4. Vector Geometry

4.1 Vectors and Lines

In this chapter we study the geometry of 3-dimensional space. We view a point in 3-space as an arrow from the origin to that point. Doing so provides a “picture” of the point that is truly worth a thousand words. We used this idea earlier, in Section 2.6, to describe rotations, reflections, and projections of the plane $\mathbb{R}^2$. We now apply the same techniques to 3-space to examine similar transformations of $\mathbb{R}^3$. Moreover, the method enables us to completely describe all lines and planes in space.

Vectors in $\mathbb{R}^3$

Introduce a coordinate system in 3-dimensional space in the usual way. First choose a point O called the *origin*, then choose three mutually perpendicular lines through O, called the x, y, and z *axes*, and establish a number scale on each axis with zero at the origin. Given a point P in 3-space we associate three numbers x, y, and z with P, as described in Figure 4.1.1. These numbers are called the *coordinates* of P, and we denote the point as (x, y, z), or $P(x, y, z)$ to emphasize the label P. The result is called a *cartesian*[1] coordinate system for 3-space, and the resulting description of 3-space is called *cartesian geometry*.

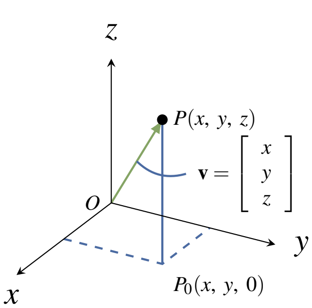


Figure 4.1.1

As in the plane, we introduce vectors by identifying each point $P(x, y, z)$ with the vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$, represented by the **arrow** from the origin to P as in Figure 4.1.1. Informally, we say that the point P *has vector* $\mathbf{v}$, and that vector $\mathbf{v}$ *has point* P. In this way 3-space is identified with $\mathbb{R}^3$, and this identification will be made throughout this chapter, often without comment. In particular, the terms “vector” and “point” are interchangeable.[2] The resulting description of 3-space is called **vector geometry**. Note that the origin is $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

[1]Named after René Descartes who introduced the idea in 1637.

[2]Recall that we defined $\mathbb{R}^n$ as the set of all ordered n-tuples of real numbers, and reserved the right to denote them as rows or as columns.

Length and Direction

We are going to discuss two fundamental geometric properties of vectors in $\mathbb{R}^3$: length and direction. First, if $\mathbf{v}$ is a vector with point P, the **length** $\|\mathbf{v}\|$ of vector $\mathbf{v}$ is defined to be the distance from the origin to P, that is the length of the arrow representing $\mathbf{v}$. The following properties of length will be used frequently.

Theorem 4.1.1

Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *be a vector.*

1. $\|\mathbf{v}\| = \sqrt{x^2 + y^2 + z^2}$. [3]
2. $\mathbf{v} = \mathbf{0}$ *if and only if* $\|\mathbf{v}\| = 0$
3. $\|a\mathbf{v}\| = |a|\|\mathbf{v}\|$ *for all scalars* a. [4]

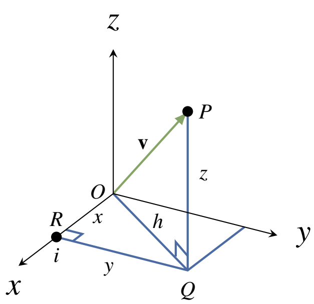


Figure 4.1.2

Proof. Let $\mathbf{v}$ have point $P(x,\ y,\ z)$.

1. In Figure 4.1.2, $\|\mathbf{v}\|$ is the hypotenuse of the right triangle OQP, and so $\|\mathbf{v}\|^2 = h^2 + z^2$ by Pythagoras' theorem.[5] But h is the hypotenuse of the right triangle ORQ, so $h^2 = x^2 + y^2$. Now (1) follows by eliminating h^2 and taking positive square roots.

2. If $\|\mathbf{v}\| = 0$, then $x^2 + y^2 + z^2 = 0$ by (1). Because squares of real numbers are nonnegative, it follows that $x = y = z = 0$, and hence that $\mathbf{v} = \mathbf{0}$. The converse is because $\|\mathbf{0}\| = 0$.

3. We have $a\mathbf{v} = \begin{bmatrix} ax & ay & az \end{bmatrix}^T$ so (1) gives

$$\|a\mathbf{v}\|^2 = (ax)^2 + (ay)^2 + (az)^2 = a^2\|\mathbf{v}\|^2$$

Hence $\|a\mathbf{v}\| = \sqrt{a^2}\|\mathbf{v}\|$, and we are done because $\sqrt{a^2} = |a|$ for any real number a. □

Of course the $\mathbb{R}^2$-version of Theorem 4.1.1 also holds.

[3]When we write $\sqrt{p}$ we mean the positive square root of p.

[4]Recall that the absolute value $|a|$ of a real number is defined by $|a| = \begin{cases} a \text{ if } a \geq 0 \\ -a \text{ if } a < 0 \end{cases}$.

[5]Pythagoras' theorem states that if a and b are sides of right triangle with hypotenuse c, then $a^2 + b^2 = c^2$. A proof is given at the end of this section.

Example 4.1.1

If $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ then $\|\mathbf{v}\| = \sqrt{4+1+9} = \sqrt{14}$. Similarly if $\mathbf{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ in 2-space then $\|\mathbf{v}\| = \sqrt{9+16} = 5$.

When we view two nonzero vectors as arrows emanating from the origin, it is clear geometrically what we mean by saying that they have the same or opposite **direction**. This leads to a fundamental new description of vectors.

Theorem 4.1.2

Let $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{w} \neq \mathbf{0}$ be vectors in $\mathbb{R}^3$. Then $\mathbf{v} = \mathbf{w}$ as matrices if and only if $\mathbf{v}$ and $\mathbf{w}$ have the same direction and the same length.[6]

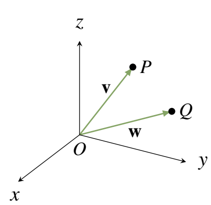


Figure 4.1.3

Proof. If $\mathbf{v} = \mathbf{w}$, they clearly have the same direction and length. Conversely, let $\mathbf{v}$ and $\mathbf{w}$ be vectors with points $P(x, y, z)$ and $Q(x_1, y_1, z_1)$ respectively. If $\mathbf{v}$ and $\mathbf{w}$ have the same length and direction then, geometrically, P and Q must be the same point (see Figure 4.1.3). Hence $x = x_1$, $y = y_1$, and $z = z_1$, that is

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \mathbf{w}.$$ □

A characterization of a vector in terms of its length and direction only is called an **intrinsic** description of the vector. The point to note is that such a description does *not* depend on the choice of coordinate system in $\mathbb{R}^3$. Such descriptions are important in applications because physical laws are often stated in terms of vectors, and these laws cannot depend on the particular coordinate system used to describe the situation.

Geometric Vectors

If A and B are distinct points in space, the arrow from A to B has length and direction.

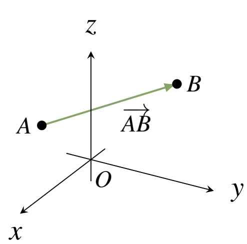


Figure 4.1.4

[6]It is Theorem 4.1.2 that gives vectors their power in science and engineering because many physical quantities are determined by their length and magnitude (and are called **vector quantities**). For example, saying that an airplane is flying at 200 km/h does not describe where it is going; the direction must also be specified. The speed and direction comprise the **velocity** of the airplane, a vector quantity.

Hence:

Definition 4.1 Geometric Vectors

Suppose that A and B are any two points in $\mathbb{R}^3$. In Figure 4.1.4 the line segment from A to B is denoted $\overrightarrow{AB}$ and is called the **geometric vector** *from A to B. Point A is called the* **tail** *of $\overrightarrow{AB}$, B is called the* **tip** *of $\overrightarrow{AB}$, and the* **length** *of $\overrightarrow{AB}$ is denoted $\|\overrightarrow{AB}\|$.*

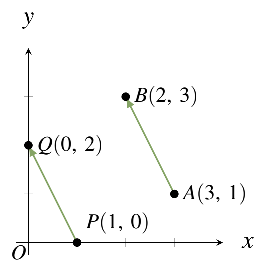


Figure 4.1.5

Note that if $\mathbf{v}$ is any vector in $\mathbb{R}^3$ with point P then $\mathbf{v} = \overrightarrow{OP}$ is itself a geometric vector where O is the origin. Referring to $\overrightarrow{AB}$ as a "vector" seems justified by Theorem 4.1.2 because it has a direction (from A to B) and a length $\|\overrightarrow{AB}\|$. However there appears to be a problem because two geometric vectors can have the same length and direction even if the tips and tails are different. For example $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ in Figure 4.1.5 have the same length $\sqrt{5}$ and the same direction (1 unit left and 2 units up) so, by Theorem 4.1.2, they are the same vector! The best way to understand this apparent paradox is to see $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ as different *representations* of the same[7] underlying vector $\left[\begin{array}{r} -1 \\ 2 \end{array}\right]$. Once it is clarified, this phenomenon is a great benefit because, thanks to Theorem 4.1.2, it means that the same geometric vector can be positioned anywhere in space; what is important is the length and direction, not the location of the tip and tail. This ability to move geometric vectors about is very useful as we shall soon see.

The Parallelogram Law

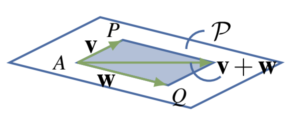


Figure 4.1.6

We now give an intrinsic description of the sum of two vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$, that is a description that depends only on the lengths and directions of $\mathbf{v}$ and $\mathbf{w}$ and not on the choice of coordinate system. Using Theorem 4.1.2 we can think of these vectors as having a common tail A. If their tips are P and Q respectively, then they both lie in a plane $\mathcal{P}$ containing A, P, and Q, as shown in Figure 4.1.6. The vectors $\mathbf{v}$ and $\mathbf{w}$ create a parallelogram[8] in $\mathcal{P}$, shaded in Figure 4.1.6, called the parallelogram **determined** by $\mathbf{v}$ and $\mathbf{w}$.

If we now choose a coordinate system in the plane $\mathcal{P}$ with A as origin, then the parallelogram law in the plane (Section 2.6) shows that their sum $\mathbf{v}+\mathbf{w}$ is the diagonal of the parallelogram they determine with tail A. This is an intrinsic description of the sum $\mathbf{v}+\mathbf{w}$ because it makes no reference to coordinates. This discussion proves:

[7]Fractions provide another example of quantities that can be the same but *look* different. For example $\frac{6}{9}$ and $\frac{14}{21}$ certainly appear different, but they are equal fractions—both equal $\frac{2}{3}$ in "lowest terms".

[8]Recall that a parallelogram is a four-sided figure whose opposite sides are parallel and of equal length.

The Parallelogram Law

In the parallelogram determined by two vectors $\mathbf{v}$ *and* $\mathbf{w}$*, the vector* $\mathbf{v}+\mathbf{w}$ *is the diagonal with the same tail as* $\mathbf{v}$ *and* $\mathbf{w}$*.*

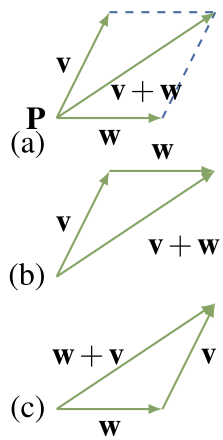


Figure 4.1.7

Because a vector can be positioned with its tail at any point, the parallelogram law leads to another way to view vector addition. In Figure 4.1.7(a) the sum $\mathbf{v}+\mathbf{w}$ of two vectors $\mathbf{v}$ and $\mathbf{w}$ is shown as given by the parallelogram law. If $\mathbf{w}$ is moved so its tail coincides with the tip of $\mathbf{v}$ (Figure 4.1.7(b)) then the sum $\mathbf{v}+\mathbf{w}$ is seen as "first $\mathbf{v}$ and then $\mathbf{w}$. Similarly, moving the tail of $\mathbf{v}$ to the tip of $\mathbf{w}$ shows in Figure 4.1.7(c) that $\mathbf{v}+\mathbf{w}$ is "first $\mathbf{w}$ and then $\mathbf{v}$." This will be referred to as the **tip-to-tail rule**, and it gives a graphic illustration of why $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$.

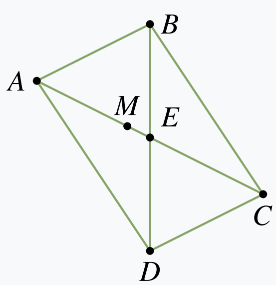
Since $\overrightarrow{AB}$ denotes the vector from a point A to a point B, the tip-to-tail rule takes the easily remembered form

$$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$$

for any points A, B, and C. The next example uses this to derive a theorem in geometry without using coordinates.

Example 4.1.2

Show that the diagonals of a parallelogram bisect each other.



Solution. Let the parallelogram have vertices A, B, C, and D, as shown; let E denote the intersection of the two diagonals; and let M denote the midpoint of diagonal AC. We must show that $M=E$ and that this is the midpoint of diagonal BD. This is accomplished by showing that $\overrightarrow{BM}=\overrightarrow{MD}$. (Then the fact that these vectors have the same direction means that $M=E$, and the fact that they have the same length means that $M=E$ is the midpoint of BD.) Now $\overrightarrow{AM}=\overrightarrow{MC}$ because M is the midpoint of AC, and $\overrightarrow{BA}=\overrightarrow{CD}$ because the figure is a parallelogram. Hence

$$\overrightarrow{BM}=\overrightarrow{BA}+\overrightarrow{AM}=\overrightarrow{CD}+\overrightarrow{MC}=\overrightarrow{MC}+\overrightarrow{CD}=\overrightarrow{MD}$$

where the first and last equalities use the tip-to-tail rule of vector addition.

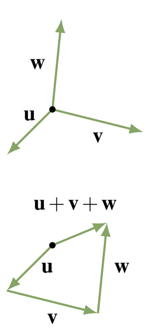


Figure 4.1.8

One reason for the importance of the tip-to-tail rule is that it means two or more vectors can be added by placing them tip-to-tail in sequence. This gives a useful "picture" of the sum of several vectors, and is illustrated for three vectors in Figure 4.1.8 where $\mathbf{u}+\mathbf{v}+\mathbf{w}$ is viewed as first $\mathbf{u}$, then $\mathbf{v}$, then $\mathbf{w}$.

There is a simple geometrical way to visualize the (matrix) **difference** $\mathbf{v}-\mathbf{w}$ of two vectors. If $\mathbf{v}$ and $\mathbf{w}$ are positioned so that they have a common tail A (see Figure 4.1.9), and if B and C are their respective tips, then the tip-to-tail rule gives $\mathbf{w} + \overrightarrow{CB} = \mathbf{v}$. Hence $\mathbf{v} - \mathbf{w} = \overrightarrow{CB}$ is the vector from the tip of $\mathbf{w}$ to the tip of $\mathbf{v}$. Thus both $\mathbf{v} - \mathbf{w}$ and $\mathbf{v} + \mathbf{w}$ appear as diagonals in the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$ (see Figure 4.1.9). We record this for reference.

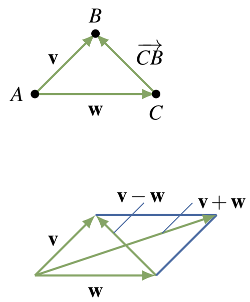


Figure 4.1.9

Theorem 4.1.3

If $\mathbf{v}$ and $\mathbf{w}$ have a common tail, then $\mathbf{v} - \mathbf{w}$ is the vector from the tip of $\mathbf{w}$ to the tip of $\mathbf{v}$.

One of the most useful applications of vector subtraction is that it gives a simple formula for the vector from one point to another, and for the distance between the points.

Theorem 4.1.4

Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two points. Then:

1. $\overrightarrow{P_1P_2} = \left[\begin{array}{c} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{array} \right]$.

2. *The distance between P_1 and P_2 is* $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

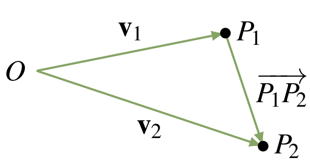


Figure 4.1.10

<u>Proof.</u> If O is the origin, write

$$\mathbf{v}_1 = \overrightarrow{OP}_1 = \left[\begin{array}{c} x_1 \\ y_1 \\ z_1 \end{array} \right] \text{ and } \mathbf{v}_2 = \overrightarrow{OP}_2 = \left[\begin{array}{c} x_2 \\ y_2 \\ z_2 \end{array} \right]$$

as in Figure 4.1.10.

Then Theorem 4.1.3 gives $\overrightarrow{P_1P_2} = \mathbf{v}_2 - \mathbf{v}_1$, and (1) follows. But the distance between P_1 and P_2 is $\|\overrightarrow{P_1P_2}\|$, so (2) follows from (1) and Theorem 4.1.1. □

Of course the $\mathbb{R}^2$-version of Theorem 4.1.4 is also valid: If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points in $\mathbb{R}^2$, then $\overrightarrow{P_1P_2} = \left[\begin{array}{c} x_2 - x_1 \\ y_2 - y_1 \end{array} \right]$, and the distance between P_1 and P_2 is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Example 4.1.3

The distance between $P_1(2, -1, 3)$ and $P_2(1, 1, 4)$ is $\sqrt{(-1)^2 + (2)^2 + (1)^2} = \sqrt{6}$, and the vector from P_1 to P_2 is $\overrightarrow{P_1P_2} = \left[\begin{array}{r} -1 \\ 2 \\ 1 \end{array} \right]$.

As for the parallelogram law, the intrinsic rule for finding the length and direction of a scalar multiple of a vector in $\mathbb{R}^3$ follows easily from the same situation in $\mathbb{R}^2$.

Scalar Multiple Law

If a is a real number and $\mathbf{v} \neq \mathbf{0}$ is a vector then:

1. *The length of $a\mathbf{v}$ is $\|a\mathbf{v}\| = |a|\|\mathbf{v}\|$.*
2. *If[9] $a\mathbf{v} \neq \mathbf{0}$, the direction of $a\mathbf{v}$ is* $\begin{cases} \text{the same as } \mathbf{v} \text{ if } a > 0, \\ \text{opposite to } \mathbf{v} \text{ if } a < 0. \end{cases}$

Proof.

1. This is part of Theorem 4.1.1.
2. Let O denote the origin in $\mathbb{R}^3$, let $\mathbf{v}$ have point P, and choose any plane containing O and P. If we set up a coordinate system in this plane with O as origin, then $\mathbf{v} = \overrightarrow{OP}$ so the result in (2) follows from the scalar multiple law in the plane (Section 2.6). □

Figure 4.1.11 gives several examples of scalar multiples of a vector $\mathbf{v}$.

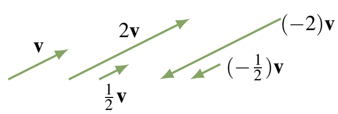


Figure 4.1.11

Consider a line L through the origin, let P be any point on L other than the origin O, and let $\mathbf{p} = \overrightarrow{OP}$. If $t \neq 0$, then $t\mathbf{p}$ is a point on L because it has direction the same or opposite as that of $\mathbf{p}$. Moreover $t > 0$ or $t < 0$ according as the point $t\mathbf{p}$ lies on the same or opposite side of the origin as P. This is illustrated in Figure 4.1.12.

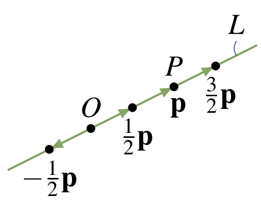


Figure 4.1.12

A vector $\mathbf{u}$ is called a **unit vector** if $\|\mathbf{u}\| = 1$. Then $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are unit vectors, called the **coordinate** vectors. We discuss them in more detail in Section 4.2.

Example 4.1.4

If $\mathbf{v} \neq \mathbf{0}$ show that $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ is the unique unit vector in the same direction as $\mathbf{v}$.

Solution. The vectors in the same direction as $\mathbf{v}$ are the scalar multiples $a\mathbf{v}$ where $a > 0$. But $\|a\mathbf{v}\| = |a|\|\mathbf{v}\| = a\|\mathbf{v}\|$ when $a > 0$, so $a\mathbf{v}$ is a unit vector if and only if $a = \frac{1}{\|\mathbf{v}\|}$.

The next example shows how to find the coordinates of a point on the line segment between two given points. The technique is important and will be used again below.

[9]Since the zero vector has no direction, we deal only with the case $a\mathbf{v} \neq \mathbf{0}$.

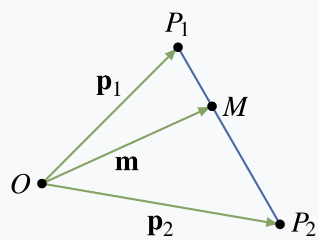
Example 4.1.5

Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the vectors of two points P_1 and P_2. If M is the point one third the way from P_1 to P_2, show that the vector $\mathbf{m}$ of M is given by

$$\mathbf{m} = \tfrac{2}{3}\mathbf{p}_1 + \tfrac{1}{3}\mathbf{p}_2$$

Conclude that if $P_1 = P_1(x_1,\ y_1,\ z_1)$ and $P_2 = P_2(x_2,\ y_2,\ z_2)$, then M has coordinates

$$M = M\left(\tfrac{2}{3}x_1 + \tfrac{1}{3}x_2,\ \tfrac{2}{3}y_1 + \tfrac{1}{3}y_2,\ \tfrac{2}{3}z_1 + \tfrac{1}{3}z_2\right)$$



Solution. The vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{m}$ are shown in the diagram. We have $\overrightarrow{P_1M} = \frac{1}{3}\overrightarrow{P_1P_2}$ because $\overrightarrow{P_1M}$ is in the same direction as $\overrightarrow{P_1P_2}$ and $\frac{1}{3}$ as long. By Theorem 4.1.3 we have $\overrightarrow{P_1P_2} = \mathbf{p}_2 - \mathbf{p}_1$, so tip-to-tail addition gives

$$\mathbf{m} = \mathbf{p}_1 + \overrightarrow{P_1M} = \mathbf{p}_1 + \tfrac{1}{3}(\mathbf{p}_2 - \mathbf{p}_1) = \tfrac{2}{3}\mathbf{p}_1 + \tfrac{1}{3}\mathbf{p}_2$$

as required. For the coordinates, we have $\mathbf{p}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, so

$$\mathbf{m} = \tfrac{2}{3}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \tfrac{1}{3}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}x_1 + \frac{1}{3}x_2 \\ \frac{2}{3}y_1 + \frac{1}{3}y_2 \\ \frac{2}{3}z_1 + \frac{1}{3}z_2 \end{bmatrix}$$

by matrix addition. The last statement follows.

Note that in Example 4.1.5 $\mathbf{m} = \frac{2}{3}\mathbf{p}_1 + \frac{1}{3}\mathbf{p}_2$ is a "weighted average" of $\mathbf{p}_1$ and $\mathbf{p}_2$ with more weight on $\mathbf{p}_1$ because $\mathbf{m}$ is closer to $\mathbf{p}_1$.

The point M halfway between points P_1 and P_2 is called the **midpoint** between these points. In the same way, the vector $\mathbf{m}$ of M is

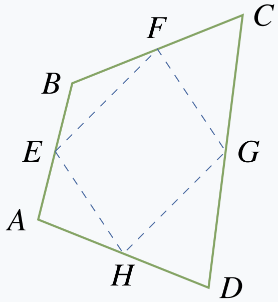
$$\mathbf{m} = \tfrac{1}{2}\mathbf{p}_1 + \tfrac{1}{2}\mathbf{p}_2 = \tfrac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2)$$

as the reader can verify, so $\mathbf{m}$ is the "average" of $\mathbf{p}_1$ and $\mathbf{p}_2$ in this case.

Example 4.1.6

Show that the midpoints of the four sides of any quadrilateral are the vertices of a parallelogram. Here a quadrilateral is any figure with four vertices and straight sides.

Solution. Suppose that the vertices of the quadrilateral are A, B, C, and D (in that order) and that E, F, G, and H are the midpoints of the sides as shown in the diagram. It suffices to show $\overrightarrow{EF} = \overrightarrow{HG}$ (because then sides EF and HG are parallel and of equal length).



Now the fact that E is the midpoint of AB means that $\overrightarrow{EB} = \frac{1}{2}\overrightarrow{AB}$. Similarly, $\overrightarrow{BF} = \frac{1}{2}\overrightarrow{BC}$, so

$$\overrightarrow{EF} = \overrightarrow{EB} + \overrightarrow{BF} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC}) = \frac{1}{2}\overrightarrow{AC}$$

A similar argument shows that $\overrightarrow{HG} = \frac{1}{2}\overrightarrow{AC}$ too, so $\overrightarrow{EF} = \overrightarrow{HG}$ as required.

Definition 4.2 Parallel Vectors in $\mathbb{R}^3$

Two nonzero vectors are called ***parallel*** *if they have the same or opposite direction.*

Many geometrical propositions involve this notion, so the following theorem will be referred to repeatedly.

Theorem 4.1.5

Two nonzero vectors $\mathbf{v}$ *and* $\mathbf{w}$ *are parallel if and only if one is a scalar multiple of the other.*

Proof. If one of them is a scalar multiple of the other, they are parallel by the scalar multiple law.

Conversely, assume that $\mathbf{v}$ and $\mathbf{w}$ are parallel and write $d = \frac{\|\mathbf{v}\|}{\|\mathbf{w}\|}$ for convenience. Then $\mathbf{v}$ and $\mathbf{w}$ have the same or opposite direction. If they have the same direction we show that $\mathbf{v} = d\mathbf{w}$ by showing that $\mathbf{v}$ and $d\mathbf{w}$ have the same length and direction. In fact, $\|d\mathbf{w}\| = |d|\|\mathbf{w}\| = \|\mathbf{v}\|$ by Theorem 4.1.1; as to the direction, $d\mathbf{w}$ and $\mathbf{w}$ have the same direction because $d > 0$, and this is the direction of $\mathbf{v}$ by assumption. Hence $\mathbf{v} = d\mathbf{w}$ in this case by Theorem 4.1.2. In the other case, $\mathbf{v}$ and $\mathbf{w}$ have opposite direction and a similar argument shows that $\mathbf{v} = -d\mathbf{w}$. We leave the details to the reader. □

Example 4.1.7

Given points $P(2, -1, 4)$, $Q(3, -1, 3)$, $A(0, 2, 1)$, and $B(1, 3, 0)$, determine if $\overrightarrow{PQ}$ and $\overrightarrow{AB}$ are parallel.

Solution. By Theorem 4.1.3, $\overrightarrow{PQ} = (1, 0, -1)$ and $\overrightarrow{AB} = (1, 1, -1)$. If $\overrightarrow{PQ} = t\overrightarrow{AB}$ then $(1, 0, -1) = (t, t, -t)$, so $1 = t$ and $0 = t$, which is impossible. Hence $\overrightarrow{PQ}$ is *not* a scalar multiple of $\overrightarrow{AB}$, so these vectors are not parallel by Theorem 4.1.5.

Lines in Space

These vector techniques can be used to give a very simple way of describing straight lines in space. In order to do this, we first need a way to specify the orientation of such a line, much as the slope does in the plane.

Definition 4.3 Direction Vector of a Line

With this in mind, we call a nonzero vector $\mathbf{d} \neq \mathbf{0}$ a ***direction vector*** *for the line if it is parallel to $\overrightarrow{AB}$ for some pair of distinct points A and B on the line.*

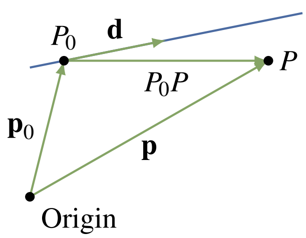


Figure 4.1.13

Of course it is then parallel to $\overrightarrow{CD}$ for *any* distinct points C and D on the line. In particular, any nonzero scalar multiple of $\mathbf{d}$ will also serve as a direction vector of the line.

We use the fact that there is exactly one line that passes through a particular point $P_0(x_0, y_0, z_0)$ and has a given direction vector $\mathbf{d} = \left[\begin{array}{c} a \\ b \\ c \end{array}\right]$. We want to describe this line by giving a condition on x, y, and z that the point $P(x, y, z)$ lies on this line. Let $\mathbf{p}_0 = \left[\begin{array}{c} x_0 \\ y_0 \\ z_0 \end{array}\right]$ and $\mathbf{p} = \left[\begin{array}{c} x \\ y \\ z \end{array}\right]$ denote the vectors of P_0 and P, respectively (see Figure 4.1.13). Then

$$\mathbf{p} = \mathbf{p}_0 + \overrightarrow{P_0P}$$

Hence P lies on the line if and only if $\overrightarrow{P_0P}$ is parallel to $\mathbf{d}$—that is, if and only if $\overrightarrow{P_0P} = t\mathbf{d}$ for some scalar t by Theorem 4.1.5. Thus $\mathbf{p}$ is the vector of a point on the line if and only if $\mathbf{p} = \mathbf{p}_0 + t\mathbf{d}$ for some scalar t. This discussion is summed up as follows.

Vector Equation of a Line

The line parallel to $\mathbf{d} \neq \mathbf{0}$ through the point with vector $\mathbf{p}_0$ is given by

$$\mathbf{p} = \mathbf{p}_0 + t\mathbf{d} \quad t \text{ any scalar}$$

In other words, the point P with vector $\mathbf{p}$ is on this line if and only if a real number t exists such that $\mathbf{p} = \mathbf{p}_0 + t\mathbf{d}$.

In component form the vector equation becomes

$$\left[\begin{array}{c} x \\ y \\ z \end{array}\right] = \left[\begin{array}{c} x_0 \\ y_0 \\ z_0 \end{array}\right] + t\left[\begin{array}{c} a \\ b \\ c \end{array}\right]$$

Equating components gives a different description of the line.

Parametric Equations of a Line

The line through $P_0(x_0, y_0, z_0)$ *with direction vector* $\mathbf{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \mathbf{0}$ *is given by*

$$\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \quad t \text{ any scalar} \\ z &= z_0 + tc \end{aligned}$$

In other words, the point $P(x, y, z)$ *is on this line if and only if a real number* t *exists such that* $x = x_0 + ta$, $y = y_0 + tb$, *and* $z = z_0 + tc$.

Example 4.1.8

Find the equations of the line through the points $P_0(2, 0, 1)$ and $P_1(4, -1, 1)$.

Solution. Let $\mathbf{d} = \overrightarrow{P_0P_1} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ denote the vector from P_0 to P_1. Then $\mathbf{d}$ is parallel to the line (P_0 and P_1 are *on* the line), so $\mathbf{d}$ serves as a direction vector for the line. Using P_0 as the point on the line leads to the parametric equations

$$\begin{aligned} x &= 2 + 2t \\ y &= -t \qquad t \text{ a parameter} \\ z &= 1 \end{aligned}$$

Note that if P_1 is used (rather than P_0), the equations are

$$\begin{aligned} x &= 4 + 2s \\ y &= -1 - s \quad s \text{ a parameter} \\ z &= 1 \end{aligned}$$

These are different from the preceding equations, but this is merely the result of a change of parameter. In fact, $s = t - 1$.

Example 4.1.9

Find the equations of the line through $P_0(3, -1, 2)$ parallel to the line with equations

$$\begin{aligned} x &= -1 + 2t \\ y &= 1 + t \\ z &= -3 + 4t \end{aligned}$$

Solution. The coefficients of t give a direction vector $\mathbf{d} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$ of the given line. Because the line we seek is parallel to this line, $\mathbf{d}$ also serves as a direction vector for the new line. It passes through P_0, so the parametric equations are

$$\begin{aligned} x &= 3 + 2t \\ y &= -1 + t \\ z &= 2 + 4t \end{aligned}$$

Example 4.1.10

Determine whether the following lines intersect and, if so, find the point of intersection.

$$\begin{array}{ll} x = 1 - 3t & x = -1 + s \\ y = 2 + 5t & y = 3 - 4s \\ z = 1 + t & z = 1 - s \end{array}$$

Solution. Suppose $P(x,\ y,\ z)$ with vector $\mathbf{p}$ lies on both lines. Then

$$\begin{bmatrix} 1 - 3t \\ 2 + 5t \\ 1 + t \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 + s \\ 3 - 4s \\ 1 - s \end{bmatrix} \text{ for some } t \text{ and } s,$$

where the first (second) equation is because P lies on the first (second) line. Hence the lines intersect if and only if the three equations

$$\begin{aligned} 1 - 3t &= -1 + s \\ 2 + 5t &= 3 - 4s \\ 1 + t &= 1 - s \end{aligned}$$

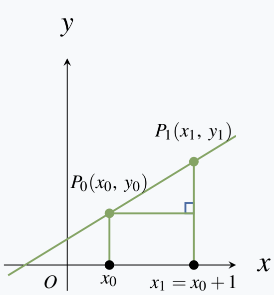
have a solution. In this case, $t = 1$ and $s = -1$ satisfy all three equations, so the lines *do* intersect and the point of intersection is

$$\mathbf{p} = \begin{bmatrix} 1 - 3t \\ 2 + 5t \\ 1 + t \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix}$$

using $t = 1$. Of course, this point can also be found from $\mathbf{p} = \begin{bmatrix} -1 + s \\ 3 - 4s \\ 1 - s \end{bmatrix}$ using $s = -1$.

Example 4.1.11

Show that the line through $P_0(x_0,\ y_0)$ with slope m has direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ m \end{bmatrix}$ and equation $y - y_0 = m(x - x_0)$. This equation is called the *point-slope* formula.



Solution. Let $P_1(x_1,\ y_1)$ be the point on the line one unit to the right of P_0 (see the diagram). Hence $x_1 = x_0 + 1$. Then $\mathbf{d} = \overrightarrow{P_0P_1}$ serves as direction vector of the line, and $\mathbf{d} = \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ y_1 - y_0 \end{bmatrix}$. But the slope m can be computed as follows:

$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{1} = y_1 - y_0$$

Hence $\mathbf{d} = \begin{bmatrix} 1 \\ m \end{bmatrix}$ and the parametric equations are $x = x_0 + t$, $y = y_0 + mt$. Eliminating t gives $y - y_0 = mt = m(x - x_0)$, as asserted.

Note that the vertical line through $P_0(x_0,\ y_0)$ has a direction vector $\mathbf{d} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ that is *not* of the form $\begin{bmatrix} 1 \\ m \end{bmatrix}$ for any m. This result confirms that the notion of slope makes no sense in this case. However, the vector method gives parametric equations for the line:

$$\begin{aligned} x &= x_0 \\ y &= y_0 + t \end{aligned}$$

Because y is arbitrary here (t is arbitrary), this is usually written simply as $x = x_0$.

Pythagoras' Theorem

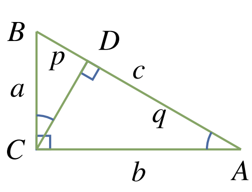


Figure 4.1.14

The Pythagorean theorem was known earlier, but Pythagoras (c. 550 B.C.) is credited with giving the first rigorous, logical, deductive proof of the result. The proof we give depends on a basic property of similar triangles: ratios of corresponding sides are equal.

Theorem 4.1.6: Pythagoras' Theorem

Given a right-angled triangle with hypotenuse c and sides a and b, then $a^2 + b^2 = c^2$.

Proof. Let A, B, and C be the vertices of the triangle as in Figure 4.1.14. Draw a perpendicular line from C to the point D on the hypotenuse, and let p and q be the lengths of BD and DA respectively. Then DBC

and CBA are similar triangles so $\frac{p}{a} = \frac{a}{c}$. This means $a^2 = pc$. In the same way, the similarity of DCA and CBA gives $\frac{q}{b} = \frac{b}{c}$, whence $b^2 = qc$. But then

$$a^2 + b^2 = pc + qc = (p+q)c = c^2$$

because $p + q = c$. This proves Pythagoras' theorem[10]. □

Exercises for 4.1

Exercise 4.1.1 Compute $\|\mathbf{v}\|$ if $\mathbf{v}$ equals:

a. $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ b. $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

c. $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ d. $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

e. $2\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ f. $-3\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

Exercise 4.1.2 Find a unit vector in the direction of:

a. $\begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix}$ b. $\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$

Exercise 4.1.3

a. Find a unit vector in the direction from $\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

b. If $\mathbf{u} \neq \mathbf{0}$, for which values of a is $a\mathbf{u}$ a unit vector?

Exercise 4.1.4 Find the distance between the following pairs of points.

a. $\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ b. $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

c. $\begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$ d. $\begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

Exercise 4.1.5 Use vectors to show that the line joining the midpoints of two sides of a triangle is parallel to the third side and half as long.

Exercise 4.1.6 Let A, B, and C denote the three vertices of a triangle.

a. If E is the midpoint of side BC, show that

$$\overrightarrow{AE} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$$

b. If F is the midpoint of side AC, show that

$$\overrightarrow{FE} = \frac{1}{2}\overrightarrow{AB}$$

Exercise 4.1.7 Determine whether $\mathbf{u}$ and $\mathbf{v}$ are parallel in each of the following cases.

a. $\mathbf{u} = \begin{bmatrix} -3 \\ -6 \\ 3 \end{bmatrix}$; $\mathbf{v} = \begin{bmatrix} 5 \\ 10 \\ -5 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$; $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

c. $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$; $\mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$; $\mathbf{v} = \begin{bmatrix} -8 \\ 0 \\ 4 \end{bmatrix}$

[10]There is an intuitive geometrical proof of Pythagoras' theorem in Example B.3.

Exercise 4.1.8 Let $\mathbf{p}$ and $\mathbf{q}$ be the vectors of points P and Q, respectively, and let R be the point whose vector is $\mathbf{p}+\mathbf{q}$. Express the following in terms of $\mathbf{p}$ and $\mathbf{q}$.

a. $\overrightarrow{QP}$

b. $\overrightarrow{QR}$

c. $\overrightarrow{RP}$

d. $\overrightarrow{RO}$ where O is the origin

Exercise 4.1.9 In each case, find $\overrightarrow{PQ}$ and $\|\overrightarrow{PQ}\|$.

a. $P(1, -1, 3)$, $Q(3, 1, 0)$

b. $P(2, 0, 1)$, $Q(1, -1, 6)$

c. $P(1, 0, 1)$, $Q(1, 0, -3)$

d. $P(1, -1, 2)$, $Q(1, -1, 2)$

e. $P(1, 0, -3)$, $Q(-1, 0, 3)$

f. $P(3, -1, 6)$, $Q(1, 1, 4)$

Exercise 4.1.10 In each case, find a point Q such that $\overrightarrow{PQ}$ has (i) the same direction as $\mathbf{v}$; (ii) the opposite direction to $\mathbf{v}$.

a. $P(-1, 2, 2)$, $\mathbf{v}=\left[\begin{array}{r}1\\3\\1\end{array}\right]$

b. $P(3, 0, -1)$, $\mathbf{v}=\left[\begin{array}{r}2\\-1\\3\end{array}\right]$

Exercise 4.1.11 Let $\mathbf{u}=\left[\begin{array}{r}3\\-1\\0\end{array}\right]$, $\mathbf{v}=\left[\begin{array}{r}4\\0\\1\end{array}\right]$, and $\mathbf{w}=\left[\begin{array}{r}-1\\1\\5\end{array}\right]$. In each case, find $\mathbf{x}$ such that:

a. $3(2\mathbf{u}+\mathbf{x})+\mathbf{w}=2\mathbf{x}-\mathbf{v}$

b. $2(3\mathbf{v}-\mathbf{x})=5\mathbf{w}+\mathbf{u}-3\mathbf{x}$

Exercise 4.1.12 Let $\mathbf{u}=\left[\begin{array}{r}1\\1\\2\end{array}\right]$, $\mathbf{v}=\left[\begin{array}{r}0\\1\\2\end{array}\right]$, and $\mathbf{w}=\left[\begin{array}{r}1\\0\\-1\end{array}\right]$. In each case, find numbers a, b, and c such that $\mathbf{x}=a\mathbf{u}+b\mathbf{v}+c\mathbf{w}$.

a. $\mathbf{x}=\left[\begin{array}{r}2\\-1\\6\end{array}\right]$

b. $\mathbf{x}=\left[\begin{array}{r}1\\3\\0\end{array}\right]$

Exercise 4.1.13 Let $\mathbf{u}=\left[\begin{array}{r}3\\-1\\0\end{array}\right]$, $\mathbf{v}=\left[\begin{array}{r}4\\0\\1\end{array}\right]$, and $\mathbf{z}=\left[\begin{array}{r}1\\1\\1\end{array}\right]$. In each case, show that there are no numbers a, b, and c such that:

a. $a\mathbf{u}+b\mathbf{v}+c\mathbf{z}=\left[\begin{array}{r}1\\2\\1\end{array}\right]$

b. $a\mathbf{u}+b\mathbf{v}+c\mathbf{z}=\left[\begin{array}{r}5\\6\\-1\end{array}\right]$

Exercise 4.1.14 Given $P_1(2, 1, -2)$ and $P_2(1, -2, 0)$. Find the coordinates of the point P:

a. $\frac{1}{5}$ the way from P_1 to P_2

b. $\frac{1}{4}$ the way from P_2 to P_1

Exercise 4.1.15 Find the two points trisecting the segment between $P(2, 3, 5)$ and $Q(8, -6, 2)$.

Exercise 4.1.16 Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two points with vectors $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. If r and s are positive integers, show that the point P lying $\frac{r}{r+s}$ the way from P_1 to P_2 has vector

$$\mathbf{p}=\left(\frac{s}{r+s}\right)\mathbf{p}_1+\left(\frac{r}{r+s}\right)\mathbf{p}_2$$

Exercise 4.1.17 In each case, find the point Q:

a. $\overrightarrow{PQ}=\left[\begin{array}{r}2\\0\\-3\end{array}\right]$ and $P=P(2, -3, 1)$

b. $\overrightarrow{PQ}=\left[\begin{array}{r}-1\\4\\7\end{array}\right]$ and $P=P(1, 3, -4)$

Exercise 4.1.18 Let $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. In each case find $\mathbf{x}$:

a. $2\mathbf{u} - \|\mathbf{v}\|\mathbf{v} = \frac{3}{2}(\mathbf{u} - 2\mathbf{x})$

b. $3\mathbf{u} + 7\mathbf{v} = \|\mathbf{u}\|^2(2\mathbf{x} + \mathbf{v})$

Exercise 4.1.19 Find all vectors $\mathbf{u}$ that are parallel to $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ and satisfy $\|\mathbf{u}\| = 3\|\mathbf{v}\|$.

Exercise 4.1.20 Let P, Q, and R be the vertices of a parallelogram with adjacent sides PQ and PR. In each case, find the other vertex S.

a. $P(3, -1, -1)$, $Q(1, -2, 0)$, $R(1, -1, 2)$

b. $P(2, 0, -1)$, $Q(-2, 4, 1)$, $R(3, -1, 0)$

Exercise 4.1.21 In each case either prove the statement or give an example showing that it is false.

a. The zero vector $\mathbf{0}$ is the only vector of length 0.

b. If $\|\mathbf{v} - \mathbf{w}\| = 0$, then $\mathbf{v} = \mathbf{w}$.

c. If $\mathbf{v} = -\mathbf{v}$, then $\mathbf{v} = \mathbf{0}$.

d. If $\|\mathbf{v}\| = \|\mathbf{w}\|$, then $\mathbf{v} = \mathbf{w}$.

e. If $\|\mathbf{v}\| = \|\mathbf{w}\|$, then $\mathbf{v} = \pm\mathbf{w}$.

f. If $\mathbf{v} = t\mathbf{w}$ for some scalar t, then $\mathbf{v}$ and $\mathbf{w}$ have the same direction.

g. If $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{v} + \mathbf{w}$ are nonzero, and $\mathbf{v}$ and $\mathbf{v} + \mathbf{w}$ parallel, then $\mathbf{v}$ and $\mathbf{w}$ are parallel.

h. $\|-5\mathbf{v}\| = -5\|\mathbf{v}\|$, for all $\mathbf{v}$.

i. If $\|\mathbf{v}\| = \|2\mathbf{v}\|$, then $\mathbf{v} = \mathbf{0}$.

j. $\|\mathbf{v} + \mathbf{w}\| = \|\mathbf{v}\| + \|\mathbf{w}\|$, for all $\mathbf{v}$ and $\mathbf{w}$.

Exercise 4.1.22 Find the vector and parametric equations of the following lines.

a. The line parallel to $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and passing through $P(1, -1, 3)$.

b. The line passing through $P(3, -1, 4)$ and $Q(1, 0, -1)$.

c. The line passing through $P(3, -1, 4)$ and $Q(3, -1, 5)$.

d. The line parallel to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and passing through $P(1, 1, 1)$.

e. The line passing through $P(1, 0, -3)$ and parallel to the line with parametric equations $x = -1 + 2t$, $y = 2 - t$, and $z = 3 + 3t$.

f. The line passing through $P(2, -1, 1)$ and parallel to the line with parametric equations $x = 2 - t$, $y = 1$, and $z = t$.

g. The lines through $P(1, 0, 1)$ that meet the line with vector equation $\mathbf{p} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ at points at distance 3 from $P_0(1, 2, 0)$.

Exercise 4.1.23 In each case, verify that the points P and Q lie on the line.

a. $\begin{array}{l} x = 3 - 4t \\ y = 2 + t \\ z = 1 - t \end{array}$ $P(-1, 3, 0)$, $Q(11, 0, 3)$

b. $\begin{array}{l} x = 4 - t \\ y = 3 \\ z = 1 - 2t \end{array}$ $P(2, 3, -3)$, $Q(-1, 3, -9)$

Exercise 4.1.24 Find the point of intersection (if any) of the following pairs of lines.

a. $\begin{array}{ll} x = 3 + t & x = 4 + 2s \\ y = 1 - 2t & y = 6 + 3s \\ z = 3 + 3t & z = 1 + s \end{array}$

b. $\begin{array}{ll} x = 1 - t & x = 2s \\ y = 2 + 2t & y = 1 + s \\ z = -1 + 3t & z = 3 \end{array}$

c. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + s\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$

d. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ 12 \end{bmatrix} + s \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$

Exercise 4.1.25 Show that if a line passes through the origin, the vectors of points on the line are all scalar multiples of some fixed nonzero vector.

Exercise 4.1.26 Show that every line parallel to the z axis has parametric equations $x = x_0$, $y = y_0$, $z = t$ for some fixed numbers x_0 and y_0.

Exercise 4.1.27 Let $\mathbf{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a vector where a, b, and c are *all* nonzero. Show that the equations of the line through $P_0(x_0, y_0, z_0)$ with direction vector $\mathbf{d}$ can be written in the form

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

This is called the **symmetric form** of the equations.

Exercise 4.1.28 A parallelogram has sides AB, BC, CD, and DA. Given $A(1, -1, 2)$, $C(2, 1, 0)$, and the midpoint $M(1, 0, -3)$ of AB, find $\overrightarrow{BD}$.

Exercise 4.1.29 Find all points C on the line through $A(1, -1, 2)$ and $B = (2, 0, 1)$ such that $\|\overrightarrow{AC}\| = 2\|\overrightarrow{BC}\|$.

Exercise 4.1.30 Let A, B, C, D, E, and F be the vertices of a regular hexagon, taken in order. Show that $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = 3\overrightarrow{AD}$.

Exercise 4.1.31

a. Let P_1, P_2, P_3, P_4, P_5, and P_6 be six points equally spaced on a circle with centre C. Show that

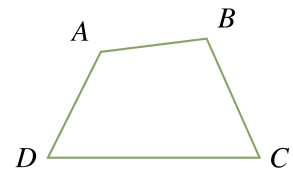
$$\overrightarrow{CP}_1 + \overrightarrow{CP}_2 + \overrightarrow{CP}_3 + \overrightarrow{CP}_4 + \overrightarrow{CP}_5 + \overrightarrow{CP}_6 = \mathbf{0}$$

b. Show that the conclusion in part (a) holds for any *even* set of points evenly spaced on the circle.

c. Show that the conclusion in part (a) holds for *three* points.

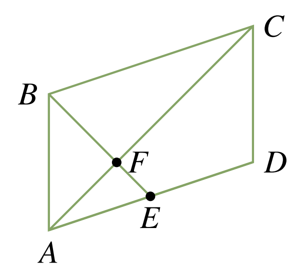
d. Do you think it works for *any* finite set of points evenly spaced around the circle?

Exercise 4.1.32 Consider a quadrilateral with vertices A, B, C, and D in order (as shown in the diagram).



If the diagonals AC and BD bisect each other, show that the quadrilateral is a parallelogram. (This is the converse of Example 4.1.2.) [*Hint*: Let E be the intersection of the diagonals. Show that $\overrightarrow{AB} = \overrightarrow{DC}$ by writing $\overrightarrow{AB} = \overrightarrow{AE} + \overrightarrow{EB}$.]

Exercise 4.1.33 Consider the parallelogram $ABCD$ (see diagram), and let E be the midpoint of side AD.



Show that BE and AC trisect each other; that is, show that the intersection point is one-third of the way from E to B and from A to C. [*Hint*: If F is one-third of the way from A to C, show that $2\overrightarrow{EF} = \overrightarrow{FB}$ and argue as in Example 4.1.2.]

Exercise 4.1.34 The line from a vertex of a triangle to the midpoint of the opposite side is called a **median** of the triangle. If the vertices of a triangle have vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, show that the point on each median that is $\frac{1}{3}$ the way from the midpoint to the vertex has vector $\frac{1}{3}(\mathbf{u} + \mathbf{v} + \mathbf{w})$. Conclude that the point C with vector $\frac{1}{3}(\mathbf{u} + \mathbf{v} + \mathbf{w})$ lies on all three medians. This point C is called the **centroid** of the triangle.

Exercise 4.1.35 Given four noncoplanar points in space, the figure with these points as vertices is called a **tetrahedron**. The line from a vertex through the centroid (see previous exercise) of the triangle formed by the remaining vertices is called a **median** of the tetrahedron. If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{x}$ are the vectors of the four vertices, show that the point on a median one-fourth the way from the centroid to the vertex has vector $\frac{1}{4}(\mathbf{u} + \mathbf{v} + \mathbf{w} + \mathbf{x})$. Conclude that the four medians are concurrent.

4.2 Projections and Planes

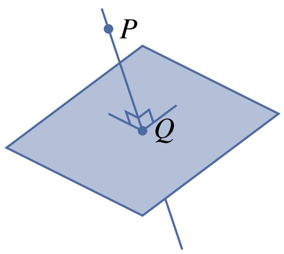


Figure 4.2.1

Any student of geometry soon realizes that the notion of perpendicular lines is fundamental. As an illustration, suppose a point P and a plane are given and it is desired to find the point Q that lies in the plane and is closest to P, as shown in Figure 4.2.1. Clearly, what is required is to find the line through P that is perpendicular to the plane and then to obtain Q as the point of intersection of this line with the plane. Finding the line *perpendicular* to the plane requires a way to determine when two vectors are perpendicular. This can be done using the idea of the dot product of two vectors.

The Dot Product and Angles

Definition 4.4 Dot Product in $\mathbb{R}^3$

Given vectors $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ *and* $\mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, *their* ***dot product*** $\mathbf{v} \cdot \mathbf{w}$ *is a number defined*

$$\mathbf{v} \cdot \mathbf{w} = x_1x_2 + y_1y_2 + z_1z_2 = \mathbf{v}^T\mathbf{w}$$

Because $\mathbf{v} \cdot \mathbf{w}$ is a number, it is sometimes called the **scalar product** of $\mathbf{v}$ and $\mathbf{w}$.[11]

Example 4.2.1

If $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$, then $\mathbf{v} \cdot \mathbf{w} = 2 \cdot 1 + (-1) \cdot 4 + 3 \cdot (-1) = -5$.

The next theorem lists several basic properties of the dot product.

Theorem 4.2.1

Let $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *denote vectors in* $\mathbb{R}^3$ *(or* $\mathbb{R}^2$*).*

1. $\mathbf{v} \cdot \mathbf{w}$ *is a real number.*
2. $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.
3. $\mathbf{v} \cdot \mathbf{0} = 0 = \mathbf{0} \cdot \mathbf{v}$.
4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$.

[11] Similarly, if $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ in $\mathbb{R}^2$, then $\mathbf{v} \cdot \mathbf{w} = x_1x_2 + y_1y_2$.

5. $(k\mathbf{v}) \cdot \mathbf{w} = k(\mathbf{w} \cdot \mathbf{v}) = \mathbf{v} \cdot (k\mathbf{w})$ *for all scalars* k.

6. $\mathbf{u} \cdot (\mathbf{v} \pm \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} \pm \mathbf{u} \cdot \mathbf{w}$

Proof. (1), (2), and (3) are easily verified, and (4) comes from Theorem 4.1.1. The rest are properties of matrix arithmetic (because $\mathbf{w} \cdot \mathbf{v} = \mathbf{v}^T\mathbf{w}$), and are left to the reader. □

The properties in Theorem 4.2.1 enable us to do calculations like

$$3\mathbf{u} \cdot (2\mathbf{v} - 3\mathbf{w} + 4\mathbf{z}) = 6(\mathbf{u} \cdot \mathbf{v}) - 9(\mathbf{u} \cdot \mathbf{w}) + 12(\mathbf{u} \cdot \mathbf{z})$$

and such computations will be used without comment below. Here is an example.

Example 4.2.2

Verify that $\|\mathbf{v} - 3\mathbf{w}\|^2 = 1$ when $\|\mathbf{v}\| = 2$, $\|\mathbf{w}\| = 1$, and $\mathbf{v} \cdot \mathbf{w} = 2$.

Solution. We apply Theorem 4.2.1 several times:

$$\begin{aligned}
\|\mathbf{v} - 3\mathbf{w}\|^2 &= (\mathbf{v} - 3\mathbf{w}) \cdot (\mathbf{v} - 3\mathbf{w}) \\
&= \mathbf{v} \cdot (\mathbf{v} - 3\mathbf{w}) - 3\mathbf{w} \cdot (\mathbf{v} - 3\mathbf{w}) \\
&= \mathbf{v} \cdot \mathbf{v} - 3(\mathbf{v} \cdot \mathbf{w}) - 3(\mathbf{w} \cdot \mathbf{v}) + 9(\mathbf{w} \cdot \mathbf{w}) \\
&= \|\mathbf{v}\|^2 - 6(\mathbf{v} \cdot \mathbf{w}) + 9\|\mathbf{w}\|^2 \\
&= 4 - 12 + 9 = 1
\end{aligned}$$

There is an intrinsic description of the dot product of two nonzero vectors in $\mathbb{R}^3$. To understand it we require the following result from trigonometry.

Law of Cosines

If a triangle has sides a, b, *and* c, *and if* θ *is the interior angle opposite* c *then*

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

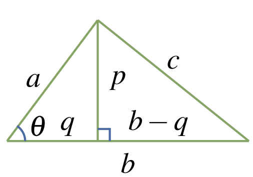


Figure 4.2.2

Proof. We prove it when is θ acute, that is $0 \leq \theta < \frac{\pi}{2}$; the obtuse case is similar. In Figure 4.2.2 we have $p = a\sin\theta$ and $q = a\cos\theta$. Hence Pythagoras' theorem gives

$$\begin{aligned}
c^2 = p^2 + (b-q)^2 &= a^2\sin^2\theta + (b - a\cos\theta)^2 \\
&= a^2(\sin^2\theta + \cos^2\theta) + b^2 - 2ab\cos\theta
\end{aligned}$$

The law of cosines follows because $\sin^2\theta + \cos^2\theta = 1$ for any angle θ. □

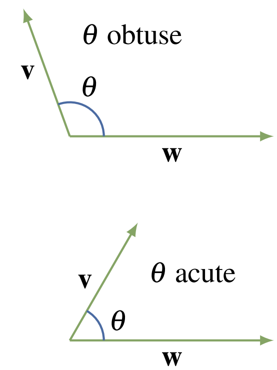


Figure 4.2.3

Note that the law of cosines reduces to Pythagoras' theorem if θ is a right angle (because $\cos\frac{\pi}{2}=0$).

Now let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors positioned with a common tail as in Figure 4.2.3. Then they determine a unique angle θ in the range

$$0 \leq \theta \leq \pi$$

This angle θ will be called the **angle between** $\mathbf{v}$ and $\mathbf{w}$. Figure 4.2.3 illustrates when θ is acute (less than $\frac{\pi}{2}$) and obtuse (greater than $\frac{\pi}{2}$). Clearly $\mathbf{v}$ and $\mathbf{w}$ are parallel if θ is either 0 or π. Note that we do not define the angle between $\mathbf{v}$ and $\mathbf{w}$ if one of these vectors is $\mathbf{0}$.

The next result gives an easy way to compute the angle between two nonzero vectors using the dot product.

Theorem 4.2.2

Let $\mathbf{v}$ *and* $\mathbf{w}$ *be nonzero vectors. If* θ *is the angle between* $\mathbf{v}$ *and* $\mathbf{w}$*, then*

$$\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$$

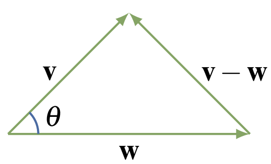


Figure 4.2.4

Proof. We calculate $\|\mathbf{v}-\mathbf{w}\|^2$ in two ways. First apply the law of cosines to the triangle in Figure 4.2.4 to obtain:

$$\|\mathbf{v}-\mathbf{w}\|^2 = \|\mathbf{v}\|^2+\|\mathbf{w}\|^2-2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$$

On the other hand, we use Theorem 4.2.1:

$$\begin{aligned}\|\mathbf{v}-\mathbf{w}\|^2 &= (\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}-\mathbf{w})\\ &= \mathbf{v}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{w}-\mathbf{w}\cdot\mathbf{v}+\mathbf{w}\cdot\mathbf{w}\\ &= \|\mathbf{v}\|^2-2(\mathbf{v}\cdot\mathbf{w})+\|\mathbf{w}\|^2\end{aligned}$$

Comparing these we see that $-2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta = -2(\mathbf{v}\cdot\mathbf{w})$, and the result follows. □

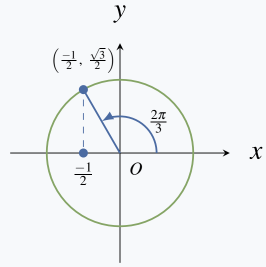
If $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors, Theorem 4.2.2 gives an intrinsic description of $\mathbf{v}\cdot\mathbf{w}$ because $\|\mathbf{v}\|$, $\|\mathbf{w}\|$, and the angle θ between $\mathbf{v}$ and $\mathbf{w}$ do not depend on the choice of coordinate system. Moreover, since $\|\mathbf{v}\|$ and $\|\mathbf{w}\|$ are nonzero ($\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors), it gives a formula for the cosine of the angle θ:

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} \tag{4.1}$$

Since $0 \leq \theta \leq \pi$, this can be used to find θ.

Example 4.2.3

Compute the angle between $\mathbf{u} = \left[\begin{array}{r} -1 \\ 1 \\ 2 \end{array}\right]$ and $\mathbf{v} = \left[\begin{array}{r} 2 \\ 1 \\ -1 \end{array}\right]$.



Solution. Compute $\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{-2+1-2}{\sqrt{6}\sqrt{6}} = -\frac{1}{2}$. Now recall that $\cos\theta$ and $\sin\theta$ are defined so that $(\cos\theta, \sin\theta)$ is the point on the unit circle determined by the angle θ (drawn counterclockwise, starting from the positive x axis). In the present case, we know that $\cos\theta = -\frac{1}{2}$ and that $0 \leq \theta \leq \pi$. Because $\cos\frac{\pi}{3} = \frac{1}{2}$, it follows that $\theta = \frac{2\pi}{3}$ (see the diagram).

If $\mathbf{v}$ and $\mathbf{w}$ are nonzero, equation (4.1) shows that $\cos\theta$ has the same sign as $\mathbf{v}\cdot\mathbf{w}$, so

$$\begin{array}{lll} \mathbf{v}\cdot\mathbf{w} > 0 & \text{if and only if} & \theta \text{ is acute } (0 \leq \theta < \frac{\pi}{2}) \\ \mathbf{v}\cdot\mathbf{w} < 0 & \text{if and only if} & \theta \text{ is obtuse } (\frac{\pi}{2} < \theta \leq 0) \\ \mathbf{v}\cdot\mathbf{w} = 0 & \text{if and only if} & \theta = \frac{\pi}{2} \end{array}$$

In this last case, the (nonzero) vectors are perpendicular. The following terminology is used in linear algebra:

Definition 4.5 Orthogonal Vectors in $\mathbb{R}^3$

*Two vectors $\mathbf{v}$ and $\mathbf{w}$ are said to be **orthogonal** if $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$ or the angle between them is $\frac{\pi}{2}$.*

Since $\mathbf{v}\cdot\mathbf{w} = 0$ if either $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$, we have the following theorem:

Theorem 4.2.3

Two vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if and only if $\mathbf{v}\cdot\mathbf{w} = 0$.

Example 4.2.4

Show that the points $P(3,\ -1,\ 1)$, $Q(4,\ 1,\ 4)$, and $R(6,\ 0,\ 4)$ are the vertices of a right triangle.

Solution. The vectors along the sides of the triangle are

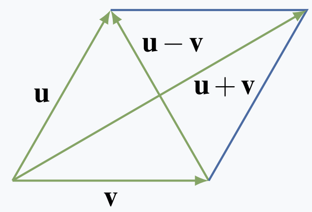
$$\overrightarrow{PQ} = \left[\begin{array}{r} 1 \\ 2 \\ 3 \end{array}\right],\ \overrightarrow{PR} = \left[\begin{array}{r} 3 \\ 1 \\ 3 \end{array}\right],\ \text{and } \overrightarrow{QR} = \left[\begin{array}{r} 2 \\ -1 \\ 0 \end{array}\right]$$

Evidently $\overrightarrow{PQ}\cdot\overrightarrow{QR} = 2-2+0 = 0$, so $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are orthogonal vectors. This means sides PQ and QR are perpendicular—that is, the angle at Q is a right angle.

Example 4.2.5 demonstrates how the dot product can be used to verify geometrical theorems involving perpendicular lines.

Example 4.2.5

A parallelogram with sides of equal length is called a **rhombus**. Show that the diagonals of a rhombus are perpendicular.



Solution. Let $\mathbf{u}$ and $\mathbf{v}$ denote vectors along two adjacent sides of a rhombus, as shown in the diagram. Then the diagonals are $\mathbf{u}-\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$, and we compute

$$\begin{aligned}(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) &= \mathbf{u}\cdot(\mathbf{u}+\mathbf{v})-\mathbf{v}\cdot(\mathbf{u}+\mathbf{v}) \\ &= \mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v} \\ &= \|\mathbf{u}\|^2-\|\mathbf{v}\|^2 \\ &= 0\end{aligned}$$

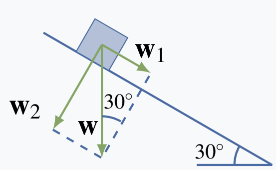
because $\|\mathbf{u}\| = \|\mathbf{v}\|$ (it is a rhombus). Hence $\mathbf{u}-\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ are orthogonal.

Projections

In applications of vectors, it is frequently useful to write a vector as the sum of two orthogonal vectors. Here is an example.

Example 4.2.6

Suppose a ten-kilogram block is placed on a flat surface inclined 30° to the horizontal as in the diagram. Neglecting friction, how much force is required to keep the block from sliding down the surface?



Solution. Let $\mathbf{w}$ denote the weight (force due to gravity) exerted on the block. Then $\|\mathbf{w}\| = 10$ kilograms and the direction of $\mathbf{w}$ is vertically down as in the diagram. The idea is to write $\mathbf{w}$ as a sum $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1$ is parallel to the inclined surface and $\mathbf{w}_2$ is perpendicular to the surface. Since there is no friction, the force required is $-\mathbf{w}_1$ because the force $\mathbf{w}_2$ has no effect parallel to the surface. As the angle between $\mathbf{w}$ and $\mathbf{w}_2$ is 30° in the diagram, we have $\frac{\|\mathbf{w}_1\|}{\|\mathbf{w}\|} = \sin 30° = \frac{1}{2}$. Hence $\|\mathbf{w}_1\| = \frac{1}{2}\|\mathbf{w}\| = \frac{1}{2}10 = 5$. Thus the required force has a magnitude of 5 kilograms weight directed up the surface.

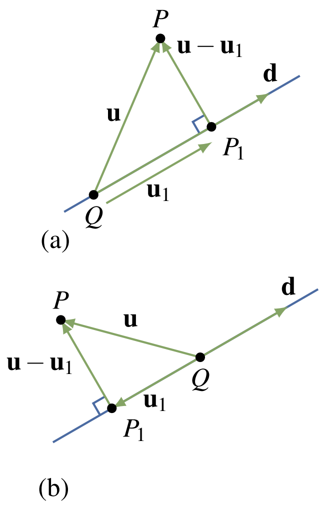


Figure 4.2.5

If a nonzero vector $\mathbf{d}$ is specified, the key idea in Example 4.2.6 is to be able to write an arbitrary vector $\mathbf{u}$ as a sum of two vectors,

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$$

where $\mathbf{u}_1$ is parallel to $\mathbf{d}$ and $\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1$ is orthogonal to $\mathbf{d}$. Suppose that $\mathbf{u}$ and $\mathbf{d} \neq \mathbf{0}$ emanate from a common tail Q (see Figure 4.2.5). Let P be the tip of $\mathbf{u}$, and let P_1 denote the foot of the perpendicular from P to the line through Q parallel to $\mathbf{d}$.

Then $\mathbf{u}_1 = \overrightarrow{QP}_1$ has the required properties:

1. $\mathbf{u}_1$ *is parallel to* $\mathbf{d}$.
2. $\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1$ *is orthogonal to* $\mathbf{d}$.
3. $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$.

Definition 4.6 Projection in $\mathbb{R}^3$

*The vector $\mathbf{u}_1 = \overrightarrow{QP}_1$ in Figure 4.2.5 is called **the projection** of $\mathbf{u}$ on $\mathbf{d}$. It is denoted*

$$\mathbf{u}_1 = \operatorname{proj}_{\mathbf{d}} \mathbf{u}$$

In Figure 4.2.5(a) the vector $\mathbf{u}_1 = \operatorname{proj}_{\mathbf{d}} \mathbf{u}$ has the same direction as $\mathbf{d}$; however, $\mathbf{u}_1$ and $\mathbf{d}$ have opposite directions if the angle between $\mathbf{u}$ and $\mathbf{d}$ is greater than $\frac{\pi}{2}$ (Figure 4.2.5(b)). Note that the projection $\mathbf{u}_1 = \operatorname{proj}_{\mathbf{d}} \mathbf{u}$ is zero if and only if $\mathbf{u}$ and $\mathbf{d}$ are orthogonal.

Calculating the projection of $\mathbf{u}$ on $\mathbf{d} \neq \mathbf{0}$ is remarkably easy.

Theorem 4.2.4

Let $\mathbf{u}$ and $\mathbf{d} \neq \mathbf{0}$ be vectors.

1. *The projection of $\mathbf{u}$ on $\mathbf{d}$ is given by $\operatorname{proj}_{\mathbf{d}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{d}}{\|\mathbf{d}\|^2} \mathbf{d}$.*
2. *The vector $\mathbf{u} - \operatorname{proj}_{\mathbf{d}} \mathbf{u}$ is orthogonal to $\mathbf{d}$.*

Proof. The vector $\mathbf{u}_1 = \operatorname{proj}_{\mathbf{d}} \mathbf{u}$ is parallel to $\mathbf{d}$ and so has the form $\mathbf{u}_1 = t\mathbf{d}$ for some scalar t. The requirement that $\mathbf{u} - \mathbf{u}_1$ and $\mathbf{d}$ are orthogonal determines t. In fact, it means that $(\mathbf{u} - \mathbf{u}_1) \cdot \mathbf{d} = 0$ by Theorem 4.2.3. If $\mathbf{u}_1 = t\mathbf{d}$ is substituted here, the condition is

$$0 = (\mathbf{u} - t\mathbf{d}) \cdot \mathbf{d} = \mathbf{u} \cdot \mathbf{d} - t(\mathbf{d} \cdot \mathbf{d}) = \mathbf{u} \cdot \mathbf{d} - t\|\mathbf{d}\|^2$$

It follows that $t = \frac{\mathbf{u} \cdot \mathbf{d}}{\|\mathbf{d}\|^2}$, where the assumption that $\mathbf{d} \neq \mathbf{0}$ guarantees that $\|\mathbf{d}\|^2 \neq 0$. □

Example 4.2.7

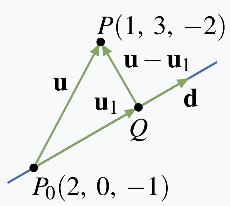
Find the projection of $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ on $\mathbf{d} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ and express $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ where $\mathbf{u}_1$ is parallel to $\mathbf{d}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{d}$.

Solution. The projection $\mathbf{u}_1$ of $\mathbf{u}$ on $\mathbf{d}$ is

$$\mathbf{u}_1 = \text{proj}_{\mathbf{d}}\, \mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d} = \frac{2+3+3}{1^2+(-1)^2+3^2}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \frac{8}{11}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

Hence $\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1 = \frac{1}{11}\begin{bmatrix} 14 \\ -25 \\ -13 \end{bmatrix}$, and this is orthogonal to $\mathbf{d}$ by Theorem 4.2.4 (alternatively, observe that $\mathbf{d}\cdot\mathbf{u}_2 = 0$). Since $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, we are done.

Example 4.2.8



Find the shortest distance (see diagram) from the point $P(1,\ 3,\ -2)$ to the line through $P_0(2,\ 0,\ -1)$ with direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Also find the point Q that lies on the line and is closest to P.

Solution. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$ denote the vector from P_0 to P, and let $\mathbf{u}_1$ denote the projection of $\mathbf{u}$ on $\mathbf{d}$. Thus

$$\mathbf{u}_1 = \frac{\mathbf{u}\cdot\mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d} = \frac{-1-3+0}{1^2+(-1)^2+0^2}\mathbf{d} = -2\mathbf{d} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$$

by Theorem 4.2.4. We see geometrically that the point Q on the line is closest to P, so the distance is

$$\|\overrightarrow{QP}\| = \|\mathbf{u} - \mathbf{u}_1\| = \left\| \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\| = \sqrt{3}$$

To find the coordinates of Q, let $\mathbf{p}_0$ and $\mathbf{q}$ denote the vectors of P_0 and Q, respectively. Then $\mathbf{p}_0 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ and $\mathbf{q} = \mathbf{p}_0 + \mathbf{u}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$. Hence $Q(0,\ 2,\ -1)$ is the required point. It can be checked that the distance from Q to P is $\sqrt{3}$, as expected.

Planes

It is evident geometrically that among all planes that are perpendicular to a given straight line there is exactly one containing any given point. This fact can be used to give a very simple description of a plane. To do this, it is necessary to introduce the following notion:

Definition 4.7 Normal Vector in a Plane

A nonzero vector $\mathbf{n}$ *is called a* ***normal*** *for a plane if it is orthogonal to every vector in the plane.*

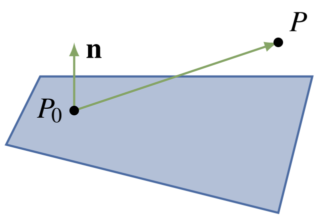


Figure 4.2.6

For example, the coordinate vector $\mathbf{k}$ is a normal for the x-y plane.

Given a point $P_0 = P_0(x_0,\ y_0,\ z_0)$ and a nonzero vector $\mathbf{n}$, there is a unique plane through P_0 with normal $\mathbf{n}$, shaded in Figure 4.2.6. A point $P = P(x,\ y,\ z)$ lies on this plane if and only if the vector $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$—that is, if and only if $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$. Because $\overrightarrow{P_0P} = \left[\begin{array}{c} x - x_0 \\ y - y_0 \\ z - z_0 \end{array} \right]$ this gives the following result:

Scalar Equation of a Plane

The plane through $P_0(x_0,\ y_0,\ z_0)$ *with normal* $\mathbf{n} = \left[\begin{array}{c} a \\ b \\ c \end{array} \right] \neq \mathbf{0}$ *as a normal vector is given by*

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

In other words, a point $P(x,\ y,\ z)$ *is on this plane if and only if* x, y, *and* z *satisfy this equation.*

Example 4.2.9

Find an equation of the plane through $P_0(1,\ -1,\ 3)$ with $\mathbf{n} = \left[\begin{array}{r} 3 \\ -1 \\ 2 \end{array} \right]$ as normal.

Solution. Here the general scalar equation becomes

$$3(x - 1) - (y + 1) + 2(z - 3) = 0$$

This simplifies to $3x - y + 2z = 10$.

If we write $d = ax_0 + by_0 + cz_0$, the scalar equation shows that every plane with normal $\mathbf{n} = \left[\begin{array}{c} a \\ b \\ c \end{array} \right]$ has a linear equation of the form

$$ax+by+cz=d \tag{4.2}$$

for some constant d. Conversely, the graph of this equation is a plane with $\mathbf{n}=\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ as a normal vector (assuming that a, b, and c are not all zero).

Example 4.2.10

Find an equation of the plane through $P_0(3,\ -1,\ 2)$ that is parallel to the plane with equation $2x-3y=6$.

Solution. The plane with equation $2x-3y=6$ has normal $\mathbf{n}=\begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$. Because the two planes are parallel, $\mathbf{n}$ serves as a normal for the plane we seek, so the equation is $2x-3y=d$ for some d by Equation 4.2. Insisting that $P_0(3,\ -1,\ 2)$ lies on the plane determines d; that is, $d=2\cdot 3-3(-1)=9$. Hence, the equation is $2x-3y=9$.

Consider points $P_0(x_0,\ y_0,\ z_0)$ and $P(x,\ y,\ z)$ with vectors $\mathbf{p}_0=\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ and $\mathbf{p}=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Given a nonzero vector $\mathbf{n}$, the scalar equation of the plane through $P_0(x_0,\ y_0,\ z_0)$ with normal $\mathbf{n}=\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ takes the vector form:

Vector Equation of a Plane

The plane with normal $\mathbf{n}\neq\mathbf{0}$ through the point with vector $\mathbf{p}_0$ is given by

$$\mathbf{n}\cdot(\mathbf{p}-\mathbf{p}_0)=0$$

In other words, the point with vector $\mathbf{p}$ is on the plane if and only if $\mathbf{p}$ satisfies this condition.

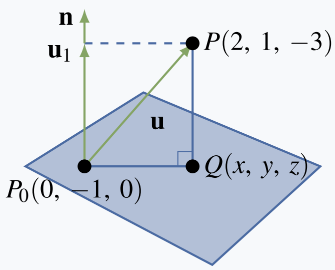
Moreover, Equation 4.2 translates as follows:

Every plane with normal $\mathbf{n}$ has vector equation $\mathbf{n}\cdot\mathbf{p}=d$ for some number d.

This is useful in the second solution of Example 4.2.11.

Example 4.2.11

Find the shortest distance from the point $P(2,\ 1,\ -3)$ to the plane with equation $3x-y+4z=1$. Also find the point Q on this plane closest to P.



Solution 1. The plane in question has normal $\mathbf{n} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$. Choose any point P_0 on the plane—say $P_0(0,\ -1,\ 0)$—and let $Q(x,\ y,\ z)$ be the point on the plane closest to P (see the diagram). The vector from P_0 to P is $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$. Now erect $\mathbf{n}$ with its tail at P_0. Then $\overrightarrow{QP} = \mathbf{u}_1$ and $\mathbf{u}_1$ is the projection of $\mathbf{u}$ on $\mathbf{n}$:

$$\mathbf{u}_1 = \frac{\mathbf{n}\cdot\mathbf{u}}{\|\mathbf{n}\|^2}\mathbf{n} = \frac{-8}{26}\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = \frac{-4}{13}\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$$

Hence the distance is $\|\overrightarrow{QP}\| = \|\mathbf{u}_1\| = \frac{4\sqrt{26}}{13}$. To calculate the point Q, let $\mathbf{q} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{p}_0 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ be the vectors of Q and P_0. Then

$$\mathbf{q} = \mathbf{p}_0 + \mathbf{u} - \mathbf{u}_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} + \frac{4}{13}\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{38}{13} \\ \frac{9}{13} \\ \frac{-23}{13} \end{bmatrix}$$

This gives the coordinates of $Q(\frac{38}{13},\ \frac{9}{13},\ \frac{-23}{13})$.

Solution 2. Let $\mathbf{q} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ be the vectors of Q and P. Then Q is on the line through P with direction vector $\mathbf{n}$, so $\mathbf{q} = \mathbf{p} + t\mathbf{n}$ for some scalar t. In addition, Q lies on the plane, so $\mathbf{n}\cdot\mathbf{q} = 1$. This determines t:

$$1 = \mathbf{n}\cdot\mathbf{q} = \mathbf{n}\cdot(\mathbf{p} + t\mathbf{n}) = \mathbf{n}\cdot\mathbf{p} + t\|\mathbf{n}\|^2 = -7 + t(26)$$

This gives $t = \frac{8}{26} = \frac{4}{13}$, so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{q} = \mathbf{p} + t\mathbf{n} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + \frac{4}{13}\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + \frac{1}{13}\begin{bmatrix} 38 \\ 9 \\ -23 \end{bmatrix}$$

as before. This determines Q (in the diagram), and the reader can verify that the required distance is $\|\overrightarrow{QP}\| = \frac{4}{13}\sqrt{26}$, as before.

The Cross Product

If P, Q, and R are three distinct points in $\mathbb{R}^3$ that are not all on some line, it is clear geometrically that there is a unique plane containing all three. The vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ both lie in this plane, so finding a normal amounts to finding a nonzero vector orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. The cross product provides a systematic way to do this.

Definition 4.8 Cross Product

Given vectors $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, *define the* ***cross product*** $\mathbf{v}_1 \times \mathbf{v}_2$ *by*

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

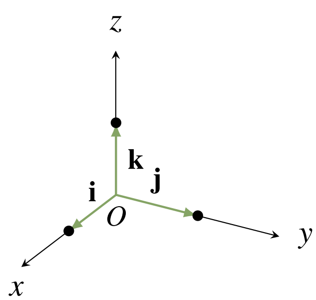


Figure 4.2.7

(Because it is a vector, $\mathbf{v}_1 \times \mathbf{v}_2$ is often called the **vector product**.) There is an easy way to remember this definition using the **coordinate vectors**:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and } \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

They are vectors of length 1 pointing along the positive x, y, and z axes, respectively, as in Figure 4.2.7. The reason for the name is that any vector can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

With this, the cross product can be described as follows:

Determinant Form of the Cross Product

If $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ *are two vectors, then*

$$\mathbf{v}_1 \times \mathbf{v}_2 = \det \begin{bmatrix} \mathbf{i} & x_1 & x_2 \\ \mathbf{j} & y_1 & y_2 \\ \mathbf{k} & z_1 & z_2 \end{bmatrix} = \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \mathbf{k}$$

where the determinant is expanded along the first column.

Example 4.2.12

If $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$, then

$$\begin{aligned} \mathbf{v}_1 \times \mathbf{v}_2 = \det \begin{bmatrix} \mathbf{i} & 2 & 1 \\ \mathbf{j} & -1 & 3 \\ \mathbf{k} & 4 & 7 \end{bmatrix} &= \begin{vmatrix} -1 & 3 \\ 4 & 7 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 4 & 7 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} \mathbf{k} \\ &= -19\mathbf{i} - 10\mathbf{j} + 7\mathbf{k} \\ &= \begin{bmatrix} -19 \\ -10 \\ 7 \end{bmatrix} \end{aligned}$$

Observe that $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$ in Example 4.2.12. This holds in general as can be verified directly by computing $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w})$ and $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w})$, and is recorded as the first part of the following theorem. It will follow from a more general result which, together with the second part, will be proved in Section 4.3 where a more detailed study of the cross product will be undertaken.

Theorem 4.2.5

Let $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^3$.

1. *$\mathbf{v} \times \mathbf{w}$ is a vector orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.*
2. *If $\mathbf{v}$ and $\mathbf{w}$ are nonzero, then $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ if and only if $\mathbf{v}$ and $\mathbf{w}$ are parallel.*

It is interesting to contrast Theorem 4.2.5(2) with the assertion (in Theorem 4.2.3) that

$$\mathbf{v} \cdot \mathbf{w} = 0 \quad \text{if and only if } \mathbf{v} \text{ and } \mathbf{w} \text{ are orthogonal.}$$

Example 4.2.13

Find the equation of the plane through $P(1, 3, -2)$, $Q(1, 1, 5)$, and $R(2, -2, 3)$.

Solution. The vectors $\overrightarrow{PQ} = \begin{bmatrix} 0 \\ -2 \\ 7 \end{bmatrix}$ and $\overrightarrow{PR} = \begin{bmatrix} 1 \\ -5 \\ 5 \end{bmatrix}$ lie in the plane, so

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \det \begin{bmatrix} \mathbf{i} & 0 & 1 \\ \mathbf{j} & -2 & -5 \\ \mathbf{k} & 7 & 5 \end{bmatrix} = 25\mathbf{i} + 7\mathbf{j} + 2\mathbf{k} = \begin{bmatrix} 25 \\ 7 \\ 2 \end{bmatrix}$$

is a normal for the plane (being orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$). Hence the plane has equation

$$25x + 7y + 2z = d \quad \text{for some number } d.$$

Since $P(1,\ 3,\ -2)$ lies in the plane we have $25 \cdot 1 + 7 \cdot 3 + 2(-2) = d$. Hence $d = 42$ and the equation is $25x + 7y + 2z = 42$. Incidentally, the same equation is obtained (verify) if $\overrightarrow{QP}$ and $\overrightarrow{QR}$, or $\overrightarrow{RP}$ and $\overrightarrow{RQ}$, are used as the vectors in the plane.

Example 4.2.14

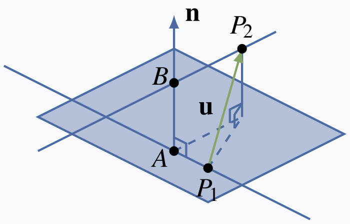
Find the shortest distance between the nonparallel lines

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Then find the points A and B on the lines that are closest together.

Solution. Direction vectors for the two lines are $\mathbf{d}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{d}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, so

$$\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2 = \det \begin{bmatrix} \mathbf{i} & 2 & 1 \\ \mathbf{j} & 0 & 1 \\ \mathbf{k} & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$$



is perpendicular to both lines. Consider the plane shaded in the diagram containing the first line with $\mathbf{n}$ as normal. This plane contains $P_1(1,\ 0,\ -1)$ and is parallel to the second line. Because $P_2(3,\ 1,\ 0)$ is on the second line, the distance in question is just the shortest distance between $P_2(3,\ 1,\ 0)$ and this plane. The vector $\mathbf{u}$ from P_1 to P_2 is $\mathbf{u} = \overrightarrow{P_1P_2} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and so, as in Example 4.2.11, the distance is the length of the projection of $\mathbf{u}$ on $\mathbf{n}$.

$$\text{distance} = \left\| \frac{\mathbf{u}\cdot\mathbf{n}}{\|\mathbf{n}\|^2}\mathbf{n} \right\| = \frac{|\mathbf{u}\cdot\mathbf{n}|}{\|\mathbf{n}\|} = \frac{3}{\sqrt{14}} = \frac{3\sqrt{14}}{14}$$

Note that it is necessary that $\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2$ be nonzero for this calculation to be possible. As is shown later (Theorem 4.3.4), this is guaranteed by the fact that $\mathbf{d}_1$ and $\mathbf{d}_2$ are *not* parallel.
The points A and B have coordinates $A(1+2t,\ 0,\ t-1)$ and $B(3+s,\ 1+s,\ -s)$ for some s and t, so $\overrightarrow{AB} = \begin{bmatrix} 2+s-2t \\ 1+s \\ 1-s-t \end{bmatrix}$. This vector is orthogonal to both $\mathbf{d}_1$ and $\mathbf{d}_2$, and the conditions $\overrightarrow{AB}\cdot\mathbf{d}_1 = 0$ and $\overrightarrow{AB}\cdot\mathbf{d}_2 = 0$ give equations $5t - s = 5$ and $t - 3s = 2$. The solution is $s = \frac{-5}{14}$ and $t = \frac{13}{14}$, so the points are $A(\frac{40}{14},\ 0,\ \frac{-1}{14})$ and $B(\frac{37}{14},\ \frac{9}{14},\ \frac{5}{14})$. We have $\|\overrightarrow{AB}\| = \frac{3\sqrt{14}}{14}$, as before.

Exercises for 4.2

Exercise 4.2.1 Compute $\mathbf{u} \cdot \mathbf{v}$ where:

a. $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v} = \mathbf{u}$

c. $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 6 \\ -7 \\ -5 \end{bmatrix}$

e. $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

f. $\mathbf{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \mathbf{v} = \mathbf{0}$

Exercise 4.2.2 Find the angle between the following pairs of vectors.

a. $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -6 \\ 2 \\ 0 \end{bmatrix}$

c. $\mathbf{u} = \begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$

e. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

f. $\mathbf{u} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 5\sqrt{2} \\ -7 \\ -1 \end{bmatrix}$

Exercise 4.2.3 Find all real numbers x such that:

a. $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ -2 \\ 1 \end{bmatrix}$ are orthogonal.

b. $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ x \\ 2 \end{bmatrix}$ are at an angle of $\frac{\pi}{3}$.

Exercise 4.2.4 Find all vectors $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ orthogonal to both:

a. $\mathbf{u}_1 = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

b. $\mathbf{u}_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

c. $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix}$

d. $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Exercise 4.2.5 Find two orthogonal vectors that are both orthogonal to $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

Exercise 4.2.6 Consider the triangle with vertices $P(2, 0, -3)$, $Q(5, -2, 1)$, and $R(7, 5, 3)$.

a. Show that it is a right-angled triangle.

b. Find the lengths of the three sides and verify the Pythagorean theorem.

Exercise 4.2.7 Show that the triangle with vertices $A(4,\ -7,\ 9)$, $B(6,\ 4,\ 4)$, and $C(7,\ 10,\ -6)$ is not a right-angled triangle.

Exercise 4.2.8 Find the three internal angles of the triangle with vertices:

a. $A(3,\ 1,\ -2)$, $B(3,\ 0,\ -1)$, and $C(5,\ 2,\ -1)$

b. $A(3,\ 1,\ -2)$, $B(5,\ 2,\ -1)$, and $C(4,\ 3,\ -3)$

Exercise 4.2.9 Show that the line through $P_0(3,\ 1,\ 4)$ and $P_1(2,\ 1,\ 3)$ is perpendicular to the line through $P_2(1,\ -1,\ 2)$ and $P_3(0,\ 5,\ 3)$.

Exercise 4.2.10 In each case, compute the projection of **u** on **v**.

a. $\mathbf{u} = \begin{bmatrix} 5 \\ 7 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$

c. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -6 \\ 4 \\ 2 \end{bmatrix}$

Exercise 4.2.11 In each case, write $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to **v** and $\mathbf{u}_2$ is orthogonal to **v**.

a. $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$

c. $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -6 \\ 4 \\ -1 \end{bmatrix}$

Exercise 4.2.12 Calculate the distance from the point P to the line in each case and find the point Q on the line closest to P.

a. $P(3,\ 2-1)$

line: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$

b. $P(1,\ -1,\ 3)$

line: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$

Exercise 4.2.13 Compute $\mathbf{u} \times \mathbf{v}$ where:

a. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -6 \\ 2 \\ 0 \end{bmatrix}$

c. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$

Exercise 4.2.14 Find an equation of each of the following planes.

a. Passing through $A(2,\ 1,\ 3)$, $B(3,\ -1,\ 5)$, and $C(1,\ 2,\ -3)$.

b. Passing through $A(1,\ -1,\ 6)$, $B(0,\ 0,\ 1)$, and $C(4,\ 7,\ -11)$.

c. Passing through $P(2,\ -3,\ 5)$ and parallel to the plane with equation $3x - 2y - z = 0$.

d. Passing through $P(3,\ 0,\ -1)$ and parallel to the plane with equation $2x - y + z = 3$.

e. Containing $P(3,\ 0,\ -1)$ and the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

f. Containing $P(2, 1, 0)$ and the line
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

g. Containing the lines
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

h. Containing the lines $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

i. Each point of which is equidistant from $P(2, -1, 3)$ and $Q(1, 1, -1)$.

j. Each point of which is equidistant from $P(0, 1, -1)$ and $Q(2, -1, -3)$.

Exercise 4.2.15 In each case, find a vector equation of the line.

a. Passing through $P(3, -1, 4)$ and perpendicular to the plane $3x - 2y - z = 0$.

b. Passing through $P(2, -1, 3)$ and perpendicular to the plane $2x + y = 1$.

c. Passing through $P(0, 0, 0)$ and perpendicular to the lines $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$.

d. Passing through $P(1, 1, -1)$, and perpendicular to the lines
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \text{ and}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ -2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}.$$

e. Passing through $P(2, 1, -1)$, intersecting the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, and perpendicular to that line.

f. Passing through $P(1, 1, 2)$, intersecting the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and perpendicular to that line.

Exercise 4.2.16 In each case, find the shortest distance from the point P to the plane and find the point Q on the plane closest to P.

a. $P(2, 3, 0)$; plane with equation $5x + y + z = 1$.

b. $P(3, 1, -1)$; plane with equation $2x + y - z = 6$.

Exercise 4.2.17

a. Does the line through $P(1, 2, -3)$ with direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ lie in the plane $2x - y - z = 3$? Explain.

b. Does the plane through $P(4, 0, 5)$, $Q(2, 2, 1)$, and $R(1, -1, 2)$ pass through the origin? Explain.

Exercise 4.2.18 Show that every plane containing $P(1, 2, -1)$ and $Q(2, 0, 1)$ must also contain $R(-1, 6, -5)$.

Exercise 4.2.19 Find the equations of the line of intersection of the following planes.

a. $2x - 3y + 2z = 5$ and $x + 2y - z = 4$.

b. $3x + y - 2z = 1$ and $x + y + z = 5$.

Exercise 4.2.20 In each case, find all points of intersection of the given plane and the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$.

a. $x - 3y + 2z = 4$

b. $2x - y - z = 5$

c. $3x - y + z = 8$

d. $-x - 4y - 3z = 6$

Exercise 4.2.21 Find the equation of *all* planes:

a. Perpendicular to the line
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

b. Perpendicular to the line
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}.$$

c. Containing the origin.

d. Containing $P(3, 2, -4)$.

e. Containing $P(1, 1, -1)$ and $Q(0, 1, 1)$.

f. Containing $P(2, -1, 1)$ and $Q(1, 0, 0)$.

g. Containing the line
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

h. Containing the line
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}.$$

Exercise 4.2.22 If a plane contains two distinct points P_1 and P_2, show that it contains every point on the line through P_1 and P_2.

Exercise 4.2.23 Find the shortest distance between the following pairs of parallel lines.

a. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix};$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

b. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix};$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Exercise 4.2.24 Find the shortest distance between the following pairs of nonparallel lines and find the points on the lines that are closest together.

a. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix};$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

b. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix};$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

c. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix};$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

d. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix};$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Exercise 4.2.25 Show that two lines in the plane with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$. [*Hint*: Example 4.1.11.]

Exercise 4.2.26

a. Show that, of the four diagonals of a cube, no pair is perpendicular.

b. Show that each diagonal is perpendicular to the face diagonals it does not meet.

Exercise 4.2.27 Given a rectangular solid with sides of lengths 1, 1, and $\sqrt{2}$, find the angle between a diagonal and one of the longest sides.

Exercise 4.2.28 Consider a rectangular solid with sides of lengths a, b, and c. Show that it has two orthogonal diagonals if and only if the sum of two of a^2, b^2, and c^2 equals the third.

Exercise 4.2.29 Let A, B, and $C(2, -1, 1)$ be the vertices of a triangle where $\overrightarrow{AB}$ is parallel to $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\overrightarrow{AC}$ is parallel to $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, and angle $C = 90°$. Find the equation of the line through B and C.

Exercise 4.2.30 If the diagonals of a parallelogram have equal length, show that the parallelogram is a rectangle.

Exercise 4.2.31 Given $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in component form, show that the projections of $\mathbf{v}$ on $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are $x\mathbf{i}$, $y\mathbf{j}$, and $z\mathbf{k}$, respectively.

Exercise 4.2.32

a. Can $\mathbf{u} \cdot \mathbf{v} = -7$ if $\|\mathbf{u}\| = 3$ and $\|\mathbf{v}\| = 2$? Defend your answer.

b. Find $\mathbf{u} \cdot \mathbf{v}$ if $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, $\|\mathbf{v}\| = 6$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\frac{2\pi}{3}$.

Exercise 4.2.33 Show $(\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}-\mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$ for any vectors $\mathbf{u}$ and $\mathbf{v}$.

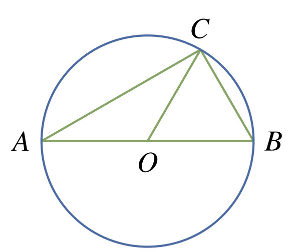
Exercise 4.2.34

a. Show $\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2)$ for any vectors $\mathbf{u}$ and $\mathbf{v}$.

b. What does this say about parallelograms?

Exercise 4.2.35 Show that if the diagonals of a parallelogram are perpendicular, it is necessarily a rhombus. [*Hint*: Example 4.2.5.]

Exercise 4.2.36 Let A and B be the end points of a diameter of a circle (see the diagram). If C is any point on the circle, show that AC and BC are perpendicular. [*Hint*: Express $\overrightarrow{AB} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = 0$ and $\overrightarrow{BC}$ in terms of $\mathbf{u} = \overrightarrow{OA}$ and $\mathbf{v} = \overrightarrow{OC}$, where O is the centre.]



Exercise 4.2.37 Show that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, if and only if $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Exercise 4.2.38 Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be pairwise orthogonal vectors.

a. Show that $\|\mathbf{u}+\mathbf{v}+\mathbf{w}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$.

b. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are all the same length, show that they all make the same angle with $\mathbf{u}+\mathbf{v}+\mathbf{w}$.

Exercise 4.2.39

a. Show that $\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ is orthogonal to every vector along the line $ax+by+c = 0$.

b. Show that the shortest distance from $P_0(x_0, y_0)$ to the line is $\frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$.

   [*Hint*: If P_1 is on the line, project $\mathbf{u} = \overrightarrow{P_1P_0}$ on $\mathbf{n}$.]

Exercise 4.2.40 Assume $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors that are not parallel. Show that $\mathbf{w} = \|\mathbf{u}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{u}$ is a nonzero vector that bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

Exercise 4.2.41 Let α, β, and γ be the angles a vector $\mathbf{v} \neq \mathbf{0}$ makes with the positive x, y, and z axes, respectively. Then $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the **direction cosines** of the vector $\mathbf{v}$.

a. If $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, show that $\cos\alpha = \frac{a}{\|\mathbf{v}\|}$, $\cos\beta = \frac{b}{\|\mathbf{v}\|}$, and $\cos\gamma = \frac{c}{\|\mathbf{v}\|}$.

b. Show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Exercise 4.2.42 Let $\mathbf{v} \neq \mathbf{0}$ be any nonzero vector and suppose that a vector $\mathbf{u}$ can be written as $\mathbf{u} = \mathbf{p}+\mathbf{q}$, where $\mathbf{p}$ is parallel to $\mathbf{v}$ and $\mathbf{q}$ is orthogonal to $\mathbf{v}$. Show that $\mathbf{p}$ must equal the projection of $\mathbf{u}$ on $\mathbf{v}$. [*Hint*: Argue as in the proof of Theorem 4.2.4.]

Exercise 4.2.43 Let $\mathbf{v} \neq \mathbf{0}$ be a nonzero vector and let $a \neq 0$ be a scalar. If $\mathbf{u}$ is any vector, show that the projection of $\mathbf{u}$ on $\mathbf{v}$ equals the projection of $\mathbf{u}$ on $a\mathbf{v}$.

Exercise 4.2.44

a. Show that the **Cauchy-Schwarz inequality** $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$ holds for all vectors $\mathbf{u}$ and $\mathbf{v}$. [*Hint*: $|\cos\theta| \leq 1$ for all angles θ.]

b. Show that $|\mathbf{u}\cdot\mathbf{v}| = \|\mathbf{u}\|\|\mathbf{v}\|$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel.

[*Hint*: When is $\cos\theta = \pm 1$?]

c. Show that $|x_1x_2 + y_1y_2 + z_1z_2| \leq \sqrt{x_1^2+y_1^2+z_1^2}\sqrt{x_2^2+y_2^2+z_2^2}$ holds for all numbers x_1, x_2, y_1, y_2, z_1, and z_2.

d. Show that $|xy + yz + zx| \leq x^2 + y^2 + z^2$ for all x, y, and z.

e. Show that $(x+y+z)^2 \leq 3(x^2+y^2+z^2)$ holds for all x, y, and z.

Exercise 4.2.45 Prove that the **triangle inequality** $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ holds for all vectors $\mathbf{u}$ and $\mathbf{v}$. [*Hint*: Consider the triangle with $\mathbf{u}$ and $\mathbf{v}$ as two sides.]

4.3 More on the Cross Product

The cross product $\mathbf{v}\times\mathbf{w}$ of two $\mathbb{R}^3$-vectors $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ was defined in Section 4.2 where we observed that it can be best remembered using a determinant:

$$\mathbf{v}\times\mathbf{w} = \det\begin{bmatrix} \mathbf{i} & x_1 & x_2 \\ \mathbf{j} & y_1 & y_2 \\ \mathbf{k} & z_1 & z_2 \end{bmatrix} = \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}\mathbf{k} \tag{4.3}$$

Here $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{k} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are the coordinate vectors, and the determinant is expanded along the first column. We observed (but did not prove) in Theorem 4.2.5 that $\mathbf{v}\times\mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$. This follows easily from the next result.

Theorem 4.3.1

If $\mathbf{u} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, *and* $\mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, *then* $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \det\begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{bmatrix}$.

Proof. Recall that $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ is computed by multiplying corresponding components of $\mathbf{u}$ and $\mathbf{v}\times\mathbf{w}$ and then adding. Using equation (4.3), the result is:

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = x_0\left(\begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix}\right) + y_0\left(-\begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix}\right) + z_0\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}\right) = \det\begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{bmatrix}$$

where the last determinant is expanded along column 1. □

The result in Theorem 4.3.1 can be succinctly stated as follows: If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three vectors in $\mathbb{R}^3$, then

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \det\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}$$

where $\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}$ denotes the matrix with $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as its columns. Now it is clear that $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$ because the determinant of a matrix is zero if two columns are identical.

Because of (4.3) and Theorem 4.3.1, several of the following properties of the cross product follow from properties of determinants (they can also be verified directly).

Theorem 4.3.2

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ denote arbitrary vectors in $\mathbb{R}^3$.

1. *$\mathbf{u} \times \mathbf{v}$ is a vector.*
2. *$\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.*
3. *$\mathbf{u} \times \mathbf{0} = \mathbf{0} = \mathbf{0} \times \mathbf{u}$.*
4. *$\mathbf{u} \times \mathbf{u} = \mathbf{0}$.*
5. *$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.*
6. *$(k\mathbf{u}) \times \mathbf{v} = k(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (k\mathbf{v})$ for any scalar k.*
7. *$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$.*
8. *$(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u})$.*

Proof. (1) is clear; (2) follows from Theorem 4.3.1; and (3) and (4) follow because the determinant of a matrix is zero if one column is zero or if two columns are identical. If two columns are interchanged, the determinant changes sign, and this proves (5). The proofs of (6), (7), and (8) are left as Exercise 4.3.15. □

We now come to a fundamental relationship between the dot and cross products.

Theorem 4.3.3: Lagrange Identity[12]

If $\mathbf{u}$ and $\mathbf{v}$ are any two vectors in $\mathbb{R}^3$, then

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

Proof. Given $\mathbf{u}$ and $\mathbf{v}$, introduce a coordinate system and write $\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ in component form. Then all the terms in the identity can be computed in terms of the components. The detailed proof is left as Exercise 4.3.14. □

An expression for the magnitude of the vector $\mathbf{u} \times \mathbf{v}$ can be easily obtained from the Lagrange identity. If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, substituting $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$ into the Lagrange identity gives

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \cos^2\theta = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \sin^2\theta$$

[12]Joseph Louis Lagrange (1736–1813) was born in Italy and spent his early years in Turin. At the age of 19 he solved a famous problem by inventing an entirely new method, known today as the calculus of variations, and went on to become one of the greatest mathematicians of all time. His work brought a new level of rigour to analysis and his *Mécanique Analytique* is a masterpiece in which he introduced methods still in use. In 1766 he was appointed to the Berlin Academy by Frederik the Great who asserted that the “greatest mathematician in Europe” should be at the court of the “greatest king in Europe.” After the death of Frederick, Lagrange went to Paris at the invitation of Louis XVI. He remained there throughout the revolution and was made a count by Napoleon.

using the fact that $1-\cos^2\theta = \sin^2\theta$. But $\sin\theta$ is nonnegative on the range $0 \leq \theta \leq \pi$, so taking the positive square root of both sides gives

$$\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$$

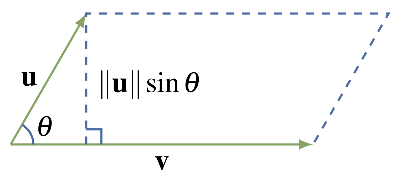


Figure 4.3.1

This expression for $\|\mathbf{u}\times\mathbf{v}\|$ makes no reference to a coordinate system and, moreover, it has a nice geometrical interpretation. The parallelogram determined by the vectors $\mathbf{u}$ and $\mathbf{v}$ has base length $\|\mathbf{v}\|$ and altitude $\|\mathbf{u}\|\sin\theta$ (see Figure 4.3.1). Hence the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$ is

$$(\|\mathbf{u}\|\sin\theta)\|\mathbf{v}\| = \|\mathbf{u}\times\mathbf{v}\|$$

This proves the first part of Theorem 4.3.4.

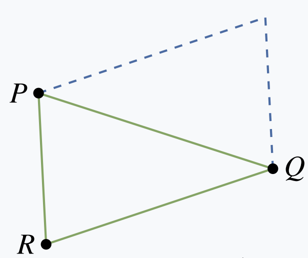
Theorem 4.3.4

If $\mathbf{u}$ and $\mathbf{v}$ are two nonzero vectors and θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

1. $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta =$ *the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.*
2. *$\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u}\times\mathbf{v} = \mathbf{0}$.*

Proof of (2). By (1), $\mathbf{u}\times\mathbf{v} = \mathbf{0}$ if and only if the area of the parallelogram is zero. By Figure 4.3.1 the area vanishes if and only if $\mathbf{u}$ and $\mathbf{v}$ have the same or opposite direction—that is, if and only if they are parallel. □

Example 4.3.1



Find the area of the triangle with vertices $P(2,\ 1,\ 0)$, $Q(3,\ -1,\ 1)$, and $R(1,\ 0,\ 1)$.

Solution. We have $\overrightarrow{RP} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\overrightarrow{RQ} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$. The area of the triangle is half the area of the parallelogram (see the diagram), and so equals $\frac{1}{2}\|\overrightarrow{RP}\times\overrightarrow{RQ}\|$. We have

$$\overrightarrow{RP}\times\overrightarrow{RQ} = \det\begin{bmatrix} \mathbf{i} & 1 & 2 \\ \mathbf{j} & 1 & -1 \\ \mathbf{k} & -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$$

so the area of the triangle is $\frac{1}{2}\|\overrightarrow{RP}\times\overrightarrow{RQ}\| = \frac{1}{2}\sqrt{1+4+9} = \frac{1}{2}\sqrt{14}$.

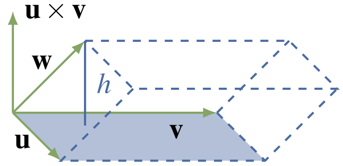


Figure 4.3.2

If three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are given, they determine a "squashed" rectangular solid called a **parallelepiped** (Figure 4.3.2), and it is often useful to be able to find the volume of such a solid. The base of the solid is the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$, so it has area $A = \|\mathbf{u}\times\mathbf{v}\|$ by Theorem 4.3.4. The height of the solid is the length h of the projection of $\mathbf{w}$ on $\mathbf{u}\times\mathbf{v}$. Hence

$$h = \left|\frac{\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})}{\|\mathbf{u}\times\mathbf{v}\|^2}\right| \|\mathbf{u}\times\mathbf{v}\| = \frac{|\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})|}{\|\mathbf{u}\times\mathbf{v}\|} = \frac{|\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})|}{A}$$

Thus the volume of the parallelepiped is $hA = |\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})|$. This proves

Theorem 4.3.5

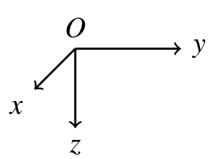
The volume of the parallelepiped determined by three vectors $\mathbf{w}$, $\mathbf{u}$, and $\mathbf{v}$ (Figure 4.3.2) is given by $|\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})|$.

Example 4.3.2

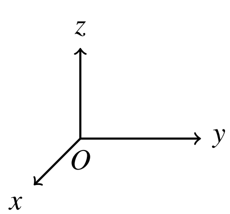
Find the volume of the parallelepiped determined by the vectors

$$\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \ \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Solution. By Theorem 4.3.1, $\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) = \det \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = -3$. Hence the volume is $|\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})| = |-3| = 3$ by Theorem 4.3.5.



Left-hand system



Right-hand system

Figure 4.3.3

We can now give an intrinsic description of the cross product $\mathbf{u}\times\mathbf{v}$. Its magnitude $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$ is coordinate-free. If $\mathbf{u}\times\mathbf{v} \neq \mathbf{0}$, its direction is very nearly determined by the fact that it is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ and so points along the line normal to the plane determined by $\mathbf{u}$ and $\mathbf{v}$. It remains only to decide which of the two possible directions is correct.

Before this can be done, the basic issue of how coordinates are assigned must be clarified. When coordinate axes are chosen in space, the procedure is as follows: An origin is selected, two perpendicular lines (the x and y axes) are chosen through the origin, and a positive direction on each of these axes is selected quite arbitrarily. Then the line through the origin normal to this x-y plane is called the z axis, but there is a choice of which direction on this axis is the positive one. The two possibilities are shown in Figure 4.3.3, and it is a standard convention that cartesian coordinates are always **right-hand coordinate systems**. The reason for this terminology is that, in such a system, if the z axis is grasped in the right hand with the thumb pointing in the positive z direction, then the fingers curl around from the positive x axis to the positive y axis (through a right angle).

Suppose now that $\mathbf{u}$ and $\mathbf{v}$ are given and that θ is the angle between them (so $0 \leq \theta \leq \pi$). Then the direction of $\|\mathbf{u} \times \mathbf{v}\|$ is given by the right-hand rule.

Right-hand Rule

If the vector $\mathbf{u} \times \mathbf{v}$ is grasped in the right hand and the fingers curl around from $\mathbf{u}$ to $\mathbf{v}$ through the angle θ, the thumb points in the direction for $\mathbf{u} \times \mathbf{v}$.

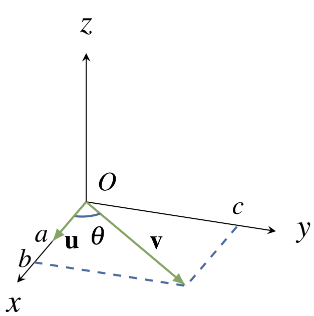


Figure 4.3.4

To indicate why this is true, introduce coordinates in $\mathbb{R}^3$ as follows: Let $\mathbf{u}$ and $\mathbf{v}$ have a common tail O, choose the origin at O, choose the x axis so that $\mathbf{u}$ points in the positive x direction, and then choose the y axis so that $\mathbf{v}$ is in the x-y plane and the positive y axis is on the same side of the x axis as $\mathbf{v}$. Then, in this system, $\mathbf{u}$ and $\mathbf{v}$ have component form $\mathbf{u} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} b \\ c \\ 0 \end{bmatrix}$ where $a > 0$ and $c > 0$. The situation is depicted in Figure 4.3.4. The right-hand rule asserts that $\mathbf{u} \times \mathbf{v}$ should point in the positive z direction. But our definition of $\mathbf{u} \times \mathbf{v}$ gives

$$\mathbf{u} \times \mathbf{v} = \det \begin{bmatrix} \mathbf{i} & a & b \\ \mathbf{j} & 0 & c \\ \mathbf{k} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ ac \end{bmatrix} = (ac)\mathbf{k}$$

and $(ac)\mathbf{k}$ has the positive z direction because $ac > 0$.

Exercises for 4.3

Exercise 4.3.1 If $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the coordinate vectors, verify that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

Exercise 4.3.2 Show that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ need not equal $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ by calculating both when

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Exercise 4.3.3 Find two unit vectors orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ if:

a. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

Exercise 4.3.4 Find the area of the triangle with the following vertices.

a. $A(3, -1, 2)$, $B(1, 1, 0)$, and $C(1, 2, -1)$

b. $A(3, 0, 1)$, $B(5, 1, 0)$, and $C(7, 2, -1)$

c. $A(1, 1, -1)$, $B(2, 0, 1)$, and $C(1, -1, 3)$

d. $A(3, -1, 1)$, $B(4, 1, 0)$, and $C(2, -3, 0)$

Exercise 4.3.5 Find the volume of the parallelepiped determined by $\mathbf{w}$, $\mathbf{u}$, and $\mathbf{v}$ when:

a. $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

b. $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Exercise 4.3.6 Let P_0 be a point with vector $\mathbf{p}_0$, and let $ax+by+cz=d$ be the equation of a plane with normal $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

a. Show that the point on the plane closest to P_0 has vector $\mathbf{p}$ given by

$$\mathbf{p} = \mathbf{p}_0 + \frac{d-(\mathbf{p}_0 \cdot \mathbf{n})}{\|\mathbf{n}\|^2}\mathbf{n}.$$

[*Hint*: $\mathbf{p} = \mathbf{p}_0 + t\mathbf{n}$ for some t, and $\mathbf{p} \cdot \mathbf{n} = \mathbf{d}$.]

b. Show that the shortest distance from P_0 to the plane is $\frac{|d-(\mathbf{p}_0 \cdot \mathbf{n})|}{\|\mathbf{n}\|}$.

c. Let P_0' denote the reflection of P_0 in the plane—that is, the point on the opposite side of the plane such that the line through P_0 and P_0' is perpendicular to the plane.

Show that $\mathbf{p}_0 + 2\frac{d-(\mathbf{p}_0 \cdot \mathbf{n})}{\|\mathbf{n}\|^2}\mathbf{n}$ is the vector of P_0'.

Exercise 4.3.7 Simplify $(a\mathbf{u}+b\mathbf{v}) \times (c\mathbf{u}+d\mathbf{v})$.

Exercise 4.3.8 Show that the shortest distance from a point P to the line through P_0 with direction vector $\mathbf{d}$ is $\frac{\|\overrightarrow{P_0P} \times \mathbf{d}\|}{\|\mathbf{d}\|}$.

Exercise 4.3.9 Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero, nonorthogonal vectors. If θ is the angle between them, show that $\tan\theta = \frac{\|\mathbf{u} \times \mathbf{v}\|}{\mathbf{u} \cdot \mathbf{v}}$.

Exercise 4.3.10 Show that points A, B, and C are all on one line if and only if $\overrightarrow{AB} \times \overrightarrow{AC} = 0$

Exercise 4.3.11 Show that points A, B, C, and D are all on one plane if and only if $\overrightarrow{AB} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = 0$

Exercise 4.3.12 Use Theorem 4.3.5 to confirm that, if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are mutually perpendicular, the (rectangular) parallelepiped they determine has volume $\|\mathbf{u}\|\|\mathbf{v}\|\|\mathbf{w}\|$.

Exercise 4.3.13 Show that the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ is $\|\mathbf{u} \times \mathbf{v}\|^2$.

Exercise 4.3.14 Complete the proof of Theorem 4.3.3.

Exercise 4.3.15 Prove the following properties in Theorem 4.3.2.

a. Property 6

b. Property 7

c. Property 8

Exercise 4.3.16

a. Show that $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \times (\mathbf{w} \times \mathbf{u})$ holds for all vectors $\mathbf{w}$, $\mathbf{u}$, and $\mathbf{v}$.

b. Show that $\mathbf{v} - \mathbf{w}$ and $(\mathbf{u} \times \mathbf{v}) + (\mathbf{v} \times \mathbf{w}) + (\mathbf{w} \times \mathbf{u})$ are orthogonal.

Exercise 4.3.17 Show $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \times \mathbf{v})\mathbf{w}$. [*Hint*: First do it for $\mathbf{u} = \mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$; then write $\mathbf{u} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and use Theorem 4.3.2.]

Exercise 4.3.18 Prove the **Jacobi identity**:

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$$

[*Hint*: The preceding exercise.]

Exercise 4.3.19 Show that

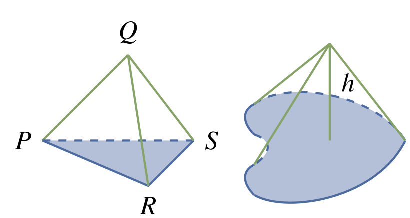
$$(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{z}) = \det \begin{bmatrix} \mathbf{u} \cdot \mathbf{w} & \mathbf{u} \cdot \mathbf{z} \\ \mathbf{v} \cdot \mathbf{w} & \mathbf{v} \cdot \mathbf{z} \end{bmatrix}$$

[*Hint*: Exercises 4.3.16 and 4.3.17.]

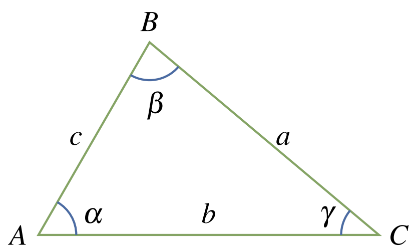
Exercise 4.3.20 Let P, Q, R, and S be four points, not all on one plane, as in the diagram. Show that the volume of the pyramid they determine is

$$\tfrac{1}{6}|\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS})|.$$

[*Hint*: The volume of a cone with base area A and height h as in the diagram below right is $\frac{1}{3}Ah$.]



Exercise 4.3.21 Consider a triangle with vertices A, B, and C, as in the diagram below. Let α, β, and γ denote the angles at A, B, and C, respectively, and let a, b, and c denote the lengths of the sides opposite A, B, and C, respectively. Write $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, and $\mathbf{w} = \overrightarrow{CA}$.



a. Deduce that $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$.

b. Show that $\mathbf{u} \times \mathbf{v} = \mathbf{w} \times \mathbf{u} = \mathbf{v} \times \mathbf{w}$. [*Hint*: Compute $\mathbf{u} \times (\mathbf{u} + \mathbf{v} + \mathbf{w})$ and $\mathbf{v} \times (\mathbf{u} + \mathbf{v} + \mathbf{w})$.]

c. Deduce the **law of sines**:

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$

Exercise 4.3.22 Show that the (shortest) distance between two planes $\mathbf{n} \cdot \mathbf{p} = d_1$ and $\mathbf{n} \cdot \mathbf{p} = d_2$ with $\mathbf{n}$ as normal is $\frac{|d_2 - d_1|}{\|\mathbf{n}\|}$.

Exercise 4.3.23 Let A and B be points other than the origin, and let $\mathbf{a}$ and $\mathbf{b}$ be their vectors. If $\mathbf{a}$ and $\mathbf{b}$ are not parallel, show that the plane through A, B, and the origin is given by

$$\left\{ P(x,\ y,\ z) \mid \left[\begin{array}{c} x \\ y \\ z \end{array} \right] = s\mathbf{a} + t\mathbf{b} \text{ for some } s \text{ and } t \right\}$$

Exercise 4.3.24 Let A be a 2×3 matrix of rank 2 with rows $\mathbf{r}_1$ and $\mathbf{r}_2$. Show that

$$P = \{XA \mid X = [xy]; x,\ y \text{ arbitrary}\}$$

is the plane through the origin with normal $\mathbf{r}_1 \times \mathbf{r}_2$.

Exercise 4.3.25 Given the cube with vertices $P(x,\ y,\ z)$, where each of x, y, and z is either 0 or 2, consider the plane perpendicular to the diagonal through $P(0,\ 0,\ 0)$ and $P(2,\ 2,\ 2)$ and bisecting it.

a. Show that the plane meets six of the edges of the cube and bisects them.

b. Show that the six points in (a) are the vertices of a regular hexagon.

4.4 Linear Operators on $\mathbb{R}^3$

Recall that a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called *linear* if $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ and $T(a\mathbf{x}) = aT(\mathbf{x})$ holds for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and all scalars a. In this case we showed (in Theorem 2.6.2) that there exists an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$, and we say that T is the **matrix transformation induced** by A.

Definition 4.9 Linear Operator on $\mathbb{R}^n$

A linear transformation

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

*is called a **linear operator** on $\mathbb{R}^n$.*

In Section 2.6 we investigated three important linear operators on $\mathbb{R}^2$: rotations about the origin, reflections in a line through the origin, and projections on this line.

In this section we investigate the analogous operators on $\mathbb{R}^3$: Rotations about a line through the origin, reflections in a plane through the origin, and projections onto a plane or line through the origin in $\mathbb{R}^3$. In every case we show that the operator is linear, and we find the matrices of all the reflections and projections.

To do this we must prove that these reflections, projections, and rotations are actually *linear* operators on $\mathbb{R}^3$. In the case of reflections and rotations, it is convenient to examine a more general situation. A transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is said to be **distance preserving** if the distance between $T(\mathbf{v})$ and $T(\mathbf{w})$ is the same as the distance between $\mathbf{v}$ and $\mathbf{w}$ for all $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$; that is,

$$\|T(\mathbf{v}) - T(\mathbf{w})\| = \|\mathbf{v} - \mathbf{w}\| \text{ for all } \mathbf{v} \text{ and } \mathbf{w} \text{ in } \mathbb{R}^3 \tag{4.4}$$

Clearly reflections and rotations are distance preserving, and both carry $\mathbf{0}$ to $\mathbf{0}$, so the following theorem shows that they are both linear.

Theorem 4.4.1

If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is distance preserving, and if $T(\mathbf{0}) = \mathbf{0}$, then T is linear.

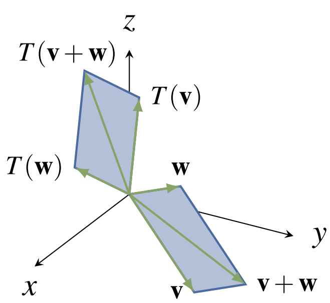


Figure 4.4.1

Proof. Since $T(\mathbf{0}) = \mathbf{0}$, taking $\mathbf{w} = \mathbf{0}$ in (4.4) shows that $\|T(\mathbf{v})\| = \|\mathbf{v}\|$ for all $\mathbf{v}$ in $\mathbb{R}^3$, that is T preserves length. Also, $\|T(\mathbf{v}) - T(\mathbf{w})\|^2 = \|\mathbf{v} - \mathbf{w}\|^2$ by (4.4). Since $\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 - 2\mathbf{v} \cdot \mathbf{w} + \|\mathbf{w}\|^2$ always holds, it follows that $T(\mathbf{v}) \cdot T(\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$ for all $\mathbf{v}$ and $\mathbf{w}$. Hence (by Theorem 4.2.2) the angle between $T(\mathbf{v})$ and $T(\mathbf{w})$ is the same as the angle between $\mathbf{v}$ and $\mathbf{w}$ for all (nonzero) vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$.

With this we can show that T is linear. Given nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$, the vector $\mathbf{v} + \mathbf{w}$ is the diagonal of the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$. By the preceding paragraph, the effect of T is to carry this *entire parallelogram* to the parallelogram determined by $T(\mathbf{v})$ and $T(\mathbf{w})$, with diagonal $T(\mathbf{v} + \mathbf{w})$. But this diagonal is $T(\mathbf{v}) + T(\mathbf{w})$ by the parallelogram law (see Figure 4.4.1).

In other words, $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$. A similar argument shows that $T(a\mathbf{v}) = aT(\mathbf{v})$ for all scalars a, proving that T is indeed linear. □

Distance-preserving linear operators are called **isometries**, and we return to them in Section 10.4.

Reflections and Projections

In Section 2.6 we studied the reflection $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ in the line $y = mx$ and projection $P_m : \mathbb{R}^2 \to \mathbb{R}^2$ on the same line. We found (in Theorems 2.6.5 and 2.6.6) that they are both linear and

$$Q_m \text{ has matrix } \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix} \quad \text{and} \quad P_m \text{ has matrix } \frac{1}{1+m^2}\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}.$$

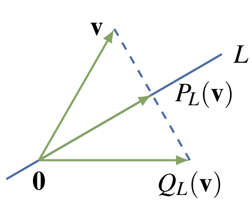


Figure 4.4.2

We now look at the analogues in $\mathbb{R}^3$.

Let L denote a line through the origin in $\mathbb{R}^3$. Given a vector $\mathbf{v}$ in $\mathbb{R}^3$, the reflection $Q_L(\mathbf{v})$ of $\mathbf{v}$ in L and the projection $P_L(\mathbf{v})$ of $\mathbf{v}$ on L are defined in Figure 4.4.2. In the same figure, we see that

$$P_L(\mathbf{v}) = \mathbf{v} + \tfrac{1}{2}[Q_L(\mathbf{v}) - \mathbf{v}] = \tfrac{1}{2}[Q_L(\mathbf{v}) + \mathbf{v}] \tag{4.5}$$

so the fact that Q_L is linear (by Theorem 4.4.1) shows that P_L is also linear.[13]

However, Theorem 4.2.4 gives us the matrix of P_L directly. In fact, if $\mathbf{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \mathbf{0}$ is a direction vector for L, and we write $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then

$$P_L(\mathbf{v}) = \frac{\mathbf{v}\cdot\mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d} = \frac{ax+by+cz}{a^2+b^2+c^2}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{a^2+b^2+c^2}\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

as the reader can verify. Note that this shows directly that P_L is a matrix transformation and so gives another proof that it is linear.

Theorem 4.4.2

Let L denote the line through the origin in $\mathbb{R}^3$ with direction vector $\mathbf{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \mathbf{0}$. Then P_L and Q_L are both linear and

$$P_L \text{ has matrix } \frac{1}{a^2+b^2+c^2}\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$$

$$Q_L \text{ has matrix } \frac{1}{a^2+b^2+c^2}\begin{bmatrix} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & b^2-a^2-c^2 & 2bc \\ 2ac & 2bc & c^2-a^2-b^2 \end{bmatrix}$$

Proof. It remains to find the matrix of Q_L. But (4.5) implies that $Q_L(\mathbf{v}) = 2P_L(\mathbf{v}) - \mathbf{v}$ for each $\mathbf{v}$ in $\mathbb{R}^3$, so if $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ we obtain (with some matrix arithmetic):

$$\begin{aligned} Q_L(\mathbf{v}) &= \left\{ \frac{2}{a^2+b^2+c^2}\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= \frac{1}{a^2+b^2+c^2}\begin{bmatrix} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & b^2-a^2-c^2 & 2bc \\ 2ac & 2bc & c^2-a^2-b^2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \end{aligned}$$

as required. □

[13]Note that Theorem 4.4.1 does *not* apply to P_L since it does not preserve distance.

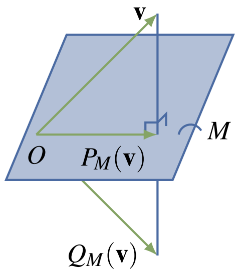


Figure 4.4.3

In $\mathbb{R}^3$ we can reflect in planes as well as lines. Let M denote a plane through the origin in $\mathbb{R}^3$. Given a vector $\mathbf{v}$ in $\mathbb{R}^3$, the reflection $Q_M(\mathbf{v})$ of $\mathbf{v}$ in M and the projection $P_M(\mathbf{v})$ of $\mathbf{v}$ on M are defined in Figure 4.4.3. As above, we have

$$P_M(\mathbf{v}) = \mathbf{v} + \tfrac{1}{2}[Q_M(\mathbf{v}) - \mathbf{v}] = \tfrac{1}{2}[Q_M(\mathbf{v}) + \mathbf{v}]$$

so the fact that Q_M is linear (again by Theorem 4.4.1) shows that P_M is also linear.

Again we can obtain the matrix directly. If $\mathbf{n}$ is a normal for the plane M, then Figure 4.4.3 shows that

$$P_M(\mathbf{v}) = \mathbf{v} - \operatorname{proj}_{\mathbf{n}} \mathbf{v} = \mathbf{v} - \tfrac{\mathbf{v}\cdot\mathbf{n}}{\|\mathbf{n}\|^2}\mathbf{n} \text{ for all vectors } \mathbf{v}.$$

If $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \mathbf{0}$ and $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, a computation like the above gives

$$\begin{aligned} P_M(\mathbf{v}) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \tfrac{ax+by+cz}{a^2+b^2+c^2} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ &= \tfrac{1}{a^2+b^2+c^2} \begin{bmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & b^2+c^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \end{aligned}$$

This proves the first part of

Theorem 4.4.3

Let M denote the plane through the origin in $\mathbb{R}^3$ with normal $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \mathbf{0}$. Then P_M and Q_M are both linear and

$$P_M \text{ has matrix } \tfrac{1}{a^2+b^2+c^2} \begin{bmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & a^2+b^2 \end{bmatrix}$$

$$Q_M \text{ has matrix } \tfrac{1}{a^2+b^2+c^2} \begin{bmatrix} b^2+c^2-a^2 & -2ab & -2ac \\ -2ab & a^2+c^2-b^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix}$$

Proof. It remains to compute the matrix of Q_M. Since $Q_M(\mathbf{v}) = 2P_M(\mathbf{v}) - \mathbf{v}$ for each $\mathbf{v}$ in $\mathbb{R}^3$, the computation is similar to the above and is left as an exercise for the reader. □

Rotations

In Section 2.6 we studied the rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ counterclockwise about the origin through the angle θ. Moreover, we showed in Theorem 2.6.4 that R_θ is linear and has matrix $\left[\begin{array}{rr} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right]$. One extension of this is given in the following example.

Example 4.4.1

Let $R_{z,\ \theta} : \mathbb{R}^3 \to \mathbb{R}^3$ denote rotation of $\mathbb{R}^3$ about the z axis through an angle θ from the positive x axis toward the positive y axis. Show that $R_{z,\ \theta}$ is linear and find its matrix.

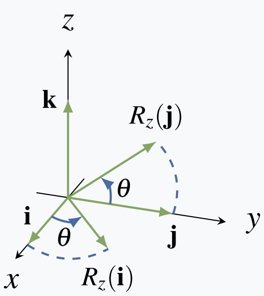


Figure 4.4.4

Solution. First R is distance preserving and so is linear by Theorem 4.4.1. Hence we apply Theorem 2.6.2 to obtain the matrix of $R_{z,\ \theta}$.

Let $\mathbf{i} = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$, $\mathbf{j} = \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right]$, and $\mathbf{k} = \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]$ denote the standard basis of $\mathbb{R}^3$; we must find $R_{z,\ \theta}(\mathbf{i})$, $R_{z,\ \theta}(\mathbf{j})$, and $R_{z,\ \theta}(\mathbf{k})$. Clearly $R_{z,\ \theta}(\mathbf{k}) = \mathbf{k}$. The effect of $R_{z,\ \theta}$ on the x-y plane is to rotate it counterclockwise through the angle θ. Hence Figure 4.4.4 gives

$$R_{z,\ \theta}(\mathbf{i}) = \left[\begin{array}{c} \cos\theta \\ \sin\theta \\ 0 \end{array}\right], \quad R_{z,\ \theta}(\mathbf{j}) = \left[\begin{array}{c} -\sin\theta \\ \cos\theta \\ 0 \end{array}\right]$$

so, by Theorem 2.6.2, $R_{z,\ \theta}$ has matrix

$$\left[\begin{array}{ccc} R_{z,\ \theta}(\mathbf{i}) & R_{z,\ \theta}(\mathbf{j}) & R_{z,\ \theta}(\mathbf{k}) \end{array}\right] = \left[\begin{array}{ccc} \cos\theta & -\sin\theta & 0 \\ sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{array}\right]$$

Example 4.4.1 begs to be generalized. Given a line L through the origin in $\mathbb{R}^3$, every rotation about L through a fixed angle is clearly distance preserving, and so is a linear operator by Theorem 4.4.1. However, giving a precise description of the matrix of this rotation is not easy and will have to wait until more techniques are available.

Transformations of Areas and Volumes

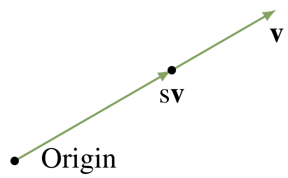


Figure 4.4.5

Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^3$. Each vector in the same direction as $\mathbf{v}$ whose length is a fraction s of the length of $\mathbf{v}$ has the form $s\mathbf{v}$ (see Figure 4.4.5).

With this, scrutiny of Figure 4.4.6 shows that a vector $\mathbf{u}$ is in the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$ if and only if it has the form $\mathbf{u} = s\mathbf{v} + t\mathbf{w}$ where $0 \le s \le 1$ and $0 \le t \le 1$. But then, if $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation, we have

$$T(s\mathbf{v} + t\mathbf{w}) = T(s\mathbf{v}) + T(t\mathbf{w}) = sT(\mathbf{v}) + tT(\mathbf{w})$$

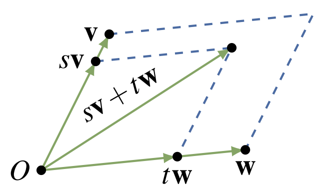


Figure 4.4.6

Hence $T(s\mathbf{v} + t\mathbf{w})$ is in the parallelogram determined by $T(\mathbf{v})$ and $T(\mathbf{w})$. Conversely, every vector in this parallelogram has the form $T(s\mathbf{v} + t\mathbf{w})$ where $s\mathbf{v} + t\mathbf{w}$ is in the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$. For this reason, the parallelogram determined by $T(\mathbf{v})$ and $T(\mathbf{w})$ is called the **image** of the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$. We record this discussion as:

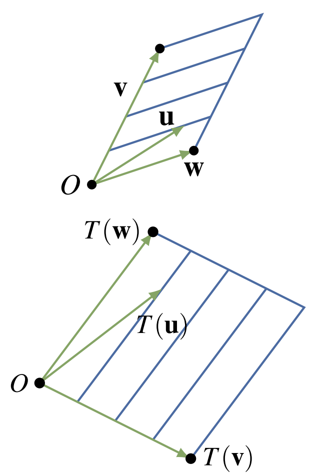


Figure 4.4.7

Theorem 4.4.4

If $T : \mathbb{R}^3 \to \mathbb{R}^3$ (or $\mathbb{R}^2 \to \mathbb{R}^2$) is a linear operator, the image of the parallelogram determined by vectors $\mathbf{v}$ and $\mathbf{w}$ is the parallelogram determined by $T(\mathbf{v})$ and $T(\mathbf{w})$.

This result is illustrated in Figure 4.4.7, and was used in Examples 2.2.15 and 2.2.16 to reveal the effect of expansion and shear transformations.

We now describe the effect of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ on the parallelepiped determined by three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$ (see the discussion preceding Theorem 4.3.5). If T has matrix A, Theorem 4.4.4 shows that this parallelepiped is carried to the parallelepiped determined by $T(\mathbf{u}) = A\mathbf{u}$, $T(\mathbf{v}) = A\mathbf{v}$, and $T(\mathbf{w}) = A\mathbf{w}$. In particular, we want to discover how the volume changes, and it turns out to be closely related to the determinant of the matrix A.

Theorem 4.4.5

Let $\operatorname{vol}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ denote the volume of the parallelepiped determined by three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$, and let $\operatorname{area}(\mathbf{p}, \mathbf{q})$ denote the area of the parallelogram determined by two vectors $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{R}^2$. Then:

1. *If A is a 3×3 matrix, then $\operatorname{vol}(A\mathbf{u}, A\mathbf{v}, A\mathbf{w}) = |\det(A)| \cdot \operatorname{vol}(\mathbf{u}, \mathbf{v}, \mathbf{w})$.*

2. *If A is a 2×2 matrix, then $\operatorname{area}(A\mathbf{p}, A\mathbf{q}) = |\det(A)| \cdot \operatorname{area}(\mathbf{p}, \mathbf{q})$.*

Proof.

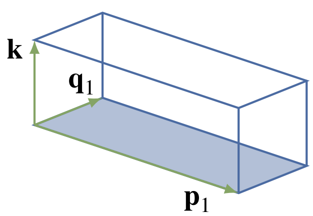
1. Let $\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}$ denote the 3×3 matrix with columns $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Then

$$\text{vol}\,(A\mathbf{u},\, A\mathbf{v},\, A\mathbf{w}) = |A\mathbf{u} \cdot (A\mathbf{v} \times A\mathbf{w})|$$

by Theorem 4.3.5. Now apply Theorem 4.3.1 twice to get

$$\begin{aligned} A\mathbf{u} \cdot (A\mathbf{v} \times A\mathbf{w}) = \det \begin{bmatrix} A\mathbf{u} & A\mathbf{v} & A\mathbf{w} \end{bmatrix} &= \det\,(A \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}) \\ &= \det\,(A) \det \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix} \\ &= \det\,(A)(\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})) \end{aligned}$$

where we used Definition 2.9 and the product theorem for determinants. Finally (1) follows from Theorem 4.3.5 by taking absolute values.



2. Given $\mathbf{p} = \begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$, $\mathbf{p}_1 = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ in $\mathbb{R}^3$. By the diagram, area $(\mathbf{p},\, \mathbf{q}) = \text{vol}\,(\mathbf{p}_1,\, \mathbf{q}_1,\, \mathbf{k})$ where $\mathbf{k}$ is the (length 1) coordinate vector along the z axis. If A is a 2×2 matrix, write $A_1 = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$ in block form, and observe that $(A\mathbf{v})_1 = (A_1\mathbf{v}_1)$ for all $\mathbf{v}$ in $\mathbb{R}^2$ and $A_1\mathbf{k} = \mathbf{k}$. Hence part (1) of this theorem shows

$$\begin{aligned} \text{area}\,(A\mathbf{p},\, A\mathbf{q}) &= \text{vol}\,(A_1\mathbf{p}_1,\, A_1\mathbf{q}_1,\, A_1\mathbf{k}) \\ &= |\det\,(A_1)|\; \text{vol}\,(\mathbf{p}_1,\, \mathbf{q}_1,\, \mathbf{k}) \\ &= |\det\,(A)|\; \text{area}\,(\mathbf{p},\, \mathbf{q}) \end{aligned}$$

as required. □

Define the **unit square** and **unit cube** to be the square and cube corresponding to the coordinate vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Then Theorem 4.4.5 gives a geometrical meaning to the determinant of a matrix A:

- If A is a 2×2 matrix, then $|\det\,(A)|$ is the area of the image of the unit square under multiplication by A;
- If A is a 3×3 matrix, then $|\det\,(A)|$ is the volume of the image of the unit cube under multiplication by A.

These results, together with the importance of areas and volumes in geometry, were among the reasons for the initial development of determinants.

Exercises for 4.4

Exercise 4.4.1 In each case show that that T is either projection on a line, reflection in a line, or rotation through an angle, and find the line or angle.

a. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} x+2y \\ 2x+4y \end{bmatrix}$

b. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x-y \\ y-x \end{bmatrix}$

c. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -x-y \\ x-y \end{bmatrix}$

d. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3x+4y \\ 4x+3y \end{bmatrix}$

e. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$

f. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x-\sqrt{3}y \\ \sqrt{3}x+y \end{bmatrix}$

Exercise 4.4.2 Determine the effect of the following transformations.

a. Rotation through $\frac{\pi}{2}$, followed by projection on the y axis, followed by reflection in the line $y = x$.

b. Projection on the line $y = x$ followed by projection on the line $y = -x$.

c. Projection on the x axis followed by reflection in the line $y = x$.

Exercise 4.4.3 In each case solve the problem by finding the matrix of the operator.

a. Find the projection of $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ on the plane with equation $3x - 5y + 2z = 0$.

b. Find the projection of $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ on the plane with equation $2x - y + 4z = 0$.

c. Find the reflection of $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ in the plane with equation $x - y + 3z = 0$.

d. Find the reflection of $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ in the plane with equation $2x + y - 5z = 0$.

e. Find the reflection of $\mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$ in the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

f. Find the projection of $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$ on the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$.

g. Find the projection of $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$ on the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$.

h. Find the reflection of $\mathbf{v} = \begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix}$ in the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$.

Exercise 4.4.4

a. Find the rotation of $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ about the z axis through $\theta = \frac{\pi}{4}$.

b. Find the rotation of $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ about the z axis through $\theta = \frac{\pi}{6}$.

Exercise 4.4.5 Find the matrix of the rotation in $\mathbb{R}^3$ about the x axis through the angle θ (from the positive y axis to the positive z axis).

Exercise 4.4.6 Find the matrix of the rotation about the y axis through the angle θ (from the positive x axis to the positive z axis).

Exercise 4.4.7 If A is 3×3, show that the image of the line in $\mathbb{R}^3$ through $\mathbf{p}_0$ with direction vector $\mathbf{d}$ is the line through $A\mathbf{p}_0$ with direction vector $A\mathbf{d}$, assuming that $A\mathbf{d} \neq \mathbf{0}$. What happens if $A\mathbf{d} = \mathbf{0}$?

Exercise 4.4.8 If A is 3×3 and invertible, show that the image of the plane through the origin with normal $\mathbf{n}$ is the plane through the origin with normal $\mathbf{n}_1 = B\mathbf{n}$ where $B = (A^{-1})^T$. [*Hint*: Use the fact that $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T\mathbf{w}$ to show that $\mathbf{n}_1 \cdot (A\mathbf{p}) = \mathbf{n} \cdot \mathbf{p}$ for each $\mathbf{p}$ in $\mathbb{R}^3$.]

Exercise 4.4.9 Let L be the line through the origin in $\mathbb{R}^2$ with direction vector $\mathbf{d} = \begin{bmatrix} a \\ b \end{bmatrix} \neq 0$.

a. If P_L denotes projection on L, show that P_L has matrix $\frac{1}{a^2+b^2}\begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$.

b. If Q_L denotes reflection in L, show that Q_L has matrix $\frac{1}{a^2+b^2}\begin{bmatrix} a^2-b^2 & 2ab \\ 2ab & b^2-a^2 \end{bmatrix}$.

Exercise 4.4.10 Let $\mathbf{n}$ be a nonzero vector in $\mathbb{R}^3$, let L be the line through the origin with direction vector $\mathbf{n}$, and let M be the plane through the origin with normal $\mathbf{n}$. Show that $P_L(\mathbf{v}) = Q_L(\mathbf{v}) + P_M(\mathbf{v})$ for all $\mathbf{v}$ in $\mathbb{R}^3$. [In this case, we say that $P_L = Q_L + P_M$.]

Exercise 4.4.11 If M is the plane through the origin in $\mathbb{R}^3$ with normal $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, show that Q_M has matrix

$$\frac{1}{a^2+b^2+c^2}\begin{bmatrix} b^2+c^2-a^2 & -2ab & -2ac \\ -2ab & a^2+c^2-b^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix}$$

4.5 An Application to Computer Graphics

Computer graphics deals with images displayed on a computer screen, and so arises in a variety of applications, ranging from word processors, to *Star Wars* animations, to video games, to wire-frame images of an airplane. These images consist of a number of points on the screen, together with instructions on how to fill in areas bounded by lines and curves. Often curves are approximated by a set of short straight-line segments, so that the curve is specified by a series of points on the screen at the end of these segments. Matrix transformations are important here because matrix images of straight line segments are again line segments.[14] Note that a colour image requires that three images are sent, one to each of the red, green, and blue phosphorus dots on the screen, in varying intensities.

Consider displaying the letter A. In reality, it is depicted on the screen, as in Figure 4.5.1, by specifying the coordinates of the 11 corners and filling in the interior.

For simplicity, we will disregard the thickness of the letter, so we require only five coordinates as in Figure 4.5.2.

[14]If $\mathbf{v}_0$ and $\mathbf{v}_1$ are vectors, the vector from $\mathbf{v}_0$ to $\mathbf{v}_1$ is $\mathbf{d} = \mathbf{v}_1 - \mathbf{v}_0$. So a vector $\mathbf{v}$ lies on the line segment between $\mathbf{v}_0$ and $\mathbf{v}_1$ if and only if $\mathbf{v} = \mathbf{v}_0 + t\mathbf{d}$ for some number t in the range $0 \leq t \leq 1$. Thus the image of this segment is the set of vectors $A\mathbf{v} = A\mathbf{v}_0 + tA\mathbf{d}$ with $0 \leq t \leq 1$, that is the image is the segment between $A\mathbf{v}_0$ and $A\mathbf{v}_1$.

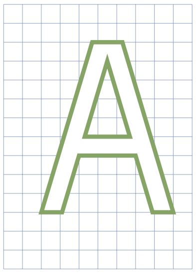


Figure 4.5.1

This simplified letter can then be stored as a data matrix

$$\begin{array}{cc} \textit{Vertex} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ D = & \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix} \end{array}$$

where the columns are the coordinates of the vertices in order. Then if we want to transform the letter by a 2×2 matrix A, we left-multiply this data matrix by A (the effect is to multiply each column by A and so transform each vertex).

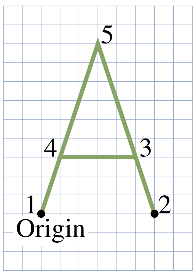


Figure 4.5.2

For example, we can slant the letter to the right by multiplying by an x-shear matrix $A = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$—see Section 2.2. The result is the letter with data matrix

$$A = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 5.6 & 1.6 & 4.8 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix}$$

which is shown in Figure 4.5.3.

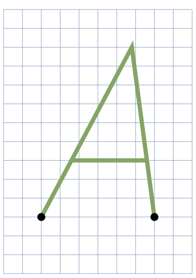


Figure 4.5.3

If we want to make this slanted matrix narrower, we can now apply an x-scale matrix $B = \begin{bmatrix} 0.8 & 0 \\ 0 & 1 \end{bmatrix}$ that shrinks the x-coordinate by 0.8. The result is the composite transformation

$$\begin{aligned} BAD &= \begin{bmatrix} 0.8 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 4.8 & 4.48 & 1.28 & 3.84 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix} \end{aligned}$$

which is drawn in Figure 4.5.4.

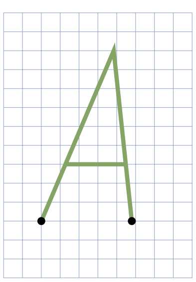


Figure 4.5.4

On the other hand, we can rotate the letter about the origin through $\frac{\pi}{6}$ (or 30°) by multiplying by the matrix $R_{\frac{\pi}{2}} = \begin{bmatrix} \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) \\ \sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{bmatrix}$. This gives

$$\begin{aligned} R_{\frac{\pi}{2}} &= \begin{bmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{bmatrix} \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 5.196 & 2.83 & -0.634 & -1.902 \\ 0 & 3 & 5.098 & 3.098 & 9.294 \end{bmatrix} \end{aligned}$$

and is plotted in Figure 4.5.5.

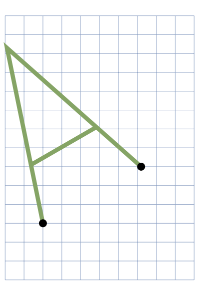


Figure 4.5.5

This poses a problem: How do we rotate at a point other than the origin? It turns out that we can do this when we have solved another more basic problem. It is clearly important to be able to translate a screen image by a fixed vector $\mathbf{w}$, that is apply the transformation $T_w : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T_w(\mathbf{v}) = \mathbf{v} + \mathbf{w}$ for all $\mathbf{v}$ in $\mathbb{R}^2$. The problem is that these translations are not matrix transformations $\mathbb{R}^2 \to \mathbb{R}^2$ because they do not carry $\mathbf{0}$ to $\mathbf{0}$ (unless $\mathbf{w} = \mathbf{0}$). However, there is a clever way around this.

The idea is to represent a point $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ as a 3×1 column $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$, called the **homogeneous coordinates** of $\mathbf{v}$. Then translation by $\mathbf{w} = \begin{bmatrix} p \\ q \end{bmatrix}$ can be achieved by multiplying by a 3×3 matrix:

$$\begin{bmatrix} 1 & 0 & p \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+p \\ y+q \\ 1 \end{bmatrix} = \begin{bmatrix} T_{\mathbf{w}}(\mathbf{v}) \\ 1 \end{bmatrix}$$

Thus, by using homogeneous coordinates we can implement the translation T_w in the top two coordinates. On the other hand, the matrix transformation induced by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is also given by a 3×3 matrix:

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \\ 1 \end{bmatrix} = \begin{bmatrix} A\mathbf{v} \\ 1 \end{bmatrix}$$

So everything can be accomplished at the expense of using 3×3 matrices and homogeneous coordinates.

Example 4.5.1

Rotate the letter A in Figure 4.5.2 through $\frac{\pi}{6}$ about the point $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

Solution. Using homogeneous coordinates for the vertices of the letter results in a data matrix with three rows:

$$K_d = \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

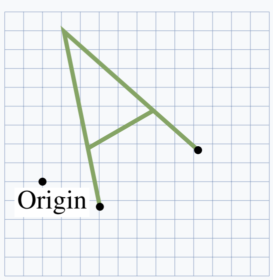


Figure 4.5.6

If we write $\mathbf{w} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, the idea is to use a composite of transformations: First translate the letter by $-\mathbf{w}$ so that the point $\mathbf{w}$ moves to the origin, then rotate this translated letter, and then translate it by $\mathbf{w}$ back to its original position. The matrix arithmetic is as follows (remember the order of composition!):

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3.036 & 8.232 & 5.866 & 2.402 & 1.134 \\ -1.33 & 1.67 & 3.768 & 1.768 & 7.964 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

This is plotted in Figure 4.5.6.

This discussion merely touches the surface of computer graphics, and the reader is referred to specialized books on the subject. Realistic graphic rendering requires an enormous number of matrix calculations. In fact, matrix multiplication algorithms are now embedded in microchip circuits, and can perform over 100 million matrix multiplications per second. This is particularly important in the field of three-dimensional graphics where the homogeneous coordinates have four components and 4×4 matrices are required.

Exercises for 4.5

Exercise 4.5.1 Consider the letter A described in Figure 4.5.2. Find the data matrix for the letter obtained by:

a. Rotating the letter through $\frac{\pi}{4}$ about the origin.

b. Rotating the letter through $\frac{\pi}{4}$ about the point $\left[\begin{array}{c} 1 \\ 2 \end{array}\right]$.

Exercise 4.5.2 Find the matrix for turning the letter A in Figure 4.5.2 upside-down in place.

Exercise 4.5.3 Find the 3×3 matrix for reflecting in the line $y = mx + b$. Use $\left[\begin{array}{c} 1 \\ m \end{array}\right]$ as direction vector for the line.

Exercise 4.5.4 Find the 3×3 matrix for rotating through the angle θ about the point $P(a, b)$.

Exercise 4.5.5 Find the reflection of the point P in the line $y = 1 + 2x$ in $\mathbb{R}^2$ if:

a. $P = P(1, 1)$

b. $P = P(1, 4)$

c. What about $P = P(1, 3)$? Explain. [*Hint*: Example 4.5.1 and Section 4.4.]

Supplementary Exercises for Chapter 4

Exercise 4.1 Suppose that $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors. If $\mathbf{u}$ and $\mathbf{v}$ are not parallel, and $a\mathbf{u} + b\mathbf{v} = a_1\mathbf{u} + b_1\mathbf{v}$, show that $a = a_1$ and $b = b_1$.

Exercise 4.2 Consider a triangle with vertices A, B, and C. Let E and F be the midpoints of sides AB and AC, respectively, and let the medians EC and FB meet at O. Write $\overrightarrow{EO} = s\overrightarrow{EC}$ and $\overrightarrow{FO} = t\overrightarrow{FB}$, where s and t are scalars. Show that $s = t = \frac{1}{3}$ by expressing $\overrightarrow{AO}$ two ways in the form $a\overrightarrow{EO} + b\overrightarrow{AC}$, and applying Exercise 4.1. Conclude that the medians of a triangle meet at the point on each that is one-third of the way from the midpoint to the vertex (and so are concurrent).

Exercise 4.3 A river flows at 1 km/h and a swimmer moves at 2 km/h (relative to the water). At what angle must he swim to go straight across? What is his resulting speed?

Exercise 4.4 A wind is blowing from the south at 75 knots, and an airplane flies heading east at 100 knots. Find the resulting velocity of the airplane.

Exercise 4.5 An airplane pilot flies at 300 km/h in a direction 30° south of east. The wind is blowing from the south at 150 km/h.

a. Find the resulting direction and speed of the airplane.

b. Find the speed of the airplane if the wind is from the west (at 150 km/h).

Exercise 4.6 A rescue boat has a top speed of 13 knots. The captain wants to go due east as fast as possible in water with a current of 5 knots due south. Find the velocity vector $\mathbf{v} = (x, y)$ that she must achieve, assuming the x and y axes point east and north, respectively, and find her resulting speed.

Exercise 4.7 A boat goes 12 knots heading north. The current is 5 knots from the west. In what direction does the boat actually move and at what speed?

Exercise 4.8 Show that the distance from a point A (with vector $\mathbf{a}$) to the plane with vector equation $\mathbf{n} \cdot \mathbf{p} = d$ is $\frac{1}{\|n\|}|\mathbf{n} \cdot \mathbf{a} - d|$.

Exercise 4.9 If two distinct points lie in a plane, show that the line through these points is contained in the plane.

Exercise 4.10 The line through a vertex of a triangle, perpendicular to the opposite side, is called an **altitude** of the triangle. Show that the three altitudes of any triangle are concurrent. (The intersection of the altitudes is called the **orthocentre** of the triangle.) [*Hint*: If P is the intersection of two of the altitudes, show that the line through P and the remaining vertex is perpendicular to the remaining side.]