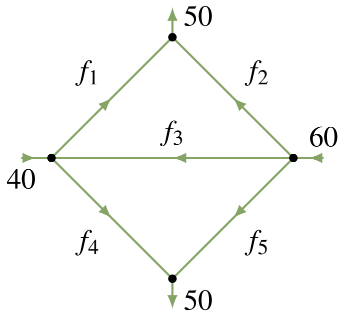
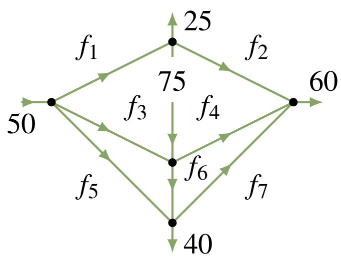
Exercises for 1.4

Exercise 1.4.1 Find the possible flows in each of the following networks of pipes.

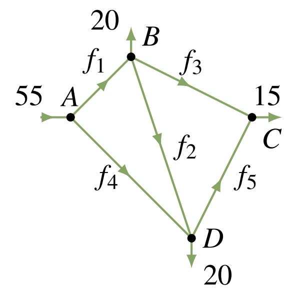
a.



b.



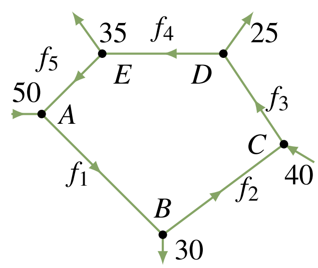
Exercise 1.4.2 A proposed network of irrigation canals is described in the accompanying diagram. At peak demand, the flows at interchanges A, B, C, and D are as shown.



a. Find the possible flows.

b. If canal BC is closed, what range of flow on AD must be maintained so that no canal carries a flow of more than 30?

Exercise 1.4.3 A traffic circle has five one-way streets, and vehicles enter and leave as shown in the accompanying diagram.



a. Compute the possible flows.

b. Which road has the heaviest flow?

1.5 An Application to Electrical Networks[7]

In an electrical network it is often necessary to find the current in amperes (A) flowing in various parts of the network. These networks usually contain resistors that retard the current. The resistors are indicated by a symbol (-\/\/\/-), and the resistance is measured in ohms (Ω). Also, the current is increased at various points by voltage sources (for example, a battery). The voltage of these sources is measured in volts (V),

[7]This section is independent of Section 1.4

and they are represented by the symbol ( ). We assume these voltage sources have no resistance. The flow of current is governed by the following principles.

Ohm's Law

The current I and the voltage drop V across a resistance R are related by the equation $V = RI$.

Kirchhoff's Laws

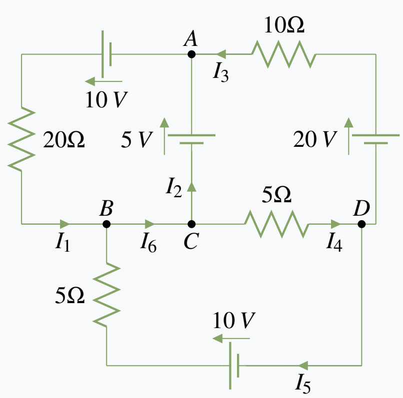
1. *(Junction Rule) The current flow into a junction equals the current flow out of that junction.*
2. *(Circuit Rule) The algebraic sum of the voltage drops (due to resistances) around any closed circuit of the network must equal the sum of the voltage increases around the circuit.*

When applying rule 2, select a direction (clockwise or counterclockwise) around the closed circuit and then consider all voltages and currents positive when in this direction and negative when in the opposite direction. This is why the term *algebraic sum* is used in rule 2. Here is an example.

Example 1.5.1

Find the various currents in the circuit shown.

Solution.



First apply the junction rule at junctions A, B, C, and D to obtain

$$\begin{array}{ll} \text{Junction } A & I_1 = I_2 + I_3 \\ \text{Junction } B & I_6 = I_1 + I_5 \\ \text{Junction } C & I_2 + I_4 = I_6 \\ \text{Junction } D & I_3 + I_5 = I_4 \end{array}$$

Note that these equations are not independent (in fact, the third is an easy consequence of the other three). Next, the circuit rule insists that the sum of the voltage increases (due to the sources) around a closed circuit must equal the sum of the voltage drops (due to resistances). By Ohm's law, the voltage loss across a resistance R (in the direction of the current I) is RI. Going counterclockwise around three closed circuits yields

$$\begin{array}{lrl} \text{Upper left} & 10 + 5 & = 20I_1 \\ \text{Upper right} & -5 + 20 & = 10I_3 + 5I_4 \\ \text{Lower} & -10 & = -20I_5 - 5I_4 \end{array}$$

Hence, disregarding the redundant equation obtained at junction C, we have six equations in the six unknowns $I_1, \ldots, I_6$. The solution is

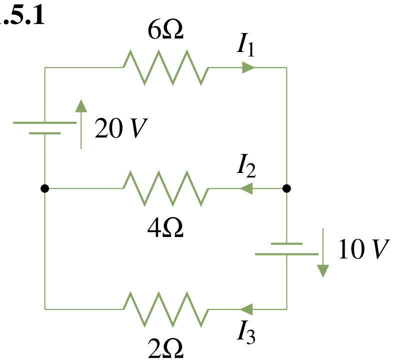
$$I_1 = \frac{15}{20} \quad I_4 = \frac{28}{20}$$
$$I_2 = \frac{-1}{20} \quad I_5 = \frac{12}{20}$$
$$I_3 = \frac{16}{20} \quad I_6 = \frac{27}{20}$$

The fact that I_2 is negative means, of course, that this current is in the opposite direction, with a magnitude of $\frac{1}{20}$ amperes.

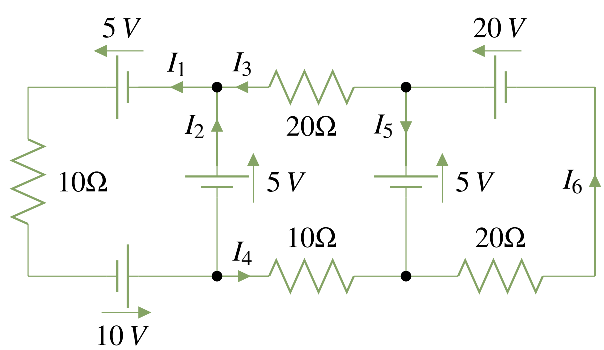
Exercises for 1.5

In Exercises 1 to 4, find the currents in the circuits.

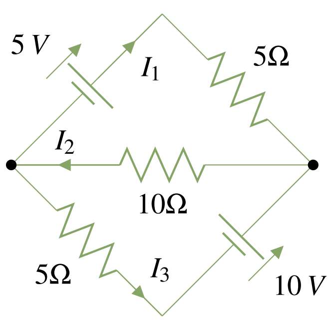
Exercise 1.5.1



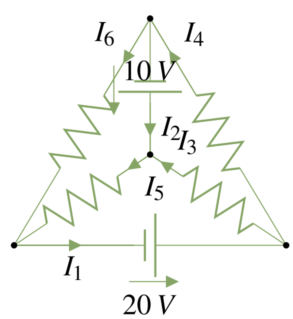
Exercise 1.5.3



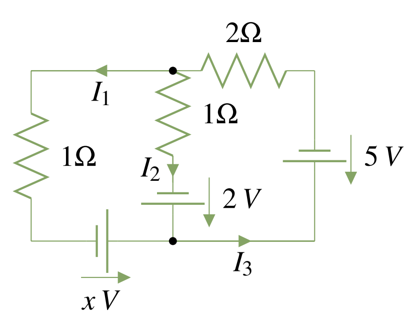
Exercise 1.5.2



Exercise 1.5.4 All resistances are 10Ω.



Exercise 1.5.5
Find the voltage x such that the current $I_1 = 0$.



1.6 An Application to Chemical Reactions

When a chemical reaction takes place a number of molecules combine to produce new molecules. Hence, when hydrogen H_2 and oxygen O_2 molecules combine, the result is water H_2O. We express this as

$$H_2 + O_2 \to H_2O$$

Individual atoms are neither created nor destroyed, so the number of hydrogen and oxygen atoms going into the reaction must equal the number coming out (in the form of water). In this case the reaction is said to be *balanced*. Note that each hydrogen molecule H_2 consists of two atoms as does each oxygen molecule O_2, while a water molecule H_2O consists of two hydrogen atoms and one oxygen atom. In the above reaction, this requires that twice as many hydrogen molecules enter the reaction; we express this as follows:

$$2H_2 + O_2 \to 2H_2O$$

This is now balanced because there are 4 hydrogen atoms and 2 oxygen atoms on each side of the reaction.

Example 1.6.1

Balance the following reaction for burning octane C_8H_{18} in oxygen O_2:

$$C_8H_{18} + O_2 \to CO_2 + H_2O$$

where CO_2 represents carbon dioxide. We must find positive integers x, y, z, and w such that

$$xC_8H_{18} + yO_2 \to zCO_2 + wH_2O$$

Equating the number of carbon, hydrogen, and oxygen atoms on each side gives $8x = z$, $18x = 2w$ and $2y = 2z + w$, respectively. These can be written as a homogeneous linear system

$$\begin{array}{rcrcrcrcl} 8x & & & - & z & & & = & 0 \\ 18x & & & & & - & 2w & = & 0 \\ & & 2y & - & 2z & - & w & = & 0 \end{array}$$

which can be solved by gaussian elimination. In larger systems this is necessary but, in such a simple situation, it is easier to solve directly. Set $w = t$, so that $x = \frac{1}{9}t$, $z = \frac{8}{9}t$, $2y = \frac{16}{9}t + t = \frac{25}{9}t$. But x, y, z, and w must be positive integers, so the smallest value of t that eliminates fractions is 18. Hence, $x = 2$, $y = 25$, $z = 16$, and $w = 18$, and the balanced reaction is

$$2C_8H_{18} + 25O_2 \to 16CO_2 + 18H_2O$$

The reader can verify that this is indeed balanced.

It is worth noting that this problem introduces a new element into the theory of linear equations: the insistence that the solution must consist of positive integers.