $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$ for all θ.

b. Show that $T_\theta \circ R_{2\phi} = T_{\theta-\phi}$ for all θ and ϕ.

Exercise 2.6.12 In each case find a rotation or reflection that equals the given transformation.

a. Reflection in the y axis followed by rotation through $\frac{\pi}{2}$.

b. Rotation through π followed by reflection in the x axis.

c. Rotation through $\frac{\pi}{2}$ followed by reflection in the line $y = x$.

d. Reflection in the x axis followed by rotation through $\frac{\pi}{2}$.

e. Reflection in the line $y = x$ followed by reflection in the x axis.

f. Reflection in the x axis followed by reflection in the line $y = x$.

Exercise 2.6.13 Let R and S be matrix transformations $\mathbb{R}^n \to \mathbb{R}^m$ induced by matrices A and B respectively. In each case, show that T is a matrix transformation and describe its matrix in terms of A and B.

a. $T(\mathbf{x}) = R(\mathbf{x}) + S(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

b. $T(\mathbf{x}) = aR(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{R}^n$ (where a is a fixed real number).

Exercise 2.6.14 Show that the following hold for all linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$:

a. $T(\mathbf{0}) = \mathbf{0}$

b. $T(-\mathbf{x}) = -T(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{R}^n$

Exercise 2.6.15 The transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x}$ in $\mathbb{R}^n$ is called the **zero transformation**.

a. Show that the zero transformation is linear and find its matrix.

b. Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ denote the columns of the $n \times n$ identity matrix. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear and $T(\mathbf{e}_i) = \mathbf{0}$ for each i, show that T is the zero transformation. [*Hint*: Theorem 2.6.1.]

Exercise 2.6.16 Write the elements of $\mathbb{R}^n$ and $\mathbb{R}^m$ as rows. If A is an $m \times n$ matrix, define $T : \mathbb{R}^m \to \mathbb{R}^n$ by $T(\mathbf{y}) = \mathbf{y}A$ for all rows $\mathbf{y}$ in $\mathbb{R}^m$. Show that:

a. T is a linear transformation.

b. the rows of A are $T(\mathbf{f}_1), T(\mathbf{f}_2), \dots, T(\mathbf{f}_m)$ where $\mathbf{f}_i$ denotes row i of I_m. [*Hint*: Show that $\mathbf{f}_iA$ is row i of A.]

Exercise 2.6.17 Let $S : \mathbb{R}^n \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear transformations with matrices A and B respectively.

a. Show that $B^2 = B$ if and only if $T^2 = T$ (where T^2 means $T \circ T$).

b. Show that $B^2 = I$ if and only if $T^2 = 1_{\mathbb{R}^n}$.

c. Show that $AB = BA$ if and only if $S \circ T = T \circ S$.

[*Hint*: Theorem 2.6.3.]

Exercise 2.6.18 Let $Q_0 : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the x axis, let $Q_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the line $y = x$, let $Q_{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the line $y = -x$, and let $R_{\frac{\pi}{2}} : \mathbb{R}^2 \to \mathbb{R}^2$ be counterclockwise rotation through $\frac{\pi}{2}$.

a. Show that $Q_1 \circ R_{\frac{\pi}{2}} = Q_0$.

b. Show that $Q_1 \circ Q_0 = R_{\frac{\pi}{2}}$.

c. Show that $R_{\frac{\pi}{2}} \circ Q_0 = Q_1$.

d. Show that $Q_0 \circ R_{\frac{\pi}{2}} = Q_{-1}$.

Exercise 2.6.19 For any slope m, show that:

a. $Q_m \circ P_m = P_m$

b. $P_m \circ Q_m = P_m$

Exercise 2.6.20 Define $T : \mathbb{R}^n \to \mathbb{R}$ by $T(x_1, x_2, \dots, x_n) = x_1 + x_2 + \cdots + x_n$. Show that T is a linear transformation and find its matrix.

Exercise 2.6.21 Given c in $\mathbb{R}$, define $T_c : \mathbb{R}^n \to \mathbb{R}$ by $T_c(\mathbf{x}) = c\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Show that T_c is a linear transformation and find its matrix.

Exercise 2.6.22 Given vectors $\mathbf{w}$ and $\mathbf{x}$ in $\mathbb{R}^n$, denote their dot product by $\mathbf{w} \cdot \mathbf{x}$.

a. Given $\mathbf{w}$ in $\mathbb{R}^n$, define $T_{\mathbf{w}} : \mathbb{R}^n \to \mathbb{R}$ by $T_{\mathbf{w}}(\mathbf{x}) = \mathbf{w} \cdot \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Show that $T_{\mathbf{w}}$ is a linear transformation.

b. Show that *every* linear transformation $T : \mathbb{R}^n \to \mathbb{R}$ is given as in (a); that is $T = T_{\mathbf{w}}$ for some $\mathbf{w}$ in $\mathbb{R}^n$.

Exercise 2.6.23 If $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$, show that there is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{y}$. [*Hint*: By Definition 2.5, find a matrix A such that $A\mathbf{x} = \mathbf{y}$.]

Exercise 2.6.24 Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m \xrightarrow{S} \mathbb{R}^k$ be two linear transformations. Show directly that $S \circ T$ is linear. That is:

a. Show that $(S \circ T)(\mathbf{x} + \mathbf{y}) = (S \circ T)\mathbf{x} + (S \circ T)\mathbf{y}$ for all $\mathbf{x}$, $\mathbf{y}$ in $\mathbb{R}^n$.

b. Show that $(S \circ T)(a\mathbf{x}) = a[(S \circ T)\mathbf{x}]$ for all $\mathbf{x}$ in $\mathbb{R}^n$ and all a in $\mathbb{R}$.

Exercise 2.6.25 Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m \xrightarrow{S} \mathbb{R}^k \xrightarrow{R} \mathbb{R}^k$ be linear. Show that $R \circ (S \circ T) = (R \circ S) \circ T$ by showing directly that $[R \circ (S \circ T)](\mathbf{x}) = [(R \circ S) \circ T)](\mathbf{x})$ holds for each vector $\mathbf{x}$ in $\mathbb{R}^n$.

2.7 LU-Factorization[15]

The solution to a system $A\mathbf{x} = \mathbf{b}$ of linear equations can be solved quickly if A can be factored as $A = LU$ where L and U are of a particularly nice form. In this section we show that gaussian elimination can be used to find such factorizations.

Triangular Matrices

As for square matrices, if $A = \left[a_{ij}\right]$ is an $m \times n$ matrix, the elements a_{11}, a_{22}, a_{33}, $\dots$ form the **main diagonal** of A. Then A is called **upper triangular** if every entry below and to the left of the main diagonal is zero. Every row-echelon matrix is upper triangular, as are the matrices

$$\left[\begin{array}{rrrr} 1 & -1 & 0 & 3 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -3 & 0 \end{array}\right] \quad \left[\begin{array}{rrrrr} 0 & 2 & 1 & 0 & 5 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{rrr} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

By analogy, a matrix A is called **lower triangular** if its transpose is upper triangular, that is if each entry above and to the right of the main diagonal is zero. A matrix is called **triangular** if it is upper or lower triangular.

Example 2.7.1

Solve the system

$$\begin{aligned} x_1 + 2x_2 - 3x_3 - x_4 + 5x_5 &= 3 \\ 5x_3 + x_4 + x_5 &= 8 \\ 2x_5 &= 6 \end{aligned}$$

where the coefficient matrix is upper triangular.

Solution. As in gaussian elimination, let the "non-leading" variables be parameters: $x_2 = s$ and $x_4 = t$. Then solve for x_5, x_3, and x_1 in that order as follows. The last equation gives

$$x_5 = \frac{6}{2} = 3$$

Substitution into the second last equation gives

$$x_3 = 1 - \frac{1}{5}t$$

Finally, substitution of both x_5 and x_3 into the first equation gives

$$x_1 = -9 - 2s + \frac{2}{5}t$$

The method used in Example 2.7.1 is called **back substitution** because later variables are substituted into earlier equations. It works because the coefficient matrix is upper triangular. Similarly, if the coeffi-

[15]This section is not used later and so may be omitted with no loss of continuity.

cient matrix is lower triangular the system can be solved by **forward substitution** where earlier variables are substituted into later equations. As observed in Section 1.2, these procedures are more numerically efficient than gaussian elimination.

Now consider a system $A\mathbf{x} = \mathbf{b}$ where A can be factored as $A = LU$ where L is lower triangular and U is upper triangular. Then the system $A\mathbf{x} = \mathbf{b}$ can be solved in two stages as follows:

1. *First solve* $L\mathbf{y} = \mathbf{b}$ *for* $\mathbf{y}$ *by forward substitution.*
2. *Then solve* $U\mathbf{x} = \mathbf{y}$ *for* $\mathbf{x}$ *by back substitution.*

Then $\mathbf{x}$ is a solution to $A\mathbf{x} = \mathbf{b}$ because $A\mathbf{x} = LU\mathbf{x} = L\mathbf{y} = \mathbf{b}$. Moreover, every solution $\mathbf{x}$ arises this way (take $\mathbf{y} = U\mathbf{x}$). Furthermore the method adapts easily for use in a computer.

This focuses attention on efficiently obtaining such factorizations $A = LU$. The following result will be needed; the proof is straightforward and is left as Exercises 2.7.7 and 2.7.8.

Lemma 2.7.1

Let A and B denote matrices.

1. *If A and B are both lower (upper) triangular, the same is true of AB.*
2. *If A is $n \times n$ and lower (upper) triangular, then A is invertible if and only if every main diagonal entry is nonzero. In this case A^{-1} is also lower (upper) triangular.*

LU-Factorization

Let A be an $m \times n$ matrix. Then A can be carried to a row-echelon matrix U (that is, upper triangular). As in Section 2.5, the reduction is

$$A \to E_1A \to E_2E_1A \to E_3E_2E_1A \to \cdots \to E_kE_{k-1}\cdots E_2E_1A = U$$

where $E_1, E_2, \dots, E_k$ are elementary matrices corresponding to the row operations used. Hence

$$A = LU$$

where $L = (E_kE_{k-1}\cdots E_2E_1)^{-1} = E_1^{-1}E_2^{-1}\cdots E_{k-1}^{-1}E_k^{-1}$. If we do not insist that U is reduced then, except for row interchanges, none of these row operations involve adding a row to a row *above* it. Thus, if no row interchanges are used, all the E_i are *lower* triangular, and so L is lower triangular (and invertible) by Lemma 2.7.1. This proves the following theorem. For convenience, let us say that A can be **lower reduced** if it can be carried to row-echelon form using no row interchanges.

Theorem 2.7.1

If A can be lower reduced to a row-echelon matrix U, then

$$A = LU$$

where L is lower triangular and invertible and U is upper triangular and row-echelon.

Definition 2.14 LU-factorization

A factorization $A = LU$ as in Theorem 2.7.1 is called an ***LU-factorization*** *of A.*

Such a factorization may not exist (Exercise 2.7.4) because A cannot be carried to row-echelon form using no row interchange. A procedure for dealing with this situation will be outlined later. However, if an LU-factorization $A = LU$ does exist, then the gaussian algorithm gives U and also leads to a procedure for finding L. Example 2.7.2 provides an illustration. For convenience, the first nonzero column from the left in a matrix A is called the **leading column** of A.

Example 2.7.2

Find an LU-factorization of $A = \begin{bmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & -1 & 3 & 3 & 2 \\ 0 & -1 & 3 & 7 & 10 \end{bmatrix}$.

Solution. We lower reduce A to row-echelon form as follows:

$$A = \begin{bmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & -1 & 3 & 3 & 2 \\ 0 & -1 & 3 & 7 & 10 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -3 & -1 & 2 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 6 & 12 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -3 & -1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

The circled columns are determined as follows: The first is the leading column of A, and is used (by lower reduction) to create the first leading 1 and create zeros below it. This completes the work on row 1, and we repeat the procedure on the matrix consisting of the remaining rows. Thus the second circled column is the leading column of this smaller matrix, which we use to create the second leading 1 and the zeros below it. As the remaining row is zero here, we are finished. Then $A = LU$ where

$$L = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 6 & 1 \end{bmatrix}$$

This matrix L is obtained from I_3 by replacing the bottom of the first two columns by the circled columns in the reduction. Note that the rank of A is 2 here, and this is the number of circled columns.

The calculation in Example 2.7.2 works in general. There is no need to calculate the elementary matrices E_i, and the method is suitable for use in a computer because the circled columns can be stored in memory as they are created. The procedure can be formally stated as follows:

LU-Algorithm

Let A be an $m \times n$ matrix of rank r, and suppose that A can be lower reduced to a row-echelon matrix U. Then $A = LU$ where the lower triangular, invertible matrix L is constructed as follows:

1. *If $A = 0$, take $L = I_m$ and $U = 0$.*
2. *If $A \neq 0$, write $A_1 = A$ and let $\mathbf{c}_1$ be the leading column of A_1. Use $\mathbf{c}_1$ to create the first leading 1 and create zeros below it (using lower reduction). When this is completed, let A_2 denote the matrix consisting of rows 2 to m of the matrix just created.*
3. *If $A_2 \neq 0$, let $\mathbf{c}_2$ be the leading column of A_2 and repeat Step 2 on A_2 to create A_3.*
4. *Continue in this way until U is reached, where all rows below the last leading 1 consist of zeros. This will happen after r steps.*
5. *Create L by placing $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r$ at the bottom of the first r columns of I_m.*

A proof of the LU-algorithm is given at the end of this section.

LU-factorization is particularly important if, as often happens in business and industry, a series of equations $A\mathbf{x} = B_1,\ A\mathbf{x} = B_2,\ \dots,\ A\mathbf{x} = B_k$, must be solved, each with the same coefficient matrix A. It is very efficient to solve the first system by gaussian elimination, simultaneously creating an LU-factorization of A, and then using the factorization to solve the remaining systems by forward and back substitution.

Example 2.7.3

Find an LU-factorization for $A = \left[\begin{array}{rrrrr} 5 & -5 & 10 & 0 & 5 \\ -3 & 3 & 2 & 2 & 1 \\ -2 & 2 & 0 & -1 & 0 \\ 1 & -1 & 10 & 2 & 5 \end{array}\right]$.

Solution. The reduction to row-echelon form is

$$\begin{bmatrix} \boxed{\begin{matrix} 5 \\ -3 \\ -2 \\ 1 \end{matrix}} & \begin{matrix} -5 \\ 3 \\ 2 \\ -1 \end{matrix} & \begin{matrix} 10 \\ 2 \\ 0 \\ 10 \end{matrix} & \begin{matrix} 0 \\ 2 \\ -1 \\ 2 \end{matrix} & \begin{matrix} 5 \\ 1 \\ 0 \\ 5 \end{matrix} \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & \boxed{\begin{matrix} 8 \\ 4 \\ 8 \end{matrix}} & \begin{matrix} 2 \\ -1 \\ 2 \end{matrix} & \begin{matrix} 4 \\ 2 \\ 4 \end{matrix} \\ 0 & 0 & & & \\ 0 & 0 & & & \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & \boxed{\begin{matrix} -2 \\ 0 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 & 0 & 0 & & \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

If U denotes this row-echelon matrix, then $A = LU$, where

$$L = \begin{bmatrix} 5 & 0 & 0 & 0 \\ -3 & 8 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 1 & 8 & 0 & 1 \end{bmatrix}$$

The next example deals with a case where no row of zeros is present in U (in fact, A is invertible).

Example 2.7.4

Find an LU-factorization for $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}$.

Solution. The reduction to row-echelon form is

$$\begin{bmatrix} \boxed{\begin{matrix} 2 \\ 1 \\ -1 \end{matrix}} & \begin{matrix} 4 \\ 1 \\ 0 \end{matrix} & \begin{matrix} 2 \\ 2 \\ 2 \end{matrix} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & \boxed{\begin{matrix} -1 \\ 2 \end{matrix}} & \begin{matrix} 1 \\ 3 \end{matrix} \\ 0 & & \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & \boxed{5} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

Hence $A = LU$ where $L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & 5 \end{bmatrix}$.

There are matrices (for example $\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]$) that have no LU-factorization and so require at least one row interchange when being carried to row-echelon form via the gaussian algorithm. However, it turns out that, if all the row interchanges encountered in the algorithm are carried out first, the resulting matrix requires no interchanges and so has an LU-factorization. Here is the precise result.

Theorem 2.7.2

Suppose an $m \times n$ matrix A is carried to a row-echelon matrix U via the gaussian algorithm. Let $P_1, P_2, \dots, P_s$ be the elementary matrices corresponding (in order) to the row interchanges used, and write $P = P_s \cdots P_2 P_1$. (If no interchanges are used take $P = I_m$.) Then:

1. *PA is the matrix obtained from A by doing these interchanges (in order) to A.*
2. *PA has an LU-factorization.*

The proof is given at the end of this section.

A matrix P that is the product of elementary matrices corresponding to row interchanges is called a **permutation matrix**. Such a matrix is obtained from the identity matrix by arranging the rows in a different order, so it has exactly one 1 in each row and each column, and has zeros elsewhere. We regard the identity matrix as a permutation matrix. The elementary permutation matrices are those obtained from I by a single row interchange, and every permutation matrix is a product of elementary ones.

Example 2.7.5

If $A = \left[\begin{array}{rrrr} 0 & 0 & -1 & 2 \\ -1 & -1 & 1 & 2 \\ 2 & 1 & -3 & 6 \\ 0 & 1 & -1 & 4 \end{array}\right]$, find a permutation matrix P such that PA has an LU-factorization, and then find the factorization.

Solution. Apply the gaussian algorithm to A:

$$A \stackrel{*}{\rightarrow} \left[\begin{array}{rrrr} -1 & -1 & 1 & 2 \\ 0 & 0 & -1 & 2 \\ 2 & 1 & -3 & 6 \\ 0 & 1 & -1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{rrrr} 1 & 1 & -1 & -2 \\ 0 & 0 & -1 & 2 \\ 0 & -1 & -1 & 10 \\ 0 & 1 & -1 & 4 \end{array}\right] \stackrel{*}{\rightarrow} \left[\begin{array}{rrrr} 1 & 1 & -1 & -2 \\ 0 & -1 & -1 & 10 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 4 \end{array}\right]$$
$$\rightarrow \left[\begin{array}{rrrr} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -2 & 14 \end{array}\right] \rightarrow \left[\begin{array}{rrrr} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 10 \end{array}\right]$$

Two row interchanges were needed (marked with $*$), first rows 1 and 2 and then rows 2 and 3. Hence, as in Theorem 2.7.2,

$$P = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

If we do these interchanges (in order) to A, the result is PA. Now apply the LU-algorithm to PA:

$$PA = \begin{bmatrix} \boxed{-1} & -1 & 1 & 2 \\ 2 & 1 & -3 & 6 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & \boxed{-1} & -1 & 10 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & \boxed{-1} & 2 \\ 0 & 0 & -2 & 14 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & \boxed{10} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U$$

Hence, $PA = LU$, where $L = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 10 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Theorem 2.7.2 provides an important general factorization theorem for matrices. If A is any $m \times n$ matrix, it asserts that there exists a permutation matrix P and an LU-factorization $PA = LU$. Moreover, it shows that either $P = I$ or $P = P_s \cdots P_2 P_1$, where $P_1, P_2, \ldots, P_s$ are the elementary permutation matrices arising in the reduction of A to row-echelon form. Now observe that $P_i^{-1} = P_i$ for each i (they are elementary row interchanges). Thus, $P^{-1} = P_1 P_2 \cdots P_s$, so the matrix A can be factored as

$$A = P^{-1}LU$$

where P^{-1} is a permutation matrix, L is lower triangular and invertible, and U is a row-echelon matrix. This is called a **PLU-factorization** of A.

The LU-factorization in Theorem 2.7.1 is not unique. For example,

$$\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

However, it is necessary here that the row-echelon matrix has a row of zeros. Recall that the rank of a matrix A is the number of nonzero rows in any row-echelon matrix U to which A can be carried by row operations. Thus, if A is $m \times n$, the matrix U has no row of zeros if and only if A has rank m.

Theorem 2.7.3

Let A be an $m \times n$ matrix that has an LU-factorization

$$A = LU$$

If A has rank m (that is, U has no row of zeros), then L and U are uniquely determined by A.

Proof. Suppose $A = MV$ is another LU-factorization of A, so M is lower triangular and invertible and V is row-echelon. Hence $LU = MV$, and we must show that $L = M$ and $U = V$. We write $N = M^{-1}L$. Then N

is lower triangular and invertible (Lemma 2.7.1) and $NU = V$, so it suffices to prove that $N = I$. If N is $m \times m$, we use induction on m. The case $m = 1$ is left to the reader. If $m > 1$, observe first that column 1 of V is N times column 1 of U. Thus if either column is zero, so is the other (N is invertible). Hence, we can assume (by deleting zero columns) that the $(1,\ 1)$-entry is 1 in both U and V.

Now we write $N = \left[\begin{array}{cc} a & 0 \\ X & N_1 \end{array}\right]$, $U = \left[\begin{array}{cc} 1 & Y \\ 0 & U_1 \end{array}\right]$, and $V = \left[\begin{array}{cc} 1 & Z \\ 0 & V_1 \end{array}\right]$ in block form. Then $NU = V$ becomes $\left[\begin{array}{cc} a & aY \\ X & XY + N_1U_1 \end{array}\right] = \left[\begin{array}{cc} 1 & Z \\ 0 & V_1 \end{array}\right]$. Hence $a = 1$, $Y = Z$, $X = 0$, and $N_1U_1 = V_1$. But $N_1U_1 = V_1$ implies $N_1 = I$ by induction, whence $N = I$. □

If A is an $m \times m$ invertible matrix, then A has rank m by Theorem 2.4.5. Hence, we get the following important special case of Theorem 2.7.3.

Corollary 2.7.1

If an invertible matrix A has an LU-factorization $A = LU$, then L and U are uniquely determined by A.

Of course, in this case U is an upper triangular matrix with 1s along the main diagonal.

Proofs of Theorems

Proof of the LU-Algorithm. If $\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_r$ are columns of lengths $m,\ m-1,\ \dots,\ m-r+1$, respectively, write $L^{(m)}(\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_r)$ for the lower triangular $m \times m$ matrix obtained from I_m by placing $\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_r$ at the bottom of the first r columns of I_m.

Proceed by induction on n. If $A = 0$ or $n = 1$, it is left to the reader. If $n > 1$, let $\mathbf{c}_1$ denote the leading column of A and let $\mathbf{k}_1$ denote the first column of the $m \times m$ identity matrix. There exist elementary matrices $E_1,\ \dots,\ E_k$ such that, in block form,

$$(E_k \cdots E_2E_1)A = \left[\begin{array}{c|c|c} 0 & \mathbf{k}_1 & \begin{array}{c} X_1 \\ \hline A_1 \end{array} \end{array}\right] \qquad \text{where } (E_k \cdots E_2E_1)\mathbf{c}_1 = \mathbf{k}_1$$

Moreover, each E_j can be taken to be lower triangular (by assumption). Write

$$G = (E_k \cdots E_2E_1)^{-1} = E_1^{-1}E_2^{-1} \cdots E_k^{-1}$$

Then G is lower triangular, and $G\mathbf{k}_1 = \mathbf{c}_1$. Also, each E_j (and so each E_j^{-1}) is the result of either multiplying row 1 of I_m by a constant or adding a multiple of row 1 to another row. Hence,

$$G = (E_1^{-1}E_2^{-1} \cdots E_k^{-1})I_m = \left[\begin{array}{c|c} \mathbf{c}_1 & \begin{array}{c} 0 \\ \hline I_{m-1} \end{array} \end{array}\right]$$

in block form. Now, by induction, let $A_1 = L_1U_1$ be an LU-factorization of A_1, where $L_1 = L^{(m-1)}[\mathbf{c}_2,\ \dots,\ \mathbf{c}_r]$ and U_1 is row-echelon. Then block multiplication gives

$$G^{-1}A = \left[\begin{array}{c|c|c} 0 & \mathbf{k}_1 & \begin{array}{c} X_1 \\ \hline L_1U_1 \end{array} \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & L_1 \end{array}\right] \left[\begin{array}{c|c|c} 0 & 1 & X_1 \\ \hline 0 & 0 & U_1 \end{array}\right]$$

Hence $A = LU$, where $U = \left[\begin{array}{c|c|c} 0 & 1 & X_1 \\ \hline 0 & 0 & U_1 \end{array}\right]$ is row-echelon and

$$L = \left[\begin{array}{c|c} \mathbf{c}_1 & \begin{array}{c} 0 \\ \hline I_{m-1} \end{array} \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & L_1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{c}_1 & \begin{array}{c} 0 \\ \hline L \end{array} \end{array}\right] = L^{(m)}\left[\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_r\right]$$

This completes the proof. □

Proof of Theorem 2.7.2. Let A be a nonzero $m \times n$ matrix and let $\mathbf{k}_j$ denote column j of I_m. There is a permutation matrix P_1 (where either P_1 is elementary or $P_1 = I_m$) such that the first nonzero column $\mathbf{c}_1$ of P_1A has a nonzero entry on top. Hence, as in the LU-algorithm,

$$L^{(m)}\left[\mathbf{c}_1\right]^{-1} \cdot P_1 \cdot A = \left[\begin{array}{c|c|c} 0 & 1 & X_1 \\ \hline 0 & 0 & A_1 \end{array}\right]$$

in block form. Then let P_2 be a permutation matrix (either elementary or I_m) such that

$$P_2 \cdot L^{(m)}\left[\mathbf{c}_1\right]^{-1} \cdot P_1 \cdot A = \left[\begin{array}{c|c|c} 0 & 1 & X_1 \\ \hline 0 & 0 & A_1' \end{array}\right]$$

and the first nonzero column $\mathbf{c}_2$ of A_1' has a nonzero entry on top. Thus,

$$L^{(m)}\left[\mathbf{k}_1,\ \mathbf{c}_2\right]^{-1} \cdot P_2 \cdot L^{(m)}\left[\mathbf{c}_1\right]^{-1} \cdot P_1 \cdot A = \left[\begin{array}{c|c|c} 0 & 1 & X_1 \\ \hline 0 & 0 & \begin{array}{c|c|c} 0 & 1 & X_2 \\ \hline 0 & 0 & A_2 \end{array} \end{array}\right]$$

in block form. Continue to obtain elementary permutation matrices $P_1,\ P_2,\ \dots,\ P_r$ and columns $\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_r$ of lengths $m,\ m-1,\ \dots,$ such that

$$(L_rP_rL_{r-1}P_{r-1}\cdots L_2P_2L_1P_1)A = U$$

where U is a row-echelon matrix and $L_j = L^{(m)}\left[\mathbf{k}_1,\ \dots,\ \mathbf{k}_{j-1},\ \mathbf{c}_j\right]^{-1}$ for each j, where the notation means the first $j-1$ columns are those of I_m. It is not hard to verify that each L_j has the form $L_j = L^{(m)}\left[\mathbf{k}_1,\ \dots,\ \mathbf{k}_{j-1},\ \mathbf{c}_j'\right]$ where $\mathbf{c}_j'$ is a column of length $m-j+1$. We now claim that each permutation matrix P_k can be "moved past" each matrix L_j to the right of it, in the sense that

$$P_kL_j = L_j'P_k$$

where $L_j' = L^{(m)}\left[\mathbf{k}_1,\ \dots,\ \mathbf{k}_{j-1},\ \mathbf{c}_j''\right]$ for some column $\mathbf{c}_j''$ of length $m-j+1$. Given that this is true, we obtain a factorization of the form

$$(L_rL_{r-1}'\cdots L_2'L_1')(P_rP_{r-1}\cdots P_2P_1)A = U$$

If we write $P = P_rP_{r-1}\cdots P_2P_1$, this shows that PA has an LU-factorization because $L_rL_{r-1}'\cdots L_2'L_1'$ is lower triangular and invertible. All that remains is to prove the following rather technical result. □

Lemma 2.7.2

Let P_k result from interchanging row k of I_m with a row below it. If $j < k$, let c_j be a column of length $m-j+1$. Then there is another column $\mathbf{c}'_j$ of length $m-j+1$ such that

$$P_k \cdot L^{(m)}\left[\mathbf{k}_1,\ \dots,\ \mathbf{k}_{j-1},\ \mathbf{c}_j\right] = L^{(m)}\left[\mathbf{k}_1,\ \dots,\ \mathbf{k}_{j-1},\ \mathbf{c}'_j\right] \cdot P_k$$

The proof is left as Exercise 2.7.11.

Exercises for 2.7

Exercise 2.7.1 Find an LU-factorization of the following matrices.

a. $\left[\begin{array}{rrrrr} 2 & 6 & -2 & 0 & 2 \\ 3 & 9 & -3 & 3 & 1 \\ -1 & -3 & 1 & -3 & 1 \end{array}\right]$

b. $\left[\begin{array}{rrr} 2 & 4 & 2 \\ 1 & -1 & 3 \\ -1 & 7 & -7 \end{array}\right]$

c. $\left[\begin{array}{rrrrr} 2 & 6 & -2 & 0 & 2 \\ 1 & 5 & -1 & 2 & 5 \\ 3 & 7 & -3 & -2 & 5 \\ -1 & -1 & 1 & 2 & 3 \end{array}\right]$

d. $\left[\begin{array}{rrrrr} -1 & -3 & 1 & 0 & -1 \\ 1 & 4 & 1 & 1 & 1 \\ 1 & 2 & -3 & -1 & 1 \\ 0 & -2 & -4 & -2 & 0 \end{array}\right]$

e. $\left[\begin{array}{rrrrrr} 2 & 2 & 4 & 6 & 0 & 2 \\ 1 & -1 & 2 & 1 & 3 & 1 \\ -2 & 2 & -4 & -1 & 1 & 6 \\ 0 & 2 & 0 & 3 & 4 & 8 \\ -2 & 4 & -4 & 1 & -2 & 6 \end{array}\right]$

f. $\left[\begin{array}{rrrrr} 2 & 2 & -2 & 4 & 2 \\ 1 & -1 & 0 & 2 & 1 \\ 3 & 1 & -2 & 6 & 3 \\ 1 & 3 & -2 & 2 & 1 \end{array}\right]$

Exercise 2.7.2 Find a permutation matrix P and an LU-factorization of PA if A is:

a. $\left[\begin{array}{rrr} 0 & 0 & 2 \\ 0 & -1 & 4 \\ 3 & 5 & 1 \end{array}\right]$

b. $\left[\begin{array}{rrr} 0 & -1 & 2 \\ 0 & 0 & 4 \\ -1 & 2 & 1 \end{array}\right]$

c. $\left[\begin{array}{rrrrr} 0 & -1 & 2 & 1 & 3 \\ -1 & 1 & 3 & 1 & 4 \\ 1 & -1 & -3 & 6 & 2 \\ 2 & -2 & -4 & 1 & 0 \end{array}\right]$

d. $\left[\begin{array}{rrrr} -1 & -2 & 3 & 0 \\ 2 & 4 & -6 & 5 \\ 1 & 1 & -1 & 3 \\ 2 & 5 & -10 & 1 \end{array}\right]$

Exercise 2.7.3 In each case use the given LU-decomposition of A to solve the system $A\mathbf{x} = \mathbf{b}$ by finding $\mathbf{y}$ such that $L\mathbf{y} = \mathbf{b}$, and then $\mathbf{x}$ such that $U\mathbf{x} = \mathbf{y}$:

a. $A = \left[\begin{array}{rrr} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 3 \end{array}\right]\left[\begin{array}{rrrr} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$;

$\mathbf{b} = \left[\begin{array}{r} 1 \\ -1 \\ 2 \end{array}\right]$

b. $A = \left[\begin{array}{rrr} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 2 & 1 \end{array}\right]\left[\begin{array}{rrrr} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$;

$\mathbf{b} = \left[\begin{array}{r} -2 \\ -1 \\ 1 \end{array}\right]$

c. $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$;

$\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$

d. $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 3 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$;

$\mathbf{b} = \begin{bmatrix} 4 \\ -6 \\ 4 \\ 5 \end{bmatrix}$

Exercise 2.7.4 Show that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = LU$ is impossible where L is lower triangular and U is upper triangular.

Exercise 2.7.5 Show that we can accomplish any row interchange by using only row operations of other types.

Exercise 2.7.6

a. Let L and L_1 be invertible lower triangular matrices, and let U and U_1 be invertible upper triangular matrices. Show that $LU = L_1U_1$ if and only if there exists an invertible diagonal matrix D such that $L_1 = LD$ and $U_1 = D^{-1}U$. [*Hint*: Scrutinize $L^{-1}L_1 = UU_1^{-1}$.]

b. Use part (a) to prove Theorem 2.7.3 in the case that A is invertible.

Exercise 2.7.7 Prove Lemma 2.7.1(1). [*Hint*: Use block multiplication and induction.]

Exercise 2.7.8 Prove Lemma 2.7.1(2). [*Hint*: Use block multiplication and induction.]

Exercise 2.7.9 A triangular matrix is called **unit triangular** if it is square and every main diagonal element is a 1.

a. If A can be carried by the gaussian algorithm to row-echelon form using no row interchanges, show that $A = LU$ where L is unit lower triangular and U is upper triangular.

b. Show that the factorization in (a.) is unique.

Exercise 2.7.10 Let $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r$ be columns of lengths $m, m-1, \dots, m-r+1$. If $\mathbf{k}_j$ denotes column j of I_m, show that $L^{(m)}[\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r] = L^{(m)}[\mathbf{c}_1]L^{(m)}[\mathbf{k}_1, \mathbf{c}_2]L^{(m)}[\mathbf{k}_1, \mathbf{k}_2, \mathbf{c}_3]\cdots$
$L^{(m)}[\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{r-1}, \mathbf{c}_r]$. The notation is as in the proof of Theorem 2.7.2. [*Hint*: Use induction on m and block multiplication.]

Exercise 2.7.11 Prove Lemma 2.7.2. [*Hint*: $P_k^{-1} = P_k$. Write $P_k = \begin{bmatrix} I_k & 0 \\ 0 & P_0 \end{bmatrix}$ in block form where P_0 is an $(m-k)\times(m-k)$ permutation matrix.]

2.8 An Application to Input-Output Economic Models[16]

In 1973 Wassily Leontief was awarded the Nobel prize in economics for his work on mathematical models.[17] Roughly speaking, an economic system in this model consists of several industries, each of which produces a product and each of which uses some of the production of the other industries. The following example is typical.

[16]The applications in this section and the next are independent and may be taken in any order.
[17]See W. W. Leontief, "The world economy of the year 2000," *Scientific American*, Sept. 1980.

Example 2.8.1

A primitive society has three basic needs: food, shelter, and clothing. There are thus three industries in the society—the farming, housing, and garment industries—that produce these commodities. Each of these industries consumes a certain proportion of the total output of each commodity according to the following table.

		OUTPUT		
		Farming	**Housing**	**Garment**
	Farming	0.4	0.2	0.3
CONSUMPTION	**Housing**	0.2	0.6	0.4
	Garment	0.4	0.2	0.3

Find the annual prices that each industry must charge for its income to equal its expenditures.

Solution. Let p_1, p_2, and p_3 be the prices charged per year by the farming, housing, and garment industries, respectively, for their total output. To see how these prices are determined, consider the farming industry. It receives p_1 for its production in any year. But it *consumes* products from all these industries in the following amounts (from row 1 of the table): 40% of the food, 20% of the housing, and 30% of the clothing. Hence, the expenditures of the farming industry are $0.4p_1 + 0.2p_2 + 0.3p_3$, so

$$0.4p_1 + 0.2p_2 + 0.3p_3 = p_1$$

A similar analysis of the other two industries leads to the following system of equations.

$$\begin{aligned} 0.4p_1 + 0.2p_2 + 0.3p_3 &= p_1 \\ 0.2p_1 + 0.6p_2 + 0.4p_3 &= p_2 \\ 0.4p_1 + 0.2p_2 + 0.3p_3 &= p_3 \end{aligned}$$

This has the matrix form $E\mathbf{p} = \mathbf{p}$, where

$$E = \left[\begin{array}{ccc} 0.4 & 0.2 & 0.3 \\ 0.2 & 0.6 & 0.4 \\ 0.4 & 0.2 & 0.3 \end{array} \right] \quad \text{and} \quad \mathbf{p} = \left[\begin{array}{c} p_1 \\ p_2 \\ p_3 \end{array} \right]$$

The equations can be written as the homogeneous system

$$(I - E)\mathbf{p} = \mathbf{0}$$

where I is the 3×3 identity matrix, and the solutions are

$$\mathbf{p} = \left[\begin{array}{c} 2t \\ 3t \\ 2t \end{array} \right]$$

where t is a parameter. Thus, the pricing must be such that the total output of the farming industry has the same value as the total output of the garment industry, whereas the total value of the housing industry must be $\frac{3}{2}$ as much.

In general, suppose an economy has n industries, each of which uses some (possibly none) of the production of every industry. We assume first that the economy is **closed** (that is, no product is exported or imported) and that all product is used. Given two industries i and j, let e_{ij} denote the proportion of the total annual output of industry j that is consumed by industry i. Then $E = \left[e_{ij}\right]$ is called the **input-output** matrix for the economy. Clearly,

$$0 \leq e_{ij} \leq 1 \quad \text{for all } i \text{ and } j \tag{2.12}$$

Moreover, all the output from industry j is used by *some* industry (the model is closed), so

$$e_{1j} + e_{2j} + \cdots + e_{ij} = 1 \quad \text{for each } j \tag{2.13}$$

This condition asserts that each column of E sums to 1. Matrices satisfying conditions (2.12) and (2.13) are called **stochastic matrices**.

As in Example 2.8.1, let p_i denote the price of the total annual production of industry i. Then p_i is the annual revenue of industry i. On the other hand, industry i spends $e_{i1}p_1 + e_{i2}p_2 + \cdots + e_{in}p_n$ annually for the product it uses ($e_{ij}p_j$ is the cost for product from industry j). The closed economic system is said to be in **equilibrium** if the annual expenditure equals the annual revenue for each industry—that is, if

$$e_{1j}p_1 + e_{2j}p_2 + \cdots + e_{ij}p_n = p_i \quad \text{for each } i = 1,\ 2,\ \dots,\ n$$

If we write $\mathbf{p} = \left[\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \end{array} \right]$, these equations can be written as the matrix equation

$$E\mathbf{p} = \mathbf{p}$$

This is called the **equilibrium condition**, and the solutions $\mathbf{p}$ are called **equilibrium price structures**. The equilibrium condition can be written as

$$(I - E)\mathbf{p} = \mathbf{0}$$

which is a system of homogeneous equations for $\mathbf{p}$. Moreover, there is always a nontrivial solution $\mathbf{p}$. Indeed, the column sums of $I - E$ are all 0 (because E is stochastic), so the row-echelon form of $I - E$ has a row of zeros. In fact, more is true:

Theorem 2.8.1

Let E be any $n \times n$ stochastic matrix. Then there is a nonzero $n \times 1$ vector $\mathbf{p}$ with nonnegative entries such that $E\mathbf{p} = \mathbf{p}$. If all the entries of E are positive, the matrix $\mathbf{p}$ can be chosen with all entries positive.

Theorem 2.8.1 guarantees the existence of an equilibrium price structure for any closed input-output system of the type discussed here. The proof is beyond the scope of this book.[18]

[18]The interested reader is referred to P. Lancaster's *Theory of Matrices* (New York: Academic Press, 1969) or to E. Seneta's *Non-negative Matrices* (New York: Wiley, 1973).

Example 2.8.2

Find the equilibrium price structures for four industries if the input-output matrix is

$$E = \begin{bmatrix} 0.6 & 0.2 & 0.1 & 0.1 \\ 0.3 & 0.4 & 0.2 & 0 \\ 0.1 & 0.3 & 0.5 & 0.2 \\ 0 & 0.1 & 0.2 & 0.7 \end{bmatrix}$$

Find the prices if the total value of business is \$1000.

Solution. If $\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$ is the equilibrium price structure, then the equilibrium condition reads $E\mathbf{p} = \mathbf{p}$. When we write this as $(I - E)\mathbf{p} = \mathbf{0}$, the methods of Chapter 1 yield the following family of solutions:

$$\mathbf{p} = \begin{bmatrix} 44t \\ 39t \\ 51t \\ 47t \end{bmatrix}$$

where t is a parameter. If we insist that $p_1 + p_2 + p_3 + p_4 = 1000$, then $t = 5.525$. Hence

$$\mathbf{p} = \begin{bmatrix} 243.09 \\ 215.47 \\ 281.76 \\ 259.67 \end{bmatrix}$$

to five figures.

The Open Model

We now assume that there is a demand for products in the **open sector** of the economy, which is the part of the economy other than the producing industries (for example, consumers). Let d_i denote the total value of the demand for product i in the open sector. If p_i and e_{ij} are as before, the value of the annual demand for product i by the producing industries themselves is $e_{i1}p_1 + e_{i2}p_2 + \cdots + e_{in}p_n$, so the total annual revenue p_i of industry i breaks down as follows:

$$p_i = (e_{i1}p_1 + e_{i2}p_2 + \cdots + e_{in}p_n) + d_i \quad \text{for each } i = 1, 2, \dots, n$$

The column $\mathbf{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$ is called the **demand matrix**, and this gives a matrix equation

$$\mathbf{p} = E\mathbf{p} + \mathbf{d}$$

or

$$(I-E)\mathbf{p}=\mathbf{d} \tag{2.14}$$

This is a system of linear equations for $\mathbf{p}$, and we ask for a solution $\mathbf{p}$ with every entry nonnegative. Note that every entry of E is between 0 and 1, but the column sums of E need not equal 1 as in the closed model.

Before proceeding, it is convenient to introduce a useful notation. If $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ are matrices of the same size, we write $A > B$ if $a_{ij} > b_{ij}$ for all i and j, and we write $A \geq B$ if $a_{ij} \geq b_{ij}$ for all i and j. Thus $P \geq 0$ means that every entry of P is nonnegative. Note that $A \geq 0$ and $B \geq 0$ implies that $AB \geq 0$.

Now, given a demand matrix $\mathbf{d} \geq \mathbf{0}$, we look for a production matrix $\mathbf{p} \geq \mathbf{0}$ satisfying equation (2.14). This certainly exists if $I-E$ is invertible and $(I-E)^{-1} \geq 0$. On the other hand, the fact that $\mathbf{d} \geq \mathbf{0}$ means any solution $\mathbf{p}$ to equation (2.14) satisfies $\mathbf{p} \geq E\mathbf{p}$. Hence, the following theorem is not too surprising.

Theorem 2.8.2

Let $E \geq 0$ be a square matrix. Then $I-E$ is invertible and $(I-E)^{-1} \geq 0$ if and only if there exists a column $\mathbf{p} > \mathbf{0}$ such that $\mathbf{p} > E\mathbf{p}$.

Heuristic Proof.

If $(I-E)^{-1} \geq 0$, the existence of $\mathbf{p} > \mathbf{0}$ with $\mathbf{p} > E\mathbf{p}$ is left as Exercise 2.8.11. Conversely, suppose such a column $\mathbf{p}$ exists. Observe that

$$(I-E)(I+E+E^2+\cdots+E^{k-1}) = I-E^k$$

holds for all $k \geq 2$. If we can show that every entry of E^k approaches 0 as k becomes large then, intuitively, the infinite matrix sum

$$U = I+E+E^2+\cdots$$

exists and $(I-E)U = I$. Since $U \geq 0$, this does it. To show that E^k approaches 0, it suffices to show that $EP < \mu P$ for some number μ with $0<\mu<1$ (then $E^kP < \mu^k P$ for all $k \geq 1$ by induction). The existence of μ is left as Exercise 2.8.12. □

The condition $\mathbf{p} > E\mathbf{p}$ in Theorem 2.8.2 has a simple economic interpretation. If $\mathbf{p}$ is a production matrix, entry i of $E\mathbf{p}$ is the total value of all product used by industry i in a year. Hence, the condition $\mathbf{p} > E\mathbf{p}$ means that, for each i, the value of product produced by industry i exceeds the value of the product it uses. In other words, each industry runs at a profit.

Example 2.8.3

If $E = \begin{bmatrix} 0.6 & 0.2 & 0.3 \\ 0.1 & 0.4 & 0.2 \\ 0.2 & 0.5 & 0.1 \end{bmatrix}$, show that $I-E$ is invertible and $(I-E)^{-1} \geq 0$.

Solution. Use $\mathbf{p} = (3,\ 2,\ 2)^T$ in Theorem 2.8.2.

If $\mathbf{p}_0 = (1,\ 1,\ 1)^T$, the entries of $E\mathbf{p}_0$ are the row sums of E. Hence $\mathbf{p}_0 > E\mathbf{p}_0$ holds if the row sums of E are all less than 1. This proves the first of the following useful facts (the second is Exercise 2.8.10).

Corollary 2.8.1

Let $E \geq 0$ be a square matrix. In each case, $I - E$ is invertible and $(I - E)^{-1} \geq 0$:

1. *All row sums of E are less than 1.*
2. *All column sums of E are less than 1.*

Exercises for 2.8

Exercise 2.8.1 Find the possible equilibrium price structures when the input-output matrices are:

a. $\left[\begin{array}{ccc} 0.1 & 0.2 & 0.3 \\ 0.6 & 0.2 & 0.3 \\ 0.3 & 0.6 & 0.4 \end{array}\right]$ b. $\left[\begin{array}{ccc} 0.5 & 0 & 0.5 \\ 0.1 & 0.9 & 0.2 \\ 0.4 & 0.1 & 0.3 \end{array}\right]$

c. $\left[\begin{array}{cccc} 0.3 & 0.1 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.1 & 0 \\ 0.3 & 0.3 & 0.2 & 0.3 \\ 0.2 & 0.3 & 0.6 & 0.7 \end{array}\right]$

d. $\left[\begin{array}{cccc} 0.5 & 0 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0 & 0.1 \\ 0.1 & 0.2 & 0.8 & 0.2 \\ 0.2 & 0.1 & 0.1 & 0.6 \end{array}\right]$

Exercise 2.8.2 Three industries A, B, and C are such that all the output of A is used by B, all the output of B is used by C, and all the output of C is used by A. Find the possible equilibrium price structures.

Exercise 2.8.3 Find the possible equilibrium price structures for three industries where the input-output matrix is $\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right]$. Discuss why there are two parameters here.

Exercise 2.8.4 Prove Theorem 2.8.1 for a 2×2 stochastic matrix E by first writing it in the form $E = \left[\begin{array}{cc} a & b \\ 1-a & 1-b \end{array}\right]$, where $0 \leq a \leq 1$ and $0 \leq b \leq 1$.

Exercise 2.8.5 If E is an $n \times n$ stochastic matrix and $\mathbf{c}$ is an $n \times 1$ matrix, show that the sum of the entries of $\mathbf{c}$ equals the sum of the entries of the $n \times 1$ matrix $E\mathbf{c}$.

Exercise 2.8.6 Let $W = \left[\begin{array}{ccccc} 1 & 1 & 1 & \cdots & 1 \end{array}\right]$. Let E and F denote $n \times n$ matrices with nonnegative entries.

a. Show that E is a stochastic matrix if and only if $WE = W$.

b. Use part (a.) to deduce that, if E and F are both stochastic matrices, then EF is also stochastic.

Exercise 2.8.7 Find a 2×2 matrix E with entries between 0 and 1 such that:

a. $I - E$ has no inverse.

b. $I - E$ has an inverse but not all entries of $(I - E)^{-1}$ are nonnegative.

Exercise 2.8.8 If E is a 2×2 matrix with entries between 0 and 1, show that $I - E$ is invertible and $(I - E)^{-1} \geq 0$ if and only if $\operatorname{tr} E < 1 + \det E$. Here, if $E = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$, then $\operatorname{tr} E = a + d$ and $\det E = ad - bc$.

Exercise 2.8.9 In each case show that $I - E$ is invertible and $(I - E)^{-1} \geq 0$.

a. $\left[\begin{array}{ccc} 0.6 & 0.5 & 0.1 \\ 0.1 & 0.3 & 0.3 \\ 0.2 & 0.1 & 0.4 \end{array}\right]$ b. $\left[\begin{array}{ccc} 0.7 & 0.1 & 0.3 \\ 0.2 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.4 \end{array}\right]$

c. $\left[\begin{array}{ccc} 0.6 & 0.2 & 0.1 \\ 0.3 & 0.4 & 0.2 \\ 0.2 & 0.5 & 0.1 \end{array}\right]$ d. $\left[\begin{array}{ccc} 0.8 & 0.1 & 0.1 \\ 0.3 & 0.1 & 0.2 \\ 0.3 & 0.3 & 0.2 \end{array}\right]$

Exercise 2.8.10 Prove that (1) implies (2) in the Corollary to Theorem 2.8.2.

Exercise 2.8.11 If $(I - E)^{-1} \geq 0$, find $\mathbf{p} > 0$ such that $\mathbf{p} > E\mathbf{p}$.

Exercise 2.8.12 If $E\mathbf{p} < \mathbf{p}$ where $E \geq 0$ and $\mathbf{p} > 0$, find a number μ such that $E\mathbf{p} < \mu\mathbf{p}$ and $0 < \mu < 1$.

[*Hint*: If $E\mathbf{p} = (q_1, \dots, q_n)^T$ and $\mathbf{p} = (p_1, \dots, p_n)^T$, take any number μ where $\max\left\{\frac{q_1}{p_1}, \dots, \frac{q_n}{p_n}\right\} < \mu < 1$.]

2.9 An Application to Markov Chains

Many natural phenomena progress through various stages and can be in a variety of states at each stage. For example, the weather in a given city progresses day by day and, on any given day, may be sunny or rainy. Here the states are "sun" and "rain," and the weather progresses from one state to another in daily stages. Another example might be a football team: The stages of its evolution are the games it plays, and the possible states are "win," "draw," and "loss."

The general setup is as follows: A real conceptual "system" is run generating a sequence of outcomes. The system evolves through a series of "stages," and at any stage it can be in any one of a finite number of "states." At any given stage, the state to which it will go at the next stage depends on the past and present history of the system—that is, on the sequence of states it has occupied to date.

Definition 2.15 Markov Chain

*A **Markov chain** is such an evolving system wherein the state to which it will go next depends only on its present state and does not depend on the earlier history of the system.*[19]

Even in the case of a Markov chain, the state the system will occupy at any stage is determined only in terms of probabilities. In other words, chance plays a role. For example, if a football team wins a particular game, we do not know whether it will win, draw, or lose the next game. On the other hand, we may know that the team tends to persist in winning streaks; for example, if it wins one game it may win the next game $\frac{1}{2}$ of the time, lose $\frac{4}{10}$ of the time, and draw $\frac{1}{10}$ of the time. These fractions are called the **probabilities** of these various possibilities. Similarly, if the team loses, it may lose the next game with probability $\frac{1}{2}$ (that is, half the time), win with probability $\frac{1}{4}$, and draw with probability $\frac{1}{4}$. The probabilities of the various outcomes after a drawn game will also be known.

We shall treat probabilities informally here: *The probability that a given event will occur is the long-run proportion of the time that the event does indeed occur*. Hence, all probabilities are numbers between 0 and 1. A probability of 0 means the event is impossible and never occurs; events with probability 1 are certain to occur.

If a Markov chain is in a particular state, the probabilities that it goes to the various states at the next stage of its evolution are called the **transition probabilities** for the chain, and they are assumed to be known quantities. To motivate the general conditions that follow, consider the following simple example. Here the system is a man, the stages are his successive lunches, and the states are the two restaurants he chooses.

Example 2.9.1

A man always eats lunch at one of two restaurants, A and B. He never eats at A twice in a row. However, if he eats at B, he is three times as likely to eat at B next time as at A. Initially, he is equally likely to eat at either restaurant.

a. What is the probability that he eats at A on the third day after the initial one?

[19] The name honours Andrei Andreyevich Markov (1856–1922) who was a professor at the university in St. Petersburg, Russia.

b. What proportion of his lunches does he eat at A?

Solution. The table of transition probabilities follows. The A column indicates that if he eats at A on one day, he never eats there again on the next day and so is certain to go to B.

		Present Lunch	
		A	B
Next	A	0	0.25
Lunch	B	1	0.75

The B column shows that, if he eats at B on one day, he will eat there on the next day $\frac{3}{4}$ of the time and switches to A only $\frac{1}{4}$ of the time.

The restaurant he visits on a given day is not determined. The most that we can expect is to know the probability that he will visit A or B on that day.

Let $\mathbf{s}_m = \begin{bmatrix} s_1^{(m)} \\ s_2^{(m)} \end{bmatrix}$ denote the *state vector* for day m. Here $s_1^{(m)}$ denotes the probability that he eats at A on day m, and $s_2^{(m)}$ is the probability that he eats at B on day m. It is convenient to let $\mathbf{s}_0$ correspond to the initial day. Because he is equally likely to eat at A or B on that initial day, $s_1^{(0)} = 0.5$ and $s_2^{(0)} = 0.5$, so $\mathbf{s}_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$. Now let

$$P = \begin{bmatrix} 0 & 0.25 \\ 1 & 0.75 \end{bmatrix}$$

denote the *transition matrix*. We claim that the relationship

$$\mathbf{s}_{m+1} = P\mathbf{s}_m$$

holds for all integers $m \geq 0$. This will be derived later; for now, we use it as follows to successively compute $\mathbf{s}_1,\ \mathbf{s}_2,\ \mathbf{s}_3,\ \dots$.

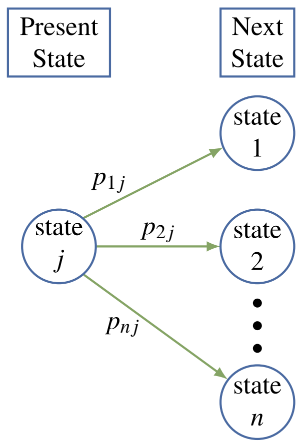
$$\begin{aligned}
\mathbf{s}_1 = P\mathbf{s}_0 &= \begin{bmatrix} 0 & 0.25 \\ 1 & 0.75 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.125 \\ 0.875 \end{bmatrix} \\
\mathbf{s}_2 = P\mathbf{s}_1 &= \begin{bmatrix} 0 & 0.25 \\ 1 & 0.75 \end{bmatrix} \begin{bmatrix} 0.125 \\ 0.875 \end{bmatrix} = \begin{bmatrix} 0.21875 \\ 0.78125 \end{bmatrix} \\
\mathbf{s}_3 = P\mathbf{s}_2 &= \begin{bmatrix} 0 & 0.25 \\ 1 & 0.75 \end{bmatrix} \begin{bmatrix} 0.21875 \\ 0.78125 \end{bmatrix} = \begin{bmatrix} 0.1953125 \\ 0.8046875 \end{bmatrix}
\end{aligned}$$

Hence, the probability that his third lunch (after the initial one) is at A is approximately 0.195, whereas the probability that it is at B is 0.805. If we carry these calculations on, the next state vectors are (to five figures):

$$\mathbf{s}_4 = \begin{bmatrix} 0.20117 \\ 0.79883 \end{bmatrix} \qquad \mathbf{s}_5 = \begin{bmatrix} 0.19971 \\ 0.80029 \end{bmatrix}$$

$$\mathbf{s}_6 = \begin{bmatrix} 0.20007 \\ 0.79993 \end{bmatrix} \qquad \mathbf{s}_7 = \begin{bmatrix} 0.19998 \\ 0.80002 \end{bmatrix}$$

Moreover, as m increases the entries of $\mathbf{s}_m$ get closer and closer to the corresponding entries of $\begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$. Hence, in the long run, he eats 20% of his lunches at A and 80% at B.



Example 2.9.1 incorporates most of the essential features of all Markov chains. The general model is as follows: The system evolves through various stages and at each stage can be in exactly one of n distinct states. It progresses through a sequence of states as time goes on. If a Markov chain is in state j at a particular stage of its development, the probability p_{ij} that it goes to state i at the next stage is called the **transition probability**. The $n \times n$ matrix $P = \left[p_{ij}\right]$ is called the **transition matrix** for the Markov chain. The situation is depicted graphically in the diagram.

We make one important assumption about the transition matrix $P = \left[p_{ij}\right]$: It does *not* depend on which stage the process is in. This assumption means that the transition probabilities are *independent of time*—that is, they do not change as time goes on. It is this assumption that distinguishes Markov chains in the literature of this subject.

Example 2.9.2

Suppose the transition matrix of a three-state Markov chain is

$$P = \left[\begin{array}{ccc} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{array}\right] = \begin{array}{c} \text{Present state} \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 0.3 & 0.1 & 0.6 \\ 0.5 & 0.9 & 0.2 \\ 0.2 & 0.0 & 0.2 \end{array}\right] \end{array} \begin{array}{l} \\ \\ 1 \\ 2 \;\; \text{Next state} \\ 3 \end{array}$$

If, for example, the system is in state 2, then column 2 lists the probabilities of where it goes next. Thus, the probability is $p_{12} = 0.1$ that it goes from state 2 to state 1, and the probability is $p_{22} = 0.9$ that it goes from state 2 to state 2. The fact that $p_{32} = 0$ means that it is impossible for it to go from state 2 to state 3 at the next stage.

Consider the jth column of the transition matrix P.

$$\left[\begin{array}{c} p_{1j} \\ p_{2j} \\ \vdots \\ p_{nj} \end{array}\right]$$

If the system is in state j at some stage of its evolution, the transition probabilities $p_{1j}, p_{2j}, \dots, p_{nj}$ represent the fraction of the time that the system will move to state 1, state 2, ..., state n, respectively, at the next stage. We assume that it has to go to *some* state at each transition, so the sum of these probabilities is 1:

$$p_{1j} + p_{2j} + \cdots + p_{nj} = 1 \quad \text{for each } j$$

Thus, the columns of P all sum to 1 and the entries of P lie between 0 and 1. Hence P is called a **stochastic matrix**.

As in Example 2.9.1, we introduce the following notation: Let $s_i^{(m)}$ denote the probability that the system is in state i after m transitions. The $n \times 1$ matrices

$$\mathbf{s}_m = \begin{bmatrix} s_1^{(m)} \\ s_2^{(m)} \\ \vdots \\ s_n^{(m)} \end{bmatrix} \qquad m = 0,\ 1,\ 2,\ \dots$$

are called the **state vectors** for the Markov chain. Note that the sum of the entries of $\mathbf{s}_m$ must equal 1 because the system must be in *some* state after m transitions. The matrix $\mathbf{s}_0$ is called the **initial state vector** for the Markov chain and is given as part of the data of the particular chain. For example, if the chain has only two states, then an initial vector $\mathbf{s}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ means that it started in state 1. If it started in state 2, the initial vector would be $\mathbf{s}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. If $\mathbf{s}_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$, it is equally likely that the system started in state 1 or in state 2.

Theorem 2.9.1

Let P be the transition matrix for an n-state Markov chain. If $\mathbf{s}_m$ is the state vector at stage m, then

$$\mathbf{s}_{m+1} = P\mathbf{s}_m$$

for each $m = 0,\ 1,\ 2,\ \dots$.

Heuristic Proof. Suppose that the Markov chain has been run N times, each time starting with the same initial state vector. Recall that p_{ij} is the proportion of the time the system goes from state j at some stage to state i at the next stage, whereas $s_i^{(m)}$ is the proportion of the time it is in state i at stage m. Hence

$$s_i^{m+1}N$$

is (approximately) the number of times the system is in state i at stage $m+1$. We are going to calculate this number another way. The system got to state i at stage $m+1$ through *some* other state (say state j) at stage m. The number of times it was *in* state j at that stage is (approximately) $s_j^{(m)}N$, so the number of times it got to state i via state j is $p_{ij}(s_j^{(m)}N)$. Summing over j gives the number of times the system is in state i (at stage $m+1$). This is the number we calculated before, so

$$s_i^{(m+1)}N = p_{i1}s_1^{(m)}N + p_{i2}s_2^{(m)}N + \cdots + p_{in}s_n^{(m)}N$$

Dividing by N gives $s_i^{(m+1)} = p_{i1}s_1^{(m)} + p_{i2}s_2^{(m)} + \cdots + p_{in}s_n^{(m)}$ for each i, and this can be expressed as the matrix equation $\mathbf{s}_{m+1} = P\mathbf{s}_m$. □

If the initial probability vector $\mathbf{s}_0$ and the transition matrix P are given, Theorem 2.9.1 gives $\mathbf{s}_1,\ \mathbf{s}_2,\ \mathbf{s}_3,\ \dots,$ one after the other, as follows:

$$\begin{aligned} \mathbf{s}_1 &= P\mathbf{s}_0 \\ \mathbf{s}_2 &= P\mathbf{s}_1 \\ \mathbf{s}_3 &= P\mathbf{s}_2 \\ &\vdots \end{aligned}$$

Hence, the state vector $\mathbf{s}_m$ is completely determined for each $m = 0,\ 1,\ 2,\ \dots$ by P and $\mathbf{s}_0$.

Example 2.9.3

A wolf pack always hunts in one of three regions R_1, R_2, and R_3. Its hunting habits are as follows:

1. If it hunts in some region one day, it is as likely as not to hunt there again the next day.
2. If it hunts in R_1, it never hunts in R_2 the next day.
3. If it hunts in R_2 or R_3, it is equally likely to hunt in each of the other regions the next day.

If the pack hunts in R_1 on Monday, find the probability that it hunts there on Thursday.

Solution. The stages of this process are the successive days; the states are the three regions. The transition matrix P is determined as follows (see the table): The first habit asserts that $p_{11} = p_{22} = p_{33} = \frac{1}{2}$. Now column 1 displays what happens when the pack starts in R_1: It never goes to state 2, so $p_{21} = 0$ and, because the column must sum to 1, $p_{31} = \frac{1}{2}$. Column 2 describes what happens if it starts in R_2: $p_{22} = \frac{1}{2}$ and p_{12} and p_{32} are equal (by habit 3), so $p_{12} = p_{32} = \frac{1}{2}$ because the column sum must equal 1. Column 3 is filled in a similar way.

	R_1	R_2	R_3
R_1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
R_2	0	$\frac{1}{2}$	$\frac{1}{4}$
R_3	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$

Now let Monday be the initial stage. Then $\mathbf{s}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ because the pack hunts in R_1 on that day. Then $\mathbf{s}_1$, $\mathbf{s}_2$, and $\mathbf{s}_3$ describe Tuesday, Wednesday, and Thursday, respectively, and we compute them using Theorem 2.9.1.

$$\mathbf{s}_1 = P\mathbf{s}_0 = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} \qquad \mathbf{s}_2 = P\mathbf{s}_1 = \begin{bmatrix} \frac{3}{8} \\ \frac{1}{8} \\ \frac{4}{8} \end{bmatrix} \qquad \mathbf{s}_3 = P\mathbf{s}_2 = \begin{bmatrix} \frac{11}{32} \\ \frac{6}{32} \\ \frac{15}{32} \end{bmatrix}$$

Hence, the probability that the pack hunts in Region R_1 on Thursday is $\frac{11}{32}$.

Steady State Vector

Another phenomenon that was observed in Example 2.9.1 can be expressed in general terms. The state vectors $\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \dots$ were calculated in that example and were found to "approach" $\mathbf{s} = \left[\begin{array}{c} 0.2 \\ 0.8 \end{array}\right]$. This means that the first component of $\mathbf{s}_m$ becomes and remains very close to 0.2 as m becomes large, whereas the second component gets close to 0.8 as m increases. When this is the case, we say that $\mathbf{s}_m$ **converges** to $\mathbf{s}$. For large m, then, there is very little error in taking $\mathbf{s}_m = \mathbf{s}$, so the long-term probability that the system is in state 1 is 0.2, whereas the probability that it is in state 2 is 0.8. In Example 2.9.1, enough state vectors were computed for the limiting vector $\mathbf{s}$ to be apparent. However, there is a better way to do this that works in most cases.

Suppose P is the transition matrix of a Markov chain, and assume that the state vectors $\mathbf{s}_m$ converge to a limiting vector $\mathbf{s}$. Then $\mathbf{s}_m$ is very close to $\mathbf{s}$ for sufficiently large m, so $\mathbf{s}_{m+1}$ is also very close to $\mathbf{s}$. Thus, the equation $\mathbf{s}_{m+1} = P\mathbf{s}_m$ from Theorem 2.9.1 is closely approximated by

$$\mathbf{s} = P\mathbf{s}$$

so it is not surprising that $\mathbf{s}$ should be a solution to this matrix equation. Moreover, it is easily solved because it can be written as a system of homogeneous linear equations

$$(I - P)\mathbf{s} = \mathbf{0}$$

with the entries of $\mathbf{s}$ as variables.

In Example 2.9.1, where $P = \left[\begin{array}{cc} 0 & 0.25 \\ 1 & 0.75 \end{array}\right]$, the general solution to $(I - P)\mathbf{s} = \mathbf{0}$ is $\mathbf{s} = \left[\begin{array}{c} t \\ 4t \end{array}\right]$, where t is a parameter. But if we insist that the entries of S sum to 1 (as must be true of all state vectors), we find $t = 0.2$ and so $\mathbf{s} = \left[\begin{array}{c} 0.2 \\ 0.8 \end{array}\right]$ as before.

All this is predicated on the existence of a limiting vector for the sequence of state vectors of the Markov chain, and such a vector may not always exist. However, it does exist in one commonly occurring situation. A stochastic matrix P is called **regular** if some power P^m of P has every entry greater than zero. The matrix $P = \left[\begin{array}{cc} 0 & 0.25 \\ 1 & 0.75 \end{array}\right]$ of Example 2.9.1 is regular (in this case, each entry of P^2 is positive), and the general theorem is as follows:

Theorem 2.9.2

Let P be the transition matrix of a Markov chain and assume that P is regular. Then there is a unique column matrix $\mathbf{s}$ satisfying the following conditions:

1. *$P\mathbf{s} = \mathbf{s}$.*
2. *The entries of $\mathbf{s}$ are positive and sum to 1.*

Moreover, condition 1 can be written as

$$(I - P)\mathbf{s} = \mathbf{0}$$

and so gives a homogeneous system of linear equations for $\mathbf{s}$*. Finally, the sequence of state vectors* $\mathbf{s}_0,\ \mathbf{s}_1,\ \mathbf{s}_2,\ \dots$ *converges to* $\mathbf{s}$ *in the sense that if* m *is large enough, each entry of* $\mathbf{s}_m$ *is closely approximated by the corresponding entry of* $\mathbf{s}$*.*

This theorem will not be proved here.[20]

If P is the regular transition matrix of a Markov chain, the column $\mathbf{s}$ satisfying conditions 1 and 2 of Theorem 2.9.2 is called the **steady-state vector** for the Markov chain. The entries of $\mathbf{s}$ are the long-term probabilities that the chain will be in each of the various states.

Example 2.9.4

A man eats one of three soups—beef, chicken, and vegetable—each day. He never eats the same soup two days in a row. If he eats beef soup on a certain day, he is equally likely to eat each of the others the next day; if he does not eat beef soup, he is twice as likely to eat it the next day as the alternative.

a. If he has beef soup one day, what is the probability that he has it again two days later?

b. What are the long-run probabilities that he eats each of the three soups?

Solution. The states here are B, C, and V, the three soups. The transition matrix P is given in the table. (Recall that, for each state, the corresponding column lists the probabilities for the next state.)

	B	C	V
B	0	$\frac{2}{3}$	$\frac{2}{3}$
C	$\frac{1}{2}$	0	$\frac{1}{3}$
V	$\frac{1}{2}$	$\frac{1}{3}$	0

If he has beef soup initially, then the initial state vector is

$$\mathbf{s}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Then two days later the state vector is $\mathbf{s}_2$. If P is the transition matrix, then

$$\mathbf{s}_1 = P\mathbf{s}_0 = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{s}_2 = P\mathbf{s}_1 = \frac{1}{6}\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

so he eats beef soup two days later with probability $\frac{2}{3}$. This answers (a.) and also shows that he eats chicken and vegetable soup each with probability $\frac{1}{6}$.

[20]The interested reader can find an elementary proof in J. Kemeny, H. Mirkil, J. Snell, and G. Thompson, *Finite Mathematical Structures* (Englewood Cliffs, N.J.: Prentice-Hall, 1958).

To find the long-run probabilities, we must find the steady-state vector $\mathbf{s}$. Theorem 2.9.2 applies because P is regular (P^2 has positive entries), so $\mathbf{s}$ satisfies $P\mathbf{s} = \mathbf{s}$. That is, $(I - P)\mathbf{s} = \mathbf{0}$ where

$$I - P = \frac{1}{6}\left[\begin{array}{rrr} 6 & -4 & -4 \\ -3 & 6 & -2 \\ -3 & -2 & 6 \end{array}\right]$$

The solution is $\mathbf{s} = \left[\begin{array}{c} 4t \\ 3t \\ 3t \end{array}\right]$, where t is a parameter, and we use $\mathbf{s} = \left[\begin{array}{c} 0.4 \\ 0.3 \\ 0.3 \end{array}\right]$ because the entries of $\mathbf{s}$ must sum to 1. Hence, in the long run, he eats beef soup 40% of the time and eats chicken soup and vegetable soup each 30% of the time.

Exercises for 2.9

Exercise 2.9.1 Which of the following stochastic matrices is regular?

a. $\left[\begin{array}{ccc} 0 & 0 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{array}\right]$

b. $\left[\begin{array}{ccc} \frac{1}{2} & 0 & \frac{1}{3} \\ \frac{1}{4} & 1 & \frac{1}{3} \\ \frac{1}{4} & 0 & \frac{1}{3} \end{array}\right]$

Exercise 2.9.2 In each case find the steady-state vector and, assuming that it starts in state 1, find the probability that it is in state 2 after 3 transitions.

a. $\left[\begin{array}{cc} 0.5 & 0.3 \\ 0.5 & 0.7 \end{array}\right]$

b. $\left[\begin{array}{cc} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{array}\right]$

c. $\left[\begin{array}{ccc} 0 & \frac{1}{2} & \frac{1}{4} \\ 1 & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$

d. $\left[\begin{array}{ccc} 0.4 & 0.1 & 0.5 \\ 0.2 & 0.6 & 0.2 \\ 0.4 & 0.3 & 0.3 \end{array}\right]$

e. $\left[\begin{array}{ccc} 0.8 & 0.0 & 0.2 \\ 0.1 & 0.6 & 0.1 \\ 0.1 & 0.4 & 0.7 \end{array}\right]$

f. $\left[\begin{array}{ccc} 0.1 & 0.3 & 0.3 \\ 0.3 & 0.1 & 0.6 \\ 0.6 & 0.6 & 0.1 \end{array}\right]$

Exercise 2.9.3 A fox hunts in three territories A, B, and C. He never hunts in the same territory on two successive days. If he hunts in A, then he hunts in C the next day. If he hunts in B or C, he is twice as likely to hunt in A the next day as in the other territory.

a. What proportion of his time does he spend in A, in B, and in C?

b. If he hunts in A on Monday (C on Monday), what is the probability that he will hunt in B on Thursday?

Exercise 2.9.4 Assume that there are three social classes—upper, middle, and lower—and that social mobility behaves as follows:

1. Of the children of upper-class parents, 70% remain upper-class, whereas 10% become middle-class and 20% become lower-class.

2. Of the children of middle-class parents, 80% remain middle-class, whereas the others are evenly split between the upper class and the lower class.

3. For the children of lower-class parents, 60% remain lower-class, whereas 30% become middle-class and 10% upper-class.

a. Find the probability that the grandchild of lower-class parents becomes upper-class.

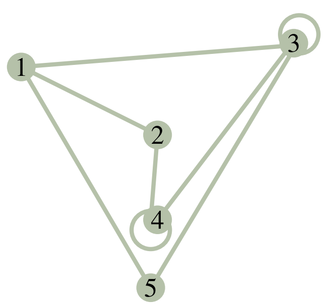
b. Find the long-term breakdown of society into classes.

Exercise 2.9.5 The prime minister says she will call an election. This gossip is passed from person to person with a probability $p \neq 0$ that the information is passed incorrectly at any stage. Assume that when a person hears the gossip he or she passes it to one person who does not know. Find the long-term probability that a person will hear that there is going to be an election.

Exercise 2.9.6 John makes it to work on time one Monday out of four. On other work days his behaviour is as follows: If he is late one day, he is twice as likely to come to work on time the next day as to be late. If he is on time one day, he is as likely to be late as not the next day. Find the probability of his being late and that of his being on time Wednesdays.

Exercise 2.9.7 Suppose you have 1¢ and match coins with a friend. At each match you either win or lose 1¢ with equal probability. If you go broke or ever get 4¢, you quit. Assume your friend never quits. If the states are 0, 1, 2, 3, and 4 representing your wealth, show that the corresponding transition matrix P is not regular. Find the probability that you will go broke after 3 matches.

Exercise 2.9.8 A mouse is put into a maze of compartments, as in the diagram. Assume that he always leaves any compartment he enters and that he is equally likely to take any tunnel entry.



a. If he starts in compartment 1, find the probability that he is in compartment 1 again after 3 moves.

b. Find the compartment in which he spends most of his time if he is left for a long time.

Exercise 2.9.9 If a stochastic matrix has a 1 on its main diagonal, show that it cannot be regular. Assume it is not 1×1.

Exercise 2.9.10 If $\mathbf{s}_m$ is the stage-m state vector for a Markov chain, show that $\mathbf{s}_{m+k} = P^k \mathbf{s}_m$ holds for all $m \geq 1$ and $k \geq 1$ (where P is the transition matrix).

Exercise 2.9.11 A stochastic matrix is **doubly stochastic** if all the row sums also equal 1. Find the steady-state vector for a doubly stochastic matrix.

Exercise 2.9.12 Consider the 2×2 stochastic matrix

$$P = \left[\begin{array}{cc} 1-p & q \\ p & 1-q \end{array}\right],$$

where $0 < p < 1$ and $0 < q < 1$.

a. Show that $\frac{1}{p+q}\left[\begin{array}{c} q \\ p \end{array}\right]$ is the steady-state vector for P.

b. Show that P^m converges to the matrix $\frac{1}{p+q}\left[\begin{array}{cc} q & q \\ p & p \end{array}\right]$ by first verifying inductively that $P^m = \frac{1}{p+q}\left[\begin{array}{cc} q & q \\ p & p \end{array}\right] + \frac{(1-p-q)^m}{p+q}\left[\begin{array}{cc} p & -q \\ -p & q \end{array}\right]$ for $m = 1, 2, \dots$. (It can be shown that the sequence of powers $P, P^2, P^3, \dots$ of any regular transition matrix converges to the matrix each of whose columns equals the steady-state vector for P.)

Supplementary Exercises for Chapter 2

Exercise 2.1 Solve for the matrix X if:

a. $PXQ = R$;

b. $XP = S$;

where $P = \left[\begin{array}{rr} 1 & 0 \\ 2 & -1 \\ 0 & 3 \end{array}\right]$, $Q = \left[\begin{array}{rrr} 1 & 1 & -1 \\ 2 & 0 & 3 \end{array}\right]$,
$R = \left[\begin{array}{rrr} -1 & 1 & -4 \\ -4 & 0 & -6 \\ 6 & 6 & -6 \end{array}\right]$, $S = \left[\begin{array}{rr} 1 & 6 \\ 3 & 1 \end{array}\right]$

Exercise 2.2 Consider

$$p(X) = X^3 - 5X^2 + 11X - 4I.$$

a. If $p(U) = \left[\begin{array}{rr} 1 & 3 \\ -1 & 0 \end{array}\right]$ compute $p(U^T)$.

b. If $p(U) = 0$ where U is $n \times n$, find U^{-1} in terms of U.

Exercise 2.3 Show that, if a (possibly nonhomogeneous) system of equations is consistent and has more variables than equations, then it must have infinitely many solutions. [*Hint*: Use Theorem 2.2.2 and Theorem 1.3.1.]

Exercise 2.4 Assume that a system $A\mathbf{x} = \mathbf{b}$ of linear equations has at least two distinct solutions $\mathbf{y}$ and $\mathbf{z}$.

a. Show that $\mathbf{x}_k = \mathbf{y} + k(\mathbf{y} - \mathbf{z})$ is a solution for every k.

b. Show that $\mathbf{x}_k = \mathbf{x}_m$ implies $k = m$. [*Hint*: See Example 2.1.7.]

c. Deduce that $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

Exercise 2.5

a. Let A be a 3×3 matrix with all entries on and below the main diagonal zero. Show that $A^3 = 0$.

b. Generalize to the $n \times n$ case and prove your answer.

Exercise 2.6 Let I_{pq} denote the $n \times n$ matrix with (p, q)-entry equal to 1 and all other entries 0. Show that:

a. $I_n = I_{11} + I_{22} + \cdots + I_{nn}$.

b. $I_{pq}I_{rs} = \left\{ \begin{array}{ll} I_{ps} & \text{if } q = r \\ 0 & \text{if } q \neq r \end{array}\right.$.

c. If $A = [a_{ij}]$ is $n \times n$, then $A = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}I_{ij}$.

d. If $A = [a_{ij}]$, then $I_{pq}AI_{rs} = a_{qr}I_{ps}$ for all p, q, r, and s.

Exercise 2.7 A matrix of the form aI_n, where a is a number, is called an $n \times n$ **scalar matrix**.

a. Show that each $n \times n$ scalar matrix commutes with every $n \times n$ matrix.

b. Show that A is a scalar matrix if it commutes with every $n \times n$ matrix. [*Hint*: See part (d.) of Exercise 2.6.]

Exercise 2.8 Let $M = \left[\begin{array}{cc} A & B \\ C & D \end{array}\right]$, where A, B, C, and D are all $n \times n$ and each commutes with all the others. If $M^2 = 0$, show that $(A + D)^3 = 0$. [*Hint*: First show that $A^2 = -BC = D^2$ and that

$$B(A + D) = 0 = C(A + D).]$$

Exercise 2.9 If A is 2×2, show that $A^{-1} = A^T$ if and only if $A = \left[\begin{array}{cc} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{array}\right]$ for some θ or
$A = \left[\begin{array}{cc} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{array}\right]$ for some θ.

[*Hint*: If $a^2 + b^2 = 1$, then $a = \cos\theta$, $b = \sin\theta$ for some θ. Use

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi.]$$

Exercise 2.10

a. If $A = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]$, show that $A^2 = I$.

b. What is wrong with the following argument? If $A^2 = I$, then $A^2 - I = 0$, so $(A - I)(A + I) = 0$, whence $A = I$ or $A = -I$.

Exercise 2.11 Let E and F be elementary matrices obtained from the identity matrix by adding multiples of row k to rows p and q. If $k \neq p$ and $k \neq q$, show that $EF = FE$.

Exercise 2.12 If A is a 2×2 real matrix, $A^2 = A$ and $A^T = A$, show that either A is one of $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, or $A = \begin{bmatrix} a & b \\ b & 1-a \end{bmatrix}$ where $a^2 + b^2 = a$, $-\frac{1}{2} \leq b \leq \frac{1}{2}$ and $b \neq 0$.

Exercise 2.13 Show that the following are equivalent for matrices P, Q:

1. P, Q, and $P+Q$ are all invertible and

$$(P+Q)^{-1} = P^{-1} + Q^{-1}$$

2. P is invertible and $Q = PG$ where $G^2 + G + I = 0$.