2.7 LU-Factorization[15]

The solution to a system $A\mathbf{x} = \mathbf{b}$ of linear equations can be solved quickly if A can be factored as $A = LU$ where L and U are of a particularly nice form. In this section we show that gaussian elimination can be used to find such factorizations.

Triangular Matrices

As for square matrices, if $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is an $m \times n$ matrix, the elements a_{11}, a_{22}, a_{33}, ... form the **main diagonal** of A. Then A is called **upper triangular** if every entry below and to the left of the main diagonal is zero. Every row-echelon matrix is upper triangular, as are the matrices

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -3 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 2 & 1 & 0 & 5 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

By analogy, a matrix A is called **lower triangular** if its transpose is upper triangular, that is if each entry above and to the right of the main diagonal is zero. A matrix is called **triangular** if it is upper or lower triangular.

Example 2.7.1

Solve the system
$$\begin{aligned} x_1 + 2x_2 - 3x_3 - x_4 + 5x_5 &= 3 \\ 5x_3 + x_4 + x_5 &= 8 \\ 2x_5 &= 6 \end{aligned}$$
where the coefficient matrix is upper triangular.

Solution. As in gaussian elimination, let the "non-leading" variables be parameters: $x_2 = s$ and $x_4 = t$. Then solve for x_5, x_3, and x_1 in that order as follows. The last equation gives

$$x_5 = \tfrac{6}{2} = 3$$

Substitution into the second last equation gives

$$x_3 = 1 - \tfrac{1}{5}t$$

Finally, substitution of both x_5 and x_3 into the first equation gives

$$x_1 = -9 - 2s + \tfrac{2}{5}t$$

The method used in Example 2.7.1 is called **back substitution** because later variables are substituted into earlier equations. It works because the coefficient matrix is upper triangular. Similarly, if the coeffi-

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[15]This section is not used later and so may be omitted with no loss of continuity.

cient matrix is lower triangular the system can be solved by **forward substitution** where earlier variables are substituted into later equations. As observed in Section 1.2, these procedures are more numerically efficient than gaussian elimination.

Now consider a system $A\mathbf{x} = \mathbf{b}$ where A can be factored as $A = LU$ where L is lower triangular and U is upper triangular. Then the system $A\mathbf{x} = \mathbf{b}$ can be solved in two stages as follows:

1. *First solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$ by forward substitution.*

2. *Then solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$ by back substitution.*

Then $\mathbf{x}$ is a solution to $A\mathbf{x} = \mathbf{b}$ because $A\mathbf{x} = LU\mathbf{x} = L\mathbf{y} = \mathbf{b}$. Moreover, every solution $\mathbf{x}$ arises this way (take $\mathbf{y} = U\mathbf{x}$). Furthermore the method adapts easily for use in a computer.

This focuses attention on efficiently obtaining such factorizations $A = LU$. The following result will be needed; the proof is straightforward and is left as Exercises 2.7.7 and 2.7.8.

> **Lemma 2.7.1**
>
> *Let A and B denote matrices.*
>
> 1. *If A and B are both lower (upper) triangular, the same is true of AB.*
>
> 2. *If A is $n \times n$ and lower (upper) triangular, then A is invertible if and only if every main diagonal entry is nonzero. In this case A^{-1} is also lower (upper) triangular.*

LU-Factorization

Let A be an $m \times n$ matrix. Then A can be carried to a row-echelon matrix U (that is, upper triangular). As in Section 2.5, the reduction is

$$A \to E_1 A \to E_2 E_1 A \to E_3 E_2 E_1 A \to \cdots \to E_k E_{k-1} \cdots E_2 E_1 A = U$$

where E_1, E_2, ..., E_k are elementary matrices corresponding to the row operations used. Hence

$$A = LU$$

where $L = (E_k E_{k-1} \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1}$. If we do not insist that U is reduced then, except for row interchanges, none of these row operations involve adding a row to a row *above* it. Thus, if no row interchanges are used, all the E_i are *lower* triangular, and so L is lower triangular (and invertible) by Lemma 2.7.1. This proves the following theorem. For convenience, let us say that A can be **lower reduced** if it can be carried to row-echelon form using no row interchanges.

> **Theorem 2.7.1**
>
> If A can be lower reduced to a row-echelon matrix U, then
>
> $$A = LU$$
>
> where L is lower triangular and invertible and U is upper triangular and row-echelon.

> **Definition 2.14 LU-factorization**
>
> A factorization $A = LU$ as in Theorem 2.7.1 is called an **LU-factorization** of A.

Such a factorization may not exist (Exercise 2.7.4) because A cannot be carried to row-echelon form using no row interchange. A procedure for dealing with this situation will be outlined later. However, if an LU-factorization $A = LU$ does exist, then the gaussian algorithm gives U and also leads to a procedure for finding L. Example 2.7.2 provides an illustration. For convenience, the first nonzero column from the left in a matrix A is called the **leading column** of A.

> **Example 2.7.2**
>
> Find an LU-factorization of $A = \begin{bmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & -1 & 3 & 3 & 2 \\ 0 & -1 & 3 & 7 & 10 \end{bmatrix}$.
>
> **Solution.** We lower reduce A to row-echelon form as follows:
>
> $$A = \begin{bmatrix} 0 & \boxed{2} & -6 & -2 & 4 \\ 0 & -1 & 3 & 3 & 2 \\ 0 & -1 & 3 & 7 & 10 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -3 & -1 & 2 \\ 0 & 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 6 & 12 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -3 & -1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$
>
> The circled columns are determined as follows: The first is the leading column of A, and is used (by lower reduction) to create the first leading 1 and create zeros below it. This completes the work on row 1, and we repeat the procedure on the matrix consisting of the remaining rows. Thus the second circled column is the leading column of this smaller matrix, which we use to create the second leading 1 and the zeros below it. As the remaining row is zero here, we are finished. Then $A = LU$ where
>
> $$L = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 6 & 1 \end{bmatrix}$$
>
> This matrix L is obtained from I_3 by replacing the bottom of the first two columns by the circled columns in the reduction. Note that the rank of A is 2 here, and this is the number of circled columns.

The calculation in Example 2.7.2 works in general. There is no need to calculate the elementary

matrices E_i, and the method is suitable for use in a computer because the circled columns can be stored in memory as they are created. The procedure can be formally stated as follows:

LU-Algorithm

Let A be an $m \times n$ matrix of rank r, and suppose that A can be lower reduced to a row-echelon matrix U. Then $A = LU$ where the lower triangular, invertible matrix L is constructed as follows:

1. *If $A = 0$, take $L = I_m$ and $U = 0$.*

2. *If $A \neq 0$, write $A_1 = A$ and let c_1 be the leading column of A_1. Use c_1 to create the first leading 1 and create zeros below it (using lower reduction). When this is completed, let A_2 denote the matrix consisting of rows 2 to m of the matrix just created.*

3. *If $A_2 \neq 0$, let c_2 be the leading column of A_2 and repeat Step 2 on A_2 to create A_3.*

4. *Continue in this way until U is reached, where all rows below the last leading 1 consist of zeros. This will happen after r steps.*

5. *Create L by placing c_1, c_2, $\ldots$, c_r at the bottom of the first r columns of I_m.*

A proof of the LU-algorithm is given at the end of this section.

LU-factorization is particularly important if, as often happens in business and industry, a series of equations $A\mathbf{x} = B_1$, $A\mathbf{x} = B_2$, $\ldots$, $A\mathbf{x} = B_k$, must be solved, each with the same coefficient matrix A. It is very efficient to solve the first system by gaussian elimination, simultaneously creating an LU-factorization of A, and then using the factorization to solve the remaining systems by forward and back substitution.

Example 2.7.3

Find an LU-factorization for $A = \begin{bmatrix} 5 & -5 & 10 & 0 & 5 \\ -3 & 3 & 2 & 2 & 1 \\ -2 & 2 & 0 & -1 & 0 \\ 1 & -1 & 10 & 2 & 5 \end{bmatrix}$.

Solution. The reduction to row-echelon form is

$$\begin{bmatrix} 5 & -5 & 10 & 0 & 5 \\ -3 & 3 & 2 & 2 & 1 \\ -2 & 2 & 0 & -1 & 0 \\ 1 & -1 & 10 & 2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 8 & 2 & 4 \\ 0 & 0 & 4 & -1 & 2 \\ 0 & 0 & 8 & 2 & 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

If U denotes this row-echelon matrix, then $A = LU$, where

$$L = \begin{bmatrix} 5 & 0 & 0 & 0 \\ -3 & 8 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 1 & 8 & 0 & 1 \end{bmatrix}$$

The next example deals with a case where no row of zeros is present in U (in fact, A is invertible).

Example 2.7.4

Find an LU-factorization for $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}$.

Solution. The reduction to row-echelon form is

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

Hence $A = LU$ where $L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & 5 \end{bmatrix}$.

There are matrices (for example $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$) that have no LU-factorization and so require at least one row interchange when being carried to row-echelon form via the gaussian algorithm. However, it turns out that, if all the row interchanges encountered in the algorithm are carried out first, the resulting matrix requires no interchanges and so has an LU-factorization. Here is the precise result.

Theorem 2.7.2

Suppose an $m \times n$ matrix A is carried to a row-echelon matrix U via the gaussian algorithm. Let P_1, P_2, ..., P_s be the elementary matrices corresponding (in order) to the row interchanges used, and write $P = P_s \cdots P_2 P_1$. (If no interchanges are used take $P = I_m$.) Then:

1. *PA is the matrix obtained from A by doing these interchanges (in order) to A.*

2. *PA has an LU-factorization.*

The proof is given at the end of this section.

A matrix P that is the product of elementary matrices corresponding to row interchanges is called a **permutation matrix**. Such a matrix is obtained from the identity matrix by arranging the rows in a different order, so it has exactly one 1 in each row and each column, and has zeros elsewhere. We regard the identity matrix as a permutation matrix. The elementary permutation matrices are those obtained from I by a single row interchange, and every permutation matrix is a product of elementary ones.

Example 2.7.5

If $A = \begin{bmatrix} 0 & 0 & -1 & 2 \\ -1 & -1 & 1 & 2 \\ 2 & 1 & -3 & 6 \\ 0 & 1 & -1 & 4 \end{bmatrix}$, find a permutation matrix P such that PA has an LU-factorization, and then find the factorization.

Solution. Apply the gaussian algorithm to A:

$$A \xrightarrow{*} \begin{bmatrix} -1 & -1 & 1 & 2 \\ 0 & 0 & -1 & 2 \\ 2 & 1 & -3 & 6 \\ 0 & 1 & -1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 0 & -1 & 2 \\ 0 & -1 & -1 & 10 \\ 0 & 1 & -1 & 4 \end{bmatrix} \xrightarrow{*} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & -1 & -1 & 10 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -2 & 14 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

Two row interchanges were needed (marked with $*$), first rows 1 and 2 and then rows 2 and 3. Hence, as in Theorem 2.7.2,

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If we do these interchanges (in order) to A, the result is PA. Now apply the LU-algorithm to PA:

$$PA = \begin{bmatrix} -1 & -1 & 1 & 2 \\ 2 & 1 & -3 & 6 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & -1 & -1 & 10 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -2 & 14 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U$$

Hence, $PA = LU$, where $L = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 10 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Theorem 2.7.2 provides an important general factorization theorem for matrices. If A is any $m \times n$ matrix, it asserts that there exists a permutation matrix P and an LU-factorization $PA = LU$. Moreover, it shows that either $P = I$ or $P = P_s \cdots P_2 P_1$, where P_1, P_2, ..., P_s are the elementary permutation matrices arising in the reduction of A to row-echelon form. Now observe that $P_i^{-1} = P_i$ for each i (they are elementary row interchanges). Thus, $P^{-1} = P_1 P_2 \cdots P_s$, so the matrix A can be factored as

$$A = P^{-1}LU$$

where P^{-1} is a permutation matrix, L is lower triangular and invertible, and U is a row-echelon matrix. This is called a **PLU-factorization** of A.

The LU-factorization in Theorem 2.7.1 is not unique. For example,

$$\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

However, it is necessary here that the row-echelon matrix has a row of zeros. Recall that the rank of a matrix A is the number of nonzero rows in any row-echelon matrix U to which A can be carried by row operations. Thus, if A is $m \times n$, the matrix U has no row of zeros if and only if A has rank m.

Theorem 2.7.3

Let A be an $m \times n$ matrix that has an LU-factorization

$$A = LU$$

If A has rank m (that is, U has no row of zeros), then L and U are uniquely determined by A.

Proof. Suppose $A = MV$ is another LU-factorization of A, so M is lower triangular and invertible and V is row-echelon. Hence $LU = MV$, and we must show that $L = M$ and $U = V$. We write $N = M^{-1}L$. Then N

is lower triangular and invertible (Lemma 2.7.1) and $NU = V$, so it suffices to prove that $N = I$. If N is $m \times m$, we use induction on m. The case $m = 1$ is left to the reader. If $m > 1$, observe first that column 1 of V is N times column 1 of U. Thus if either column is zero, so is the other (N is invertible). Hence, we can assume (by deleting zero columns) that the $(1, 1)$-entry is 1 in both U and V.

Now we write $N = \begin{bmatrix} a & 0 \\ X & N_1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & Y \\ 0 & U_1 \end{bmatrix}$, and $V = \begin{bmatrix} 1 & Z \\ 0 & V_1 \end{bmatrix}$ in block form. Then $NU = V$ becomes $\begin{bmatrix} a & aY \\ X & XY + N_1U_1 \end{bmatrix} = \begin{bmatrix} 1 & Z \\ 0 & V_1 \end{bmatrix}$. Hence $a = 1$, $Y = Z$, $X = 0$, and $N_1U_1 = V_1$. But $N_1U_1 = V_1$ implies $N_1 = I$ by induction, whence $N = I$. $\qquad \square$

If A is an $m \times m$ invertible matrix, then A has rank m by Theorem 2.4.5. Hence, we get the following important special case of Theorem 2.7.3.

Corollary 2.7.1

If an invertible matrix A has an LU-factorization $A = LU$, then L and U are uniquely determined by A.

Of course, in this case U is an upper triangular matrix with 1s along the main diagonal.

Proofs of Theorems

Proof of the LU-Algorithm. If $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_r$ are columns of lengths $m, m-1, \ldots, m-r+1$, respectively, write $L^{(m)}(\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_r)$ for the lower triangular $m \times m$ matrix obtained from I_m by placing $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_r$ at the bottom of the first r columns of I_m.

Proceed by induction on n. If $A = 0$ or $n = 1$, it is left to the reader. If $n > 1$, let $\mathbf{c}_1$ denote the leading column of A and let $\mathbf{k}_1$ denote the first column of the $m \times m$ identity matrix. There exist elementary matrices $E_1, \ldots, E_k$ such that, in block form,

$$(E_k \cdots E_2E_1)A = \begin{bmatrix} 0 & \mathbf{k}_1 & \dfrac{X_1}{A_1} \end{bmatrix} \quad \text{where } (E_k \cdots E_2E_1)\mathbf{c}_1 = \mathbf{k}_1$$

Moreover, each E_j can be taken to be lower triangular (by assumption). Write

$$G = (E_k \cdots E_2E_1)^{-1} = E_1^{-1}E_2^{-1} \cdots E_k^{-1}$$

Then G is lower triangular, and $G\mathbf{k}_1 = \mathbf{c}_1$. Also, each E_j (and so each E_j^{-1}) is the result of either multiplying row 1 of I_m by a constant or adding a multiple of row 1 to another row. Hence,

$$G = (E_1^{-1}E_2^{-1} \cdots E_k^{-1})I_m = \begin{bmatrix} \mathbf{c}_1 & \dfrac{0}{I_{m-1}} \end{bmatrix}$$

in block form. Now, by induction, let $A_1 = L_1U_1$ be an LU-factorization of A_1, where $L_1 = L^{(m-1)}[\mathbf{c}_2, \ldots, \mathbf{c}_r]$ and U_1 is row-echelon. Then block multiplication gives

$$G^{-1}A = \begin{bmatrix} 0 & \mathbf{k}_1 & \dfrac{X_1}{L_1U_1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & L_1 \end{bmatrix} \begin{bmatrix} 0 & 1 & X_1 \\ 0 & 0 & U_1 \end{bmatrix}$$

Hence $A = LU$, where $U = \begin{bmatrix} 0 & 1 & X_1 \\ 0 & 0 & U_1 \end{bmatrix}$ is row-echelon and

$$L = \begin{bmatrix} \mathbf{c}_1 & \begin{array}{c} 0 \\ I_{m-1} \end{array} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & L_1 \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 & \begin{array}{c} 0 \\ L \end{array} \end{bmatrix} = L^{(m)}[\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_r]$$

This completes the proof. ☐

Proof of Theorem 2.7.2. Let A be a nonzero $m \times n$ matrix and let $\mathbf{k}_j$ denote column j of I_m. There is a permutation matrix P_1 (where either P_1 is elementary or $P_1 = I_m$) such that the first nonzero column $\mathbf{c}_1$ of $P_1 A$ has a nonzero entry on top. Hence, as in the LU-algorithm,

$$L^{(m)}[\mathbf{c}_1]^{-1} \cdot P_1 \cdot A = \begin{bmatrix} 0 & 1 & X_1 \\ 0 & 0 & A_1 \end{bmatrix}$$

in block form. Then let P_2 be a permutation matrix (either elementary or I_m) such that

$$P_2 \cdot L^{(m)}[\mathbf{c}_1]^{-1} \cdot P_1 \cdot A = \begin{bmatrix} 0 & 1 & X_1 \\ 0 & 0 & A_1' \end{bmatrix}$$

and the first nonzero column $\mathbf{c}_2$ of A_1' has a nonzero entry on top. Thus,

$$L^{(m)}[\mathbf{k}_1, \mathbf{c}_2]^{-1} \cdot P_2 \cdot L^{(m)}[\mathbf{c}_1]^{-1} \cdot P_1 \cdot A = \begin{bmatrix} 0 & 1 & & X_1 \\ 0 & 0 & \begin{array}{cc} 0 & 1 & X_2 \\ 0 & 0 & A_2 \end{array} \end{bmatrix}$$

in block form. Continue to obtain elementary permutation matrices $P_1, P_2, \ldots, P_r$ and columns $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_r$ of lengths m, $m-1$, $\ldots$, such that

$$(L_r P_r L_{r-1} P_{r-1} \cdots L_2 P_2 L_1 P_1) A = U$$

where U is a row-echelon matrix and $L_j = L^{(m)}[\mathbf{k}_1, \ldots, \mathbf{k}_{j-1}, \mathbf{c}_j]^{-1}$ for each j, where the notation means the first $j-1$ columns are those of I_m. It is not hard to verify that each L_j has the form $L_j = L^{(m)}[\mathbf{k}_1, \ldots, \mathbf{k}_{j-1}, \mathbf{c}_j']$ where $\mathbf{c}_j'$ is a column of length $m - j + 1$. We now claim that each permutation matrix P_k can be "moved past" each matrix L_j to the right of it, in the sense that

$$P_k L_j = L_j' P_k$$

where $L_j' = L^{(m)}[\mathbf{k}_1, \ldots, \mathbf{k}_{j-1}, \mathbf{c}_j'']$ for some column $\mathbf{c}_j''$ of length $m - j + 1$. Given that this is true, we obtain a factorization of the form

$$(L_r L_{r-1}' \cdots L_2' L_1')(P_r P_{r-1} \cdots P_2 P_1) A = U$$

If we write $P = P_r P_{r-1} \cdots P_2 P_1$, this shows that PA has an LU-factorization because $L_r L_{r-1}' \cdots L_2' L_1'$ is lower triangular and invertible. All that remains is to prove the following rather technical result. ☐

Lemma 2.7.2

Let P_k result from interchanging row k of I_m with a row below it. If $j < k$, let c_j be a column of length $m - j + 1$. Then there is another column c'_j of length $m - j + 1$ such that

$$P_k \cdot L^{(m)} \left[k_1, \ \ldots, \ k_{j-1}, \ c_j \right] = L^{(m)} \left[k_1, \ \ldots, \ k_{j-1}, \ c'_j \right] \cdot P_k$$

The proof is left as Exercise 2.7.11.

Exercises for 2.7

Exercise 2.7.1 Find an LU-factorization of the following matrices.

a. $\begin{bmatrix} 2 & 6 & -2 & 0 & 2 \\ 3 & 9 & -3 & 3 & 1 \\ -1 & -3 & 1 & -3 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 4 & 2 \\ 1 & -1 & 3 \\ -1 & 7 & -7 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 6 & -2 & 0 & 2 \\ 1 & 5 & -1 & 2 & 5 \\ 3 & 7 & -3 & -2 & 5 \\ -1 & -1 & 1 & 2 & 3 \end{bmatrix}$

d. $\begin{bmatrix} -1 & -3 & 1 & 0 & -1 \\ 1 & 4 & 1 & 1 & 1 \\ 1 & 2 & -3 & -1 & 1 \\ 0 & -2 & -4 & -2 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 2 & 2 & 4 & 6 & 0 & 2 \\ 1 & -1 & 2 & 1 & 3 & 1 \\ -2 & 2 & -4 & -1 & 1 & 6 \\ 0 & 2 & 0 & 3 & 4 & 8 \\ -2 & 4 & -4 & 1 & -2 & 6 \end{bmatrix}$

f. $\begin{bmatrix} 2 & 2 & -2 & 4 & 2 \\ 1 & -1 & 0 & 2 & 1 \\ 3 & 1 & -2 & 6 & 3 \\ 1 & 3 & -2 & 2 & 1 \end{bmatrix}$

Exercise 2.7.2 Find a permutation matrix P and an LU-factorization of PA if A is:

a. $\begin{bmatrix} 0 & 0 & 2 \\ 0 & -1 & 4 \\ 3 & 5 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 4 \\ -1 & 2 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 0 & -1 & 2 & 1 & 3 \\ -1 & 1 & 3 & 1 & 4 \\ 1 & -1 & -3 & 6 & 2 \\ 2 & -2 & -4 & 1 & 0 \end{bmatrix}$

d. $\begin{bmatrix} -1 & -2 & 3 & 0 \\ 2 & 4 & -6 & 5 \\ 1 & 1 & -1 & 3 \\ 2 & 5 & -10 & 1 \end{bmatrix}$

Exercise 2.7.3 In each case use the given LU-decomposition of A to solve the system $A\mathbf{x} = \mathbf{b}$ by finding $\mathbf{y}$ such that $L\mathbf{y} = \mathbf{b}$, and then $\mathbf{x}$ such that $U\mathbf{x} = \mathbf{y}$:

a. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$;

$\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$;

$\mathbf{b} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$

c. $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$;

$\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$

d. $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 3 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$;

$\mathbf{b} = \begin{bmatrix} 4 \\ -6 \\ 4 \\ 5 \end{bmatrix}$

Exercise 2.7.4 Show that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = LU$ is impossible where L is lower triangular and U is upper triangular.

Exercise 2.7.5 Show that we can accomplish any row interchange by using only row operations of other types.

Exercise 2.7.6

a. Let L and L_1 be invertible lower triangular matrices, and let U and U_1 be invertible upper triangular matrices. Show that $LU = L_1 U_1$ if and only if there exists an invertible diagonal matrix D such that $L_1 = LD$ and $U_1 = D^{-1}U$. [*Hint*: Scrutinize $L^{-1}L_1 = UU_1^{-1}$.]

b. Use part (a) to prove Theorem 2.7.3 in the case that A is invertible.

Exercise 2.7.7 Prove Lemma 2.7.1(1). [*Hint*: Use block multiplication and induction.]

Exercise 2.7.8 Prove Lemma 2.7.1(2). [*Hint*: Use block multiplication and induction.]

Exercise 2.7.9 A triangular matrix is called **unit triangular** if it is square and every main diagonal element is a 1.

a. If A can be carried by the gaussian algorithm to row-echelon form using no row interchanges, show that $A = LU$ where L is unit lower triangular and U is upper triangular.

b. Show that the factorization in (a.) is unique.

Exercise 2.7.10 Let $\mathbf{c}_1$, $\mathbf{c}_2$, ..., $\mathbf{c}_r$ be columns of lengths m, $m-1$, ..., $m-r+1$. If $\mathbf{k}_j$ denotes column j of I_m, show that $L^{(m)}[\mathbf{c}_1, \mathbf{c}_2, ..., \mathbf{c}_r] = L^{(m)}[\mathbf{c}_1]L^{(m)}[\mathbf{k}_1, \mathbf{c}_2]L^{(m)}[\mathbf{k}_1, \mathbf{k}_2, \mathbf{c}_3] \cdots L^{(m)}[\mathbf{k}_1, \mathbf{k}_2, ..., \mathbf{k}_{r-1}, \mathbf{c}_r]$. The notation is as in the proof of Theorem 2.7.2. [*Hint*: Use induction on m and block multiplication.]

Exercise 2.7.11 Prove Lemma 2.7.2. [*Hint*: $P_k^{-1} = P_k$. Write $P_k = \begin{bmatrix} I_k & 0 \\ 0 & P_0 \end{bmatrix}$ in block form where P_0 is an $(m-k) \times (m-k)$ permutation matrix.]

2.8 An Application to Input-Output Economic Models[16]

In 1973 Wassily Leontief was awarded the Nobel prize in economics for his work on mathematical models.[17] Roughly speaking, an economic system in this model consists of several industries, each of which produces a product and each of which uses some of the production of the other industries. The following example is typical.

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[16]The applications in this section and the next are independent and may be taken in any order.

[17]See W. W. Leontief, "The world economy of the year 2000," *Scientific American,* Sept. 1980.