Exercise 6.2.20 If $\mathbf{P}_n = \text{span}\{p_1(x), p_2(x), \dots, p_k(x)\}$ and a is in $\mathbb{R}$, show that $p_i(a) \neq 0$ for some i.

Exercise 6.2.21 Let U be a subspace of a vector space V.

a. If $a\mathbf{u}$ is in U where $a \neq 0$, show that $\mathbf{u}$ is in U.

b. If $\mathbf{u}$ and $\mathbf{u}+\mathbf{v}$ are in U, show that $\mathbf{v}$ is in U.

Exercise 6.2.22 Let U be a nonempty subset of a vector space V. Show that U is a subspace of V if and only if $\mathbf{u}_1 + a\mathbf{u}_2$ lies in U for all $\mathbf{u}_1$ and $\mathbf{u}_2$ in U and all a in $\mathbb{R}$.

Exercise 6.2.23 Let $U = \{p(x) \text{ in } \mathbf{P} \mid p(3) = 0\}$ be the set in Example 6.2.4. Use the factor theorem (see Section 6.5) to show that U consists of multiples of $x-3$; that is, show that $U = \{(x-3)q(x) \mid q(x) \in \mathbf{P}\}$. Use this to show that U is a subspace of $\mathbf{P}$.

Exercise 6.2.24 Let $A_1, A_2, \dots, A_m$ denote $n \times n$ matrices. If $\mathbf{0} \neq \mathbf{y} \in \mathbb{R}^n$ and $A_1\mathbf{y} = A_2\mathbf{y} = \cdots = A_m\mathbf{y} = \mathbf{0}$, show that $\{A_1, A_2, \dots, A_m\}$ cannot span $\mathbf{M}_{nn}$.

Exercise 6.2.25 Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ be sets of vectors in a vector space, and let

$$X = \left[\begin{array}{c} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{array}\right] \quad Y = \left[\begin{array}{c} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{array}\right]$$

as in Exercise 6.1.18.

a. Show that $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subseteq \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ if and only if $AY = X$ for some $n \times n$ matrix A.

b. If $X = AY$ where A is invertible, show that $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\} = \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$.

Exercise 6.2.26 If U and W are subspaces of a vector space V, let $U \cup W = \{\mathbf{v} \mid \mathbf{v} \text{ is in } U \text{ or } \mathbf{v} \text{ is in } W\}$. Show that $U \cup W$ is a subspace if and only if $U \subseteq W$ or $W \subseteq U$.

Exercise 6.2.27 Show that $\mathbf{P}$ cannot be spanned by a finite set of polynomials.

6.3 Linear Independence and Dimension

Definition 6.4 Linear Independence and Dependence

*As in $\mathbb{R}^n$, a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ in a vector space V is called **linearly independent** (or simply **independent**) if it satisfies the following condition:*

$$\text{If} \quad s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_n\mathbf{v}_n = \mathbf{0}, \qquad \text{then} \quad s_1 = s_2 = \cdots = s_n = 0.$$

*A set of vectors that is not linearly independent is said to be **linearly dependent** (or simply **dependent**).*

The **trivial linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is the one with every coefficient zero:

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n$$

This is obviously one way of expressing $\mathbf{0}$ as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$, and they are linearly independent when it is the *only* way.

Example 6.3.1

Show that $\{1+x,\ 3x+x^2,\ 2+x-x^2\}$ is independent in $\mathbf{P}_2$.

Solution. Suppose a linear combination of these polynomials vanishes.

$$s_1(1+x)+s_2(3x+x^2)+s_3(2+x-x^2)=0$$

Equating the coefficients of 1, x, and x^2 gives a set of linear equations.

$$\begin{aligned} s_1 + + 2s_3 &= 0 \\ s_1 + 3s_2 + s_3 &= 0 \\ s_2 - s_3 &= 0 \end{aligned}$$

The only solution is $s_1 = s_2 = s_3 = 0$.

Example 6.3.2

Show that $\{\sin x,\ \cos x\}$ is independent in the vector space $\mathbf{F}[0,\ 2\pi]$ of functions defined on the interval $[0,\ 2\pi]$.

Solution. Suppose that a linear combination of these functions vanishes.

$$s_1(\sin x)+s_2(\cos x)=0$$

This must hold for *all* values of x in $[0,\ 2\pi]$ (by the definition of equality in $\mathbf{F}[0,\ 2\pi]$). Taking $x=0$ yields $s_2=0$ (because $\sin 0=0$ and $\cos 0=1$). Similarly, $s_1=0$ follows from taking $x=\frac{\pi}{2}$ (because $\sin\frac{\pi}{2}=1$ and $\cos\frac{\pi}{2}=0$).

Example 6.3.3

Suppose that $\{\mathbf{u},\ \mathbf{v}\}$ is an independent set in a vector space V. Show that $\{\mathbf{u}+2\mathbf{v},\ \mathbf{u}-3\mathbf{v}\}$ is also independent.

Solution. Suppose a linear combination of $\mathbf{u}+2\mathbf{v}$ and $\mathbf{u}-3\mathbf{v}$ vanishes:

$$s(\mathbf{u}+2\mathbf{v})+t(\mathbf{u}-3\mathbf{v})=\mathbf{0}$$

We must deduce that $s=t=0$. Collecting terms involving $\mathbf{u}$ and $\mathbf{v}$ gives

$$(s+t)\mathbf{u}+(2s-3t)\mathbf{v}=\mathbf{0}$$

Because $\{\mathbf{u},\ \mathbf{v}\}$ is independent, this yields linear equations $s+t=0$ and $2s-3t=0$. The only solution is $s=t=0$.

Example 6.3.4

Show that any set of polynomials of distinct degrees is independent.

Solution. Let $p_1, p_2, \dots, p_m$ be polynomials where $\deg(p_i) = d_i$. By relabelling if necessary, we may assume that $d_1 > d_2 > \cdots > d_m$. Suppose that a linear combination vanishes:

$$t_1 p_1 + t_2 p_2 + \cdots + t_m p_m = 0$$

where each t_i is in $\mathbb{R}$. As $\deg(p_1) = d_1$, let ax^{d_1} be the term in p_1 of highest degree, where $a \neq 0$. Since $d_1 > d_2 > \cdots > d_m$, it follows that $t_1 ax^{d_1}$ is the only term of degree d_1 in the linear combination $t_1 p_1 + t_2 p_2 + \cdots + t_m p_m = 0$. This means that $t_1 ax^{d_1} = 0$, whence $t_1 a = 0$, hence $t_1 = 0$ (because $a \neq 0$). But then $t_2 p_2 + \cdots + t_m p_m = 0$ so we can repeat the argument to show that $t_2 = 0$. Continuing, we obtain $t_i = 0$ for each i, as desired.

Example 6.3.5

Suppose that A is an $n \times n$ matrix such that $A^k = 0$ but $A^{k-1} \neq 0$. Show that $B = \{I, A, A^2, \dots, A^{k-1}\}$ is independent in $\mathbf{M}_{nn}$.

Solution. Suppose $r_0 I + r_1 A + r_2 A^2 + \cdots + r_{k-1} A^{k-1} = 0$. Multiply by A^{k-1}:

$$r_0 A^{k-1} + r_1 A^k + r_2 A^{k+1} + \cdots + r_{k-1} A^{2k-2} = 0$$

Since $A^k = 0$, all the higher powers are zero, so this becomes $r_0 A^{k-1} = 0$. But $A^{k-1} \neq 0$, so $r_0 = 0$, and we have $r_1 A^1 + r_2 A^2 + \cdots + r_{k-1} A^{k-1} = 0$. Now multiply by A^{k-2} to conclude that $r_1 = 0$. Continuing, we obtain $r_i = 0$ for each i, so B is independent.

The next example collects several useful properties of independence for reference.

Example 6.3.6

Let V denote a vector space.

1. If $\mathbf{v} \neq \mathbf{0}$ in V, then $\{\mathbf{v}\}$ is an independent set.
2. No independent set of vectors in V can contain the zero vector.

Solution.

1. Let $t\mathbf{v} = \mathbf{0}$, t in $\mathbb{R}$. If $t \neq 0$, then $\mathbf{v} = 1\mathbf{v} = \frac{1}{t}(t\mathbf{v}) = \frac{1}{t}\mathbf{0} = \mathbf{0}$, contrary to assumption. So $t = 0$.
2. If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is independent and (say) $\mathbf{v}_2 = \mathbf{0}$, then $0\mathbf{v}_1 + 1\mathbf{v}_2 + \cdots + 0\mathbf{v}_k = \mathbf{0}$ is a nontrivial linear combination that vanishes, contrary to the independence of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$.

A set of vectors is independent if $\mathbf{0}$ is a linear combination in a unique way. The following theorem shows that *every* linear combination of these vectors has uniquely determined coefficients, and so extends Theorem 5.2.1.

Theorem 6.3.1

Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a linearly independent set of vectors in a vector space V. If a vector $\mathbf{v}$ has two (ostensibly different) representations

$$\begin{aligned}\mathbf{v} &= s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_n\mathbf{v}_n \\ \mathbf{v} &= t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_n\mathbf{v}_n\end{aligned}$$

as linear combinations of these vectors, then $s_1 = t_1, s_2 = t_2, \dots, s_n = t_n$. In other words, every vector in V can be written in a unique way as a linear combination of the $\mathbf{v}_i$.

Proof. Subtracting the equations given in the theorem gives

$$(s_1 - t_1)\mathbf{v}_1 + (s_2 - t_2)\mathbf{v}_2 + \cdots + (s_n - t_n)\mathbf{v}_n = \mathbf{0}$$

The independence of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ gives $s_i - t_i = 0$ for each i, as required. □

The following theorem extends (and proves) Theorem 5.2.4, and is one of the most useful results in linear algebra.

Theorem 6.3.2: Fundamental Theorem

Suppose a vector space V can be spanned by n vectors. If any set of m vectors in V is linearly independent, then $m \leq n$.

Proof. Let $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, and suppose that $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ is an independent set in V. Then $\mathbf{u}_1 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$ where each a_i is in $\mathbb{R}$. As $\mathbf{u}_1 \neq \mathbf{0}$ (Example 6.3.6), not all of the a_i are zero, say $a_1 \neq 0$ (after relabelling the $\mathbf{v}_i$). Then $V = \text{span}\{\mathbf{u}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$ as the reader can verify. Hence, write $\mathbf{u}_2 = b_1\mathbf{u}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_n\mathbf{v}_n$. Then some $c_i \neq 0$ because $\{\mathbf{u}_1, \mathbf{u}_2\}$ is independent; so, as before, $V = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$, again after possible relabelling of the $\mathbf{v}_i$. If $m > n$, this procedure continues until all the vectors $\mathbf{v}_i$ are replaced by the vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$. In particular, $V = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$. But then $\mathbf{u}_{n+1}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ contrary to the independence of the $\mathbf{u}_i$. Hence, the assumption $m > n$ cannot be valid, so $m \leq n$ and the theorem is proved. □

If $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, and if $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ is an independent set in V, the above proof shows not only that $m \leq n$ but also that m of the (spanning) vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ can be replaced by the (independent) vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ and the resulting set will still span V. In this form the result is called the **Steinitz Exchange Lemma**.

Definition 6.5 Basis of a Vector Space

As in $\mathbb{R}^n$, a set $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ of vectors in a vector space V is called a ***basis*** *of V if it satisfies the following two conditions:*

1. $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ *is linearly independent*
2. $V = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$

Thus if a set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis, then *every* vector in V can be written as a linear combination of these vectors in a *unique* way (Theorem 6.3.1). But even more is true: Any two (finite) bases of V contain the same number of vectors.

Theorem 6.3.3: Invariance Theorem

Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ and $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ be two bases of a vector space V. Then $n = m$.

Proof. Because $V = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ and $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ is independent, it follows from Theorem 6.3.2 that $m \leq n$. Similarly $n \leq m$, so $n = m$, as asserted. □

Theorem 6.3.3 guarantees that no matter which basis of V is chosen it contains the same number of vectors as any other basis. Hence there is no ambiguity about the following definition.

Definition 6.6 Dimension of a Vector Space

If $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis of the nonzero vector space V, the number n of vectors in the basis is called the ***dimension*** *of V, and we write*

$$\dim V = n$$

The zero vector space $\{\mathbf{0}\}$ is defined to have dimension 0:

$$\dim\{\mathbf{0}\} = 0$$

In our discussion to this point we have always assumed that a basis is nonempty and hence that the dimension of the space is at least 1. However, the zero space $\{\mathbf{0}\}$ has *no* basis (by Example 6.3.6) so our insistence that $\dim\{\mathbf{0}\} = 0$ amounts to saying that the *empty* set of vectors is a basis of $\{\mathbf{0}\}$. Thus the statement that "the dimension of a vector space is the number of vectors in any basis" holds even for the zero space.

We saw in Example 5.2.9 that $\dim(\mathbb{R}^n) = n$ and, if $\mathbf{e}_j$ denotes column j of I_n, that $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis (called the standard basis). In Example 6.3.7 below, similar considerations apply to the space $\mathbf{M}_{mn}$ of all $m \times n$ matrices; the verifications are left to the reader.

Example 6.3.7

The space $\mathbf{M}_{mn}$ has dimension mn, and one basis consists of all $m \times n$ matrices with exactly one entry equal to 1 and all other entries equal to 0. We call this the **standard basis** of $\mathbf{M}_{mn}$.

Example 6.3.8

Show that $\dim \mathbf{P}_n = n+1$ and that $\{1,\ x,\ x^2,\ \dots,\ x^n\}$ is a basis, called the **standard basis** of $\mathbf{P}_n$.

Solution. Each polynomial $p(x) = a_0 + a_1x + \cdots + a_nx^n$ in $\mathbf{P}_n$ is clearly a linear combination of $1,\ x,\ \dots,\ x^n$, so $\mathbf{P}_n = \text{span}\,\{1,\ x,\ \dots,\ x^n\}$. However, if a linear combination of these vectors vanishes, $a_0 1 + a_1 x + \cdots + a_n x^n = 0$, then $a_0 = a_1 = \cdots = a_n = 0$ because x is an indeterminate. So $\{1,\ x,\ \dots,\ x^n\}$ is linearly independent and hence is a basis containing $n+1$ vectors. Thus, $\dim(\mathbf{P}_n) = n+1$.

Example 6.3.9

If $\mathbf{v} \neq \mathbf{0}$ is any nonzero vector in a vector space V, show that $\text{span}\,\{\mathbf{v}\} = \mathbb{R}\mathbf{v}$ has dimension 1.

Solution. $\{\mathbf{v}\}$ clearly spans $\mathbb{R}\mathbf{v}$, and it is linearly independent by Example 6.3.6. Hence $\{\mathbf{v}\}$ is a basis of $\mathbb{R}\mathbf{v}$, and so $\dim \mathbb{R}\mathbf{v} = 1$.

Example 6.3.10

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and consider the subspace

$$U = \{X \text{ in } \mathbf{M}_{22} \mid AX = XA\}$$

of $\mathbf{M}_{22}$. Show that $\dim U = 2$ and find a basis of U.

Solution. It was shown in Example 6.2.3 that U is a subspace for any choice of the matrix A. In the present case, if $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is in U, the condition $AX = XA$ gives $z = 0$ and $x = y + w$. Hence each matrix X in U can be written

$$X = \begin{bmatrix} y+w & y \\ 0 & w \end{bmatrix} = y\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + w\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so $U = \text{span}\, B$ where $B = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Moreover, the set B is linearly independent (verify this), so it is a basis of U and $\dim U = 2$.

Example 6.3.11

Show that the set V of all symmetric 2×2 matrices is a vector space, and find the dimension of V.

Solution. A matrix A is symmetric if $A^T = A$. If A and B lie in V, then

$$(A+B)^T = A^T + B^T = A + B \quad \text{and} \quad (kA)^T = kA^T = kA$$

using Theorem 2.1.2. Hence $A+B$ and kA are also symmetric. As the 2×2 zero matrix is also in

V, this shows that V is a vector space (being a subspace of $\mathbf{M}_{22}$). Now a matrix A is symmetric when entries directly across the main diagonal are equal, so each 2×2 symmetric matrix has the form

$$\begin{bmatrix} a & c \\ c & b \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + c \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Hence the set $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ spans V, and the reader can verify that B is linearly independent. Thus B is a basis of V, so $\dim V = 3$.

It is frequently convenient to alter a basis by multiplying each basis vector by a nonzero scalar. The next example shows that this always produces another basis. The proof is left as Exercise 6.3.22.

Example 6.3.12

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be nonzero vectors in a vector space V. Given nonzero scalars $a_1, a_2, \dots, a_n$, write $D = \{a_1\mathbf{v}_1, a_2\mathbf{v}_2, \dots, a_n\mathbf{v}_n\}$. If B is independent or spans V, the same is true of D. In particular, if B is a basis of V, so also is D.

Exercises for 6.3

Exercise 6.3.1 Show that each of the following sets of vectors is independent.

a. $\{1+x,\ 1-x,\ x+x^2\}$ in $\mathbf{P}_2$

b. $\{x^2,\ x+1,\ 1-x-x^2\}$ in $\mathbf{P}_2$

c. $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ in $\mathbf{M}_{22}$

d. $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ in $\mathbf{M}_{22}$

Exercise 6.3.2 Which of the following subsets of V are independent?

a. $V = \mathbf{P}_2$; $\{x^2+1,\ x+1,\ x\}$

b. $V = \mathbf{P}_2$; $\{x^2-x+3,\ 2x^2+x+5,\ x^2+5x+1\}$

c. $V = \mathbf{M}_{22}$; $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

d. $V = \mathbf{M}_{22}$;
$\left\{ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right\}$

e. $V = \mathbf{F}[1,\ 2]$; $\left\{\frac{1}{x},\ \frac{1}{x^2},\ \frac{1}{x^3}\right\}$

f. $V = \mathbf{F}[0,\ 1]$; $\left\{\frac{1}{x^2+x-6},\ \frac{1}{x^2-5x+6},\ \frac{1}{x^2-9}\right\}$

Exercise 6.3.3 Which of the following are independent in $\mathbf{F}[0,\ 2\pi]$?

a. $\{\sin^2 x,\ \cos^2 x\}$

b. $\{1,\ \sin^2 x,\ \cos^2 x\}$

c. $\{x,\ \sin^2 x,\ \cos^2 x\}$

Exercise 6.3.4 Find all values of a such that the following are independent in $\mathbb{R}^3$.

a. $\{(1,\ -1,\ 0),\ (a,\ 1,\ 0),\ (0,\ 2,\ 3)\}$

b. $\{(2,\ a,\ 1),\ (1,\ 0,\ 1),\ (0,\ 1,\ 3)\}$

Exercise 6.3.5 Show that the following are bases of the space V indicated.

a. $\{(1,\ 1,\ 0),\ (1,\ 0,\ 1),\ (0,\ 1,\ 1)\};\ V = \mathbb{R}^3$

b. $\{(-1,\ 1,\ 1),\ (1,\ -1,\ 1),\ (1,\ 1,\ -1)\};\ V = \mathbb{R}^3$

c. $\left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right], \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right], \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] \right\}$; $V = \mathbf{M}_{22}$

d. $\{1+x,\ x+x^2,\ x^2+x^3,\ x^3\};\ V = \mathbf{P}_3$

Exercise 6.3.6 Exhibit a basis and calculate the dimension of each of the following subspaces of $\mathbf{P}_2$.

a. $\{a(1+x)+b(x+x^2) \mid a \text{ and } b \text{ in } \mathbb{R}\}$

b. $\{a+b(x+x^2) \mid a \text{ and } b \text{ in } \mathbb{R}\}$

c. $\{p(x) \mid p(1) = 0\}$

d. $\{p(x) \mid p(x) = p(-x)\}$

Exercise 6.3.7 Exhibit a basis and calculate the dimension of each of the following subspaces of $\mathbf{M}_{22}$.

a. $\{A \mid A^T = -A\}$

b. $\left\{ A \,\middle|\, A \left[\begin{array}{rr} 1 & 1 \\ -1 & 0 \end{array}\right] = \left[\begin{array}{rr} 1 & 1 \\ -1 & 0 \end{array}\right] A \right\}$

c. $\left\{ A \,\middle|\, A \left[\begin{array}{rr} 1 & 0 \\ -1 & 0 \end{array}\right] = \left[\begin{array}{rr} 0 & 0 \\ 0 & 0 \end{array}\right] \right\}$

d. $\left\{ A \,\middle|\, A \left[\begin{array}{rr} 1 & 1 \\ -1 & 0 \end{array}\right] = \left[\begin{array}{rr} 0 & 1 \\ -1 & 1 \end{array}\right] A \right\}$

Exercise 6.3.8 Let $A = \left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right]$ and define $U = \{X \mid X \in \mathbf{M}_{22} \text{ and } AX = X\}$.

a. Find a basis of U containing A.

b. Find a basis of U not containing A.

Exercise 6.3.9 Show that the set $\mathbb{C}$ of all complex numbers is a vector space with the usual operations, and find its dimension.

Exercise 6.3.10

a. Let V denote the set of all 2×2 matrices with equal column sums. Show that V is a subspace of $\mathbf{M}_{22}$, and compute $\dim V$.

b. Repeat part (a) for 3×3 matrices.

c. Repeat part (a) for $n \times n$ matrices.

Exercise 6.3.11

a. Let $V = \{(x^2+x+1)p(x) \mid p(x) \text{ in } \mathbf{P}_2\}$. Show that V is a subspace of $\mathbf{P}_4$ and find $\dim V$. [*Hint*: If $f(x)g(x) = 0$ in $\mathbf{P}$, then $f(x) = 0$ or $g(x) = 0$.]

b. Repeat with $V = \{(x^2-x)p(x) \mid p(x) \text{ in } \mathbf{P}_3\}$, a subset of $\mathbf{P}_5$.

c. Generalize.

Exercise 6.3.12 In each case, either prove the assertion or give an example showing that it is false.

a. Every set of four nonzero polynomials in $\mathbf{P}_3$ is a basis.

b. $\mathbf{P}_2$ has a basis of polynomials $f(x)$ such that $f(0) = 0$.

c. $\mathbf{P}_2$ has a basis of polynomials $f(x)$ such that $f(0) = 1$.

d. Every basis of $\mathbf{M}_{22}$ contains a noninvertible matrix.

e. No independent subset of $\mathbf{M}_{22}$ contains a matrix A with $A^2 = 0$.

f. If $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is independent then, $a\mathbf{u}+b\mathbf{v}+c\mathbf{w} = \mathbf{0}$ for some a, b, c.

g. $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is independent if $a\mathbf{u}+b\mathbf{v}+c\mathbf{w} = \mathbf{0}$ for some a, b, c.

h. If $\{\mathbf{u},\ \mathbf{v}\}$ is independent, so is $\{\mathbf{u},\ \mathbf{u}+\mathbf{v}\}$.

i. If $\{\mathbf{u},\ \mathbf{v}\}$ is independent, so is $\{\mathbf{u},\ \mathbf{v},\ \mathbf{u}+\mathbf{v}\}$.

j. If $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is independent, so is $\{\mathbf{u},\ \mathbf{v}\}$.

k. If $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is independent, so is $\{\mathbf{u}+\mathbf{w},\ \mathbf{v}+\mathbf{w}\}$.

l. If $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is independent, so is $\{\mathbf{u}+\mathbf{v}+\mathbf{w}\}$.

m. If $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$ then $\{\mathbf{u}, \mathbf{v}\}$ is dependent if and only if one is a scalar multiple of the other.

n. If $\dim V = n$, then no set of more than n vectors can be independent.

o. If $\dim V = n$, then no set of fewer than n vectors can span V.

Exercise 6.3.13 Let $A \neq 0$ and $B \neq 0$ be $n \times n$ matrices, and assume that A is symmetric and B is skew-symmetric (that is, $B^T = -B$). Show that $\{A, B\}$ is independent.

Exercise 6.3.14 Show that every set of vectors containing a dependent set is again dependent.

Exercise 6.3.15 Show that every nonempty subset of an independent set of vectors is again independent.

Exercise 6.3.16 Let f and g be functions on $[a, b]$, and assume that $f(a) = 1 = g(b)$ and $f(b) = 0 = g(a)$. Show that $\{f, g\}$ is independent in $\mathbf{F}[a, b]$.

Exercise 6.3.17 Let $\{A_1, A_2, \dots, A_k\}$ be independent in $\mathbf{M}_{mn}$, and suppose that U and V are invertible matrices of size $m \times m$ and $n \times n$, respectively. Show that $\{UA_1V, UA_2V, \dots, UA_kV\}$ is independent.

Exercise 6.3.18 Show that $\{\mathbf{v}, \mathbf{w}\}$ is independent if and only if neither $\mathbf{v}$ nor $\mathbf{w}$ is a scalar multiple of the other.

Exercise 6.3.19 Assume that $\{\mathbf{u}, \mathbf{v}\}$ is independent in a vector space V. Write $\mathbf{u}' = a\mathbf{u} + b\mathbf{v}$ and $\mathbf{v}' = c\mathbf{u} + d\mathbf{v}$, where a, b, c, and d are numbers. Show that $\{\mathbf{u}', \mathbf{v}'\}$ is independent if and only if the matrix $\left[\begin{array}{cc} a & c \\ b & d \end{array} \right]$ is invertible. [*Hint*: Theorem 2.4.5.]

Exercise 6.3.20 If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is independent and $\mathbf{w}$ is not in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, show that:

a. $\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is independent.

b. $\{\mathbf{v}_1 + \mathbf{w}, \mathbf{v}_2 + \mathbf{w}, \dots, \mathbf{v}_k + \mathbf{w}\}$ is independent.

Exercise 6.3.21 If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is independent, show that $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \dots, \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k\}$ is also independent.

Exercise 6.3.22 Prove Example 6.3.12.

Exercise 6.3.23 Let $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ be independent. Which of the following are dependent?

a. $\{\mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{w}, \mathbf{w} - \mathbf{u}\}$

b. $\{\mathbf{u} + \mathbf{v}, \mathbf{v} + \mathbf{w}, \mathbf{w} + \mathbf{u}\}$

c. $\{\mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{w}, \mathbf{w} - \mathbf{z}, \mathbf{z} - \mathbf{u}\}$

d. $\{\mathbf{u} + \mathbf{v}, \mathbf{v} + \mathbf{w}, \mathbf{w} + \mathbf{z}, \mathbf{z} + \mathbf{u}\}$

Exercise 6.3.24 Let U and W be subspaces of V with bases $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\{\mathbf{w}_1, \mathbf{w}_2\}$ respectively. If U and W have only the zero vector in common, show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{w}_1, \mathbf{w}_2\}$ is independent.

Exercise 6.3.25 Let $\{p, q\}$ be independent polynomials. Show that $\{p, q, pq\}$ is independent if and only if $\deg p \geq 1$ and $\deg q \geq 1$.

Exercise 6.3.26 If z is a complex number, show that $\{z, z^2\}$ is independent if and only if z is not real.

Exercise 6.3.27 Let $B = \{A_1, A_2, \dots, A_n\} \subseteq \mathbf{M}_{mn}$, and write $B' = \{A_1^T, A_2^T, \dots, A_n^T\} \subseteq \mathbf{M}_{nm}$. Show that:

a. B is independent if and only if B' is independent.

b. B spans $\mathbf{M}_{mn}$ if and only if B' spans $\mathbf{M}_{nm}$.

Exercise 6.3.28 If $V = \mathbf{F}[a, b]$ as in Example 6.1.7, show that the set of constant functions is a subspace of dimension 1 (f is **constant** if there is a number c such that $f(x) = c$ for all x).

Exercise 6.3.29

a. If U is an invertible $n \times n$ matrix and $\{A_1, A_2, \dots, A_{mn}\}$ is a basis of $\mathbf{M}_{mn}$, show that $\{A_1U, A_2U, \dots, A_{mn}U\}$ is also a basis.

b. Show that part (a) fails if U is not invertible. [*Hint*: Theorem 2.4.5.]

Exercise 6.3.30 Show that $\{(a, b), (a_1, b_1)\}$ is a basis of $\mathbb{R}^2$ if and only if $\{a + bx, a_1 + b_1x\}$ is a basis of $\mathbf{P}_1$.

Exercise 6.3.31 Find the dimension of the subspace $\operatorname{span}\{1, \sin^2\theta, \cos 2\theta\}$ of $\mathbf{F}[0, 2\pi]$.

Exercise 6.3.32 Show that $\mathbf{F}[0, 1]$ is not finite dimensional.

Exercise 6.3.33 If U and W are subspaces of V, define their intersection $U \cap W$ as follows:

$$U \cap W = \{\mathbf{v} \mid \mathbf{v} \text{ is in both } U \text{ and } W\}$$

a. Show that $U \cap W$ is a subspace contained in U and W.

b. Show that $U \cap W = \{\mathbf{0}\}$ if and only if $\{\mathbf{u},\ \mathbf{w}\}$ is independent for any nonzero vectors $\mathbf{u}$ in U and $\mathbf{w}$ in W.

c. If B and D are bases of U and W, and if $U \cap W = \{\mathbf{0}\}$, show that $B \cup D = \{\mathbf{v} \mid \mathbf{v} \text{ is in } B \text{ or } D\}$ is independent.

Exercise 6.3.34 If U and W are vector spaces, let $V = \{(\mathbf{u},\ \mathbf{w}) \mid \mathbf{u} \text{ in } U \text{ and } \mathbf{w} \text{ in } W\}$.

a. Show that V is a vector space if $(\mathbf{u},\ \mathbf{w}) + (\mathbf{u}_1,\ \mathbf{w}_1) = (\mathbf{u}+\mathbf{u}_1,\ \mathbf{w}+\mathbf{w}_1)$ and $a(\mathbf{u},\ \mathbf{w}) = (a\mathbf{u},\ a\mathbf{w})$.

b. If $\dim U = m$ and $\dim W = n$, show that $\dim V = m+n$.

c. If $V_1,\ \dots,\ V_m$ are vector spaces, let

$$\begin{aligned} V &= V_1 \times \cdots \times V_m \\ &= \{(\mathbf{v}_1,\ \dots,\ \mathbf{v}_m) \mid \mathbf{v}_i \in V_i \text{ for each } i\} \end{aligned}$$

denote the space of n-tuples from the V_i with componentwise operations (see Exercise 6.1.17). If $\dim V_i = n_i$ for each i, show that $\dim V = n_1 + \cdots + n_m$.

Exercise 6.3.35 Let $\mathbf{D}_n$ denote the set of all functions f from the set $\{1,\ 2,\ \dots,\ n\}$ to $\mathbb{R}$.

a. Show that $\mathbf{D}_n$ is a vector space with pointwise addition and scalar multiplication.

b. Show that $\{S_1,\ S_2,\ \dots,\ S_n\}$ is a basis of $\mathbf{D}_n$ where, for each $k = 1,\ 2,\ \dots,\ n$, the function S_k is defined by $S_k(k) = 1$, whereas $S_k(j) = 0$ if $j \neq k$.

Exercise 6.3.36 A polynomial $p(x)$ is called **even** if $p(-x) = p(x)$ and **odd** if $p(-x) = -p(x)$. Let E_n and O_n denote the sets of even and odd polynomials in $\mathbf{P}_n$.

a. Show that E_n is a subspace of $\mathbf{P}_n$ and find $\dim E_n$.

b. Show that O_n is a subspace of $\mathbf{P}_n$ and find $\dim O_n$.

Exercise 6.3.37 Let $\{\mathbf{v}_1,\ \dots,\ \mathbf{v}_n\}$ be independent in a vector space V, and let A be an $n \times n$ matrix. Define $\mathbf{u}_1,\ \dots,\ \mathbf{u}_n$ by

$$\left[\begin{array}{c} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{array}\right] = A \left[\begin{array}{c} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{array}\right]$$

(See Exercise 6.1.18.) Show that $\{\mathbf{u}_1,\ \dots,\ \mathbf{u}_n\}$ is independent if and only if A is invertible.

6.4 Finite Dimensional Spaces

Up to this point, we have had no guarantee that an arbitrary vector space *has* a basis—and hence no guarantee that one can speak *at all* of the dimension of V. However, Theorem 6.4.1 will show that any space that is spanned by a finite set of vectors has a (finite) basis: The proof requires the following basic lemma, of interest in itself, that gives a way to enlarge a given independent set of vectors.

Lemma 6.4.1: Independent Lemma

Let $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_k\}$ be an independent set of vectors in a vector space V. If $\mathbf{u} \in V$ but[5] *$\mathbf{u} \notin \operatorname{span}\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_k\}$, then $\{\mathbf{u},\ \mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_k\}$ is also independent.*

Proof. Let $t\mathbf{u} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0}$; we must show that all the coefficients are zero. First, $t = 0$ because, otherwise, $\mathbf{u} = -\frac{t_1}{t}\mathbf{v}_1 - \frac{t_2}{t}\mathbf{v}_2 - \cdots - \frac{t_k}{t}\mathbf{v}_k$ is in $\operatorname{span}\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_k\}$, contrary to our assumption.

[5]If X is a set, we write $a \in X$ to indicate that a is an element of the set X. If a is not an element of X, we write $a \notin X$.