# Supplementary Exercises for Chapter 2

**Exercise 2.1** Solve for the matrix $X$ if:

a. $PXQ = R$;

b. $XP = S$;

where $P = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 3 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix}$,
$R = \begin{bmatrix} -1 & 1 & -4 \\ -4 & 0 & -6 \\ 6 & 6 & -6 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 6 \\ 3 & 1 \end{bmatrix}$

**Exercise 2.2** Consider

$$p(X) = X^3 - 5X^2 + 11X - 4I.$$

a. If $p(U) = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}$ compute $p(U^T)$.

b. If $p(U) = 0$ where $U$ is $n \times n$, find $U^{-1}$ in terms of $U$.

**Exercise 2.3** Show that, if a (possibly nonhomogeneous) system of equations is consistent and has more variables than equations, then it must have infinitely many solutions. [*Hint*: Use Theorem 2.2.2 and Theorem 1.3.1.]

**Exercise 2.4** Assume that a system $A\mathbf{x} = \mathbf{b}$ of linear equations has at least two distinct solutions $\mathbf{y}$ and $\mathbf{z}$.

a. Show that $\mathbf{x}_k = \mathbf{y} + k(\mathbf{y} - \mathbf{z})$ is a solution for every $k$.

b. Show that $\mathbf{x}_k = \mathbf{x}_m$ implies $k = m$. [*Hint*: See Example 2.1.7.]

c. Deduce that $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

**Exercise 2.5**

a. Let $A$ be a $3 \times 3$ matrix with all entries on and below the main diagonal zero. Show that $A^3 = 0$.

b. Generalize to the $n \times n$ case and prove your answer.

**Exercise 2.6** Let $I_{pq}$ denote the $n \times n$ matrix with $(p, q)$-entry equal to 1 and all other entries 0. Show that:

a. $I_n = I_{11} + I_{22} + \cdots + I_{nn}$.

b. $I_{pq}I_{rs} = \begin{cases} I_{ps} & \text{if } q = r \\ 0 & \text{if } q \neq r \end{cases}$.

c. If $A = [a_{ij}]$ is $n \times n$, then $A = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}I_{ij}$.

d. If $A = [a_{ij}]$, then $I_{pq}AI_{rs} = a_{qr}I_{ps}$ for all $p$, $q$, $r$, and $s$.

**Exercise 2.7** A matrix of the form $aI_n$, where $a$ is a number, is called an $n \times n$ **scalar matrix**.

a. Show that each $n \times n$ scalar matrix commutes with every $n \times n$ matrix.

b. Show that $A$ is a scalar matrix if it commutes with every $n \times n$ matrix. [*Hint*: See part (d.) of Exercise 2.6.]

**Exercise 2.8** Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where $A$, $B$, $C$, and $D$ are all $n \times n$ and each commutes with all the others. If $M^2 = 0$, show that $(A + D)^3 = 0$. [*Hint*: First show that $A^2 = -BC = D^2$ and that

$$B(A + D) = 0 = C(A + D).]$$

**Exercise 2.9** If $A$ is $2 \times 2$, show that $A^{-1} = A^T$ if and only if $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ for some $\theta$ or
$A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ for some $\theta$.

[*Hint*: If $a^2 + b^2 = 1$, then $a = \cos\theta$, $b = \sin\theta$ for some $\theta$. Use

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi.]$$

**Exercise 2.10**

a. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, show that $A^2 = I$.

b. What is wrong with the following argument? If $A^2 = I$, then $A^2 - I = 0$, so $(A - I)(A + I) = 0$, whence $A = I$ or $A = -I$.

**Exercise 2.11** Let $E$ and $F$ be elementary matrices obtained from the identity matrix by adding multiples of row $k$ to rows $p$ and $q$. If $k \neq p$ and $k \neq q$, show that $EF = FE$.

**Exercise 2.12** If $A$ is a $2 \times 2$ real matrix, $A^2 = A$ and $A^T = A$, show that either $A$ is one of $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, or $A = \begin{bmatrix} a & b \\ b & 1-a \end{bmatrix}$ where $a^2 + b^2 = a$, $-\frac{1}{2} \leq b \leq \frac{1}{2}$ and $b \neq 0$.

**Exercise 2.13** Show that the following are equivalent for matrices $P$, $Q$:

1. $P$, $Q$, and $P+Q$ are all invertible and

$$(P+Q)^{-1} = P^{-1} + Q^{-1}$$

2. $P$ is invertible and $Q = PG$ where $G^2 + G + I = 0$.