6.6 An Application to Differential Equations

Call a function $f : \mathbb{R} \to \mathbb{R}$ **differentiable** if it can be differentiated as many times as we want. If f is a differentiable function, the nth derivative $f^{(n)}$ of f is the result of differentiating n times. Thus $f^{(0)} = f$, $f^{(1)} = f'$, $f^{(2)} = f^{(1)\prime}$, ... and, in general, $f^{(n+1)} = f^{(n)\prime}$ for each $n \geq 0$. For small values of n these are often written as $f, f', f'', f''', \dots$.

If a, b, and c are numbers, the differential equations

$$f'' + af' + bf = 0 \quad \text{or} \quad f''' + af'' + bf' + cf = 0$$

are said to be of **second-order** and **third-order**, respectively. In general, an equation

$$f^{(n)} + a_{n-1}f^{(n-1)} + a_{n-2}f^{(n-2)} + \cdots + a_2f^{(2)} + a_1f^{(1)} + a_0f^{(0)} = 0, \quad a_i \text{ in } \mathbb{R} \tag{6.3}$$

is called a **differential equation of order** n. In this section we investigate the set of solutions to (6.3) and, if n is 1 or 2, find explicit solutions. Of course an acquaintance with calculus is required.

Let f and g be solutions to (6.3). Then $f + g$ is also a solution because $(f + g)^{(k)} = f^{(k)} + g^{(k)}$ for all k, and af is a solution for any a in $\mathbb{R}$ because $(af)^{(k)} = af^{(k)}$. It follows that the set of solutions to (6.3) is a vector space, and we ask for the dimension of this space.

We have already dealt with the simplest case (see Theorem 3.5.1):

Theorem 6.6.1

The set of solutions of the first-order differential equation $f' + af = 0$ is a one-dimensional vector space and $\{e^{-ax}\}$ is a basis.

There is a far-reaching generalization of Theorem 6.6.1 that will be proved in Theorem 7.4.1.

Theorem 6.6.2

The set of solutions to the nth order equation (6.3) has dimension n.

Remark

Every differential equation of order n can be converted into a system of n linear first-order equations (see Exercises 3.5.6 and 3.5.7). In the case that the matrix of this system is diagonalizable, this approach provides a proof of Theorem 6.6.2. But if the matrix is not diagonalizable, Theorem 7.4.1 is required.

Theorem 6.6.1 suggests that we look for solutions to (6.3) of the form $e^{\lambda x}$ for some number λ. This is a good idea. If we write $f(x) = e^{\lambda x}$, it is easy to verify that $f^{(k)}(x) = \lambda^k e^{\lambda x}$ for each $k \geq 0$, so substituting f in (6.3) gives

$$(\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_2\lambda^2 + a_1\lambda^1 + a_0)e^{\lambda x} = 0$$

Since $e^{\lambda x} \neq 0$ for all x, this shows that $e^{\lambda x}$ is a solution of (6.3) if and only if λ is a root of the **characteristic polynomial** $c(x)$, defined to be

$$c(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$$

This proves Theorem 6.6.3.

Theorem 6.6.3

If λ is real, the function $e^{\lambda x}$ is a solution of (6.3) if and only if λ is a root of the characteristic polynomial $c(x)$.

Example 6.6.1

Find a basis of the space U of solutions of $f''' - 2f'' - f' - 2f = 0$.

Solution. The characteristic polynomial is $x^3 - 2x^2 - x - 1 = (x-1)(x+1)(x-2)$, with roots $\lambda_1 = 1$, $\lambda_2 = -1$, and $\lambda_3 = 2$. Hence e^x, e^{-x}, and e^{2x} are all in U. Moreover they are independent (by Lemma 6.6.1 below) so, since $\dim(U) = 3$ by Theorem 6.6.2, $\{e^x, e^{-x}, e^{2x}\}$ is a basis of U.

Lemma 6.6.1

If $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct, then $\{e^{\lambda_1 x}, e^{\lambda_2 x}, \dots, e^{\lambda_k x}\}$ is linearly independent.

Proof. If $r_1 e^{\lambda_1 x} + r_2 e^{\lambda_2 x} + \cdots + r_k e^{\lambda_k x} = 0$ for all x, then $r_1 + r_2 e^{(\lambda_2 - \lambda_1)x} + \cdots + r_k e^{(\lambda_k - \lambda_1)x} = 0$; that is, $r_2 e^{(\lambda_2 - \lambda_1)x} + \cdots + r_k e^{(\lambda_k - \lambda_1)x}$ is a constant. Since the λ_i are distinct, this forces $r_2 = \cdots = r_k = 0$, whence $r_1 = 0$ also. This is what we wanted. □

Theorem 6.6.4

Let U denote the space of solutions to the second-order equation

$$f'' + af' + bf = 0$$

where a and b are real constants. Assume that the characteristic polynomial $x^2 + ax + b$ has two real roots λ and μ. Then

1. *If $\lambda \neq \mu$, then $\{e^{\lambda x}, e^{\mu x}\}$ is a basis of U.*
2. *If $\lambda = \mu$, then $\{e^{\lambda x}, xe^{\lambda x}\}$ is a basis of U.*

Proof. Since $\dim(U) = 2$ by Theorem 6.6.2, (1) follows by Lemma 6.6.1, and (2) follows because the set $\{e^{\lambda x}, xe^{\lambda x}\}$ is independent (Exercise 6.6.3). □

Example 6.6.2

Find the solution of $f'' + 4f' + 4f = 0$ that satisfies the **boundary conditions** $f(0) = 1$, $f(1) = -1$.

Solution. The characteristic polynomial is $x^2+4x+4=(x+2)^2$, so -2 is a double root. Hence $\{e^{-2x},\ xe^{-2x}\}$ is a basis for the space of solutions, and the general solution takes the form $f(x)=ce^{-2x}+dxe^{-2x}$. Applying the boundary conditions gives $1=f(0)=c$ and $-1=f(1)=(c+d)e^{-2}$. Hence $c=1$ and $d=-(1+e^2)$, so the required solution is

$$f(x)=e^{-2x}-(1+e^2)xe^{-2x}$$

One other question remains: What happens if the roots of the characteristic polynomial are not real? To answer this, we must first state precisely what $e^{\lambda x}$ means when λ is not real. If q is a real number, define

$$e^{iq}=\cos q+i\sin q$$

where $i^2=-1$. Then the relationship $e^{iq}e^{iq_1}=e^{i(q+q_1)}$ holds for all real q and q_1, as is easily verified. If $\lambda=p+iq$, where p and q are real numbers, we define

$$e^{\lambda}=e^pe^{iq}=e^p(\cos q+i\sin q)$$

Then it is a routine exercise to show that

1. $e^{\lambda}e^{\mu}=e^{\lambda+\mu}$
2. $e^{\lambda}=1$ if and only if $\lambda=0$
3. $(e^{\lambda x})'=\lambda e^{\lambda x}$

These easily imply that $f(x)=e^{\lambda x}$ is a solution to $f''+af'+bf=0$ if λ is a (possibly complex) root of the characteristic polynomial x^2+ax+b. Now write $\lambda=p+iq$ so that

$$f(x)=e^{\lambda x}=e^{px}\cos(qx)+ie^{px}\sin(qx)$$

For convenience, denote the real and imaginary parts of $f(x)$ as $u(x)=e^{px}\cos(qx)$ and $v(x)=e^{px}\sin(qx)$. Then the fact that $f(x)$ satisfies the differential equation gives

$$0=f''+af'+bf=(u''+au'+bu)+i(v''+av'+bv)$$

Equating real and imaginary parts shows that $u(x)$ and $v(x)$ are both solutions to the differential equation. This proves part of Theorem 6.6.5.

Theorem 6.6.5

Let U denote the space of solutions of the second-order differential equation

$$f''+af'+bf=0$$

where a and b are real. Suppose λ is a nonreal root of the characteristic polynomial x^2+ax+b. If $\lambda=p+iq$, where p and q are real, then

$$\{e^{px}\cos(qx),\ e^{px}\sin(qx)\}$$

is a basis of U.

Proof. The foregoing discussion shows that these functions lie in U. Because $\dim U = 2$ by Theorem 6.6.2, it suffices to show that they are linearly independent. But if

$$re^{px}\cos(qx) + se^{px}\sin(qx) = 0$$

for all x, then $r\cos(qx) + s\sin(qx) = 0$ for all x (because $e^{px} \neq 0$). Taking $x = 0$ gives $r = 0$, and taking $x = \frac{\pi}{2q}$ gives $s = 0$ ($q \neq 0$ because λ is not real). This is what we wanted. □

Example 6.6.3

Find the solution $f(x)$ to $f'' - 2f' + 2f = 0$ that satisfies $f(0) = 2$ and $f(\frac{\pi}{2}) = 0$.

Solution. The characteristic polynomial $x^2 - 2x + 2$ has roots $1 + i$ and $1 - i$. Taking $\lambda = 1 + i$ (quite arbitrarily) gives $p = q = 1$ in the notation of Theorem 6.6.5, so $\{e^x \cos x,\ e^x \sin x\}$ is a basis for the space of solutions. The general solution is thus $f(x) = e^x(r\cos x + s\sin x)$. The boundary conditions yield $2 = f(0) = r$ and $0 = f(\frac{\pi}{2}) = e^{\pi/2}s$. Thus $r = 2$ and $s = 0$, and the required solution is $f(x) = 2e^x \cos x$.

The following theorem is an important special case of Theorem 6.6.5.

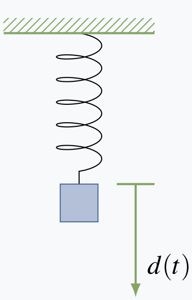
Theorem 6.6.6

If $q \neq 0$ is a real number, the space of solutions to the differential equation $f'' + q^2 f = 0$ has basis $\{\cos(qx),\ \sin(qx)\}$.

Proof. The characteristic polynomial $x^2 + q^2$ has roots qi and $-qi$, so Theorem 6.6.5 applies with $p = 0$. □

In many situations, the displacement $s(t)$ of some object at time t turns out to have an oscillating form $s(t) = c\sin(at) + d\cos(at)$. These are called **simple harmonic motions**. An example follows.

Example 6.6.4



A weight is attached to an extension spring (see diagram). If it is pulled from the equilibrium position and released, it is observed to oscillate up and down. Let $d(t)$ denote the distance of the weight below the equilibrium position t seconds later. It is known (**Hooke's law**) that the acceleration $d''(t)$ of the weight is proportional to the displacement $d(t)$ and in the opposite direction. That is,

$$d''(t) = -kd(t)$$

where $k > 0$ is called the **spring constant**. Find $d(t)$ if the maximum extension is 10 cm below the equilibrium position and find the **period** of the oscillation (time taken for the weight to make a full oscillation).

Solution. It follows from Theorem 6.6.6 (with $q^2 = k$) that

$$d(t) = r\sin(\sqrt{k}\,t) + s\cos(\sqrt{k}\,t)$$

where r and s are constants. The condition $d(0) = 0$ gives $s = 0$, so $d(t) = r\sin(\sqrt{k}\,t)$. Now the maximum value of the function $\sin x$ is 1 (when $x = \frac{\pi}{2}$), so $r = 10$ (when $t = \frac{\pi}{2\sqrt{k}}$). Hence

$$d(t) = 10\sin(\sqrt{k}\,t)$$

Finally, the weight goes through a full oscillation as $\sqrt{k}\,t$ increases from 0 to 2π. The time taken is $t = \frac{2\pi}{\sqrt{k}}$, the period of the oscillation.

Exercises for 6.6

Exercise 6.6.1 Find a solution f to each of the following differential equations satisfying the given boundary conditions.

a. $f' - 3f = 0$; $f(1) = 2$

b. $f' + f = 0$; $f(1) = 1$

c. $f'' + 2f' - 15f = 0$; $f(1) = f(0) = 0$

d. $f'' + f' - 6f = 0$; $f(0) = 0$, $f(1) = 1$

e. $f'' - 2f' + f = 0$; $f(1) = f(0) = 1$

f. $f'' - 4f' + 4f = 0$; $f(0) = 2$, $f(-1) = 0$

g. $f'' - 3af' + 2a^2f = 0$; $a \neq 0$; $f(0) = 0$, $f(1) = 1 - e^a$

h. $f'' - a^2f = 0$, $a \neq 0$; $f(0) = 1$, $f(1) = 0$

i. $f'' - 2f' + 5f = 0$; $f(0) = 1$, $f(\frac{\pi}{4}) = 0$

j. $f'' + 4f' + 5f = 0$; $f(0) = 0$, $f(\frac{\pi}{2}) = 1$

Exercise 6.6.2 If the characteristic polynomial of $f'' + af' + bf = 0$ has real roots, show that $f = 0$ is the only solution satisfying $f(0) = 0 = f(1)$.

Exercise 6.6.3 Complete the proof of Theorem 6.6.2. [*Hint*: If λ is a double root of $x^2 + ax + b$, show that $a = -2\lambda$ and $b = \lambda^2$. Hence $xe^{\lambda x}$ is a solution.]

Exercise 6.6.4

a. Given the equation $f' + af = b$, $(a \neq 0)$, make the substitution $f(x) = g(x) + b/a$ and obtain a differential equation for g. Then derive the general solution for $f' + af = b$.

b. Find the general solution to $f' + f = 2$.

Exercise 6.6.5 Consider the differential equation $f' + af' + bf = g$, where g is some fixed function. Assume that f_0 is one solution of this equation.

a. Show that the general solution is $cf_1 + df_2 + f_0$, where c and d are constants and $\{f_1, f_2\}$ is any basis for the solutions to $f'' + af' + bf = 0$.

b. Find a solution to $f'' + f' - 6f = 2x^3 - x^2 - 2x$. [*Hint*: Try $f(x) = \frac{-1}{3}x^3$.]

Exercise 6.6.6 A radioactive element decays at a rate proportional to the amount present. Suppose an initial mass of 10 grams decays to 8 grams in 3 hours.

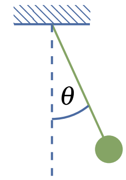
a. Find the mass t hours later.

b. Find the *half-life* of the element—the time it takes to decay to half its mass.

Exercise 6.6.7 The population $N(t)$ of a region at time t increases at a rate proportional to the population. If the population doubles in 5 years and is 3 million initially, find $N(t)$.

Exercise 6.6.8 Consider a spring, as in Example 6.6.4. If the period of the oscillation is 30 seconds, find the spring constant k.

Exercise 6.6.9 As a pendulum swings (see the diagram), let t measure the time since it was vertical. The angle $\theta = \theta(t)$ from the vertical can be shown to satisfy the equation $\theta'' + k\theta = 0$, provided that θ is small. If the maximal angle is $\theta = 0.05$ radians, find $\theta(t)$ in terms of k. If the period is 0.5 seconds, find k. [Assume that $\theta = 0$ when $t = 0$.]



Supplementary Exercises for Chapter 6

Exercise 6.1 (Requires calculus) Let V denote the space of all functions $f : \mathbb{R} \to \mathbb{R}$ for which the derivatives f' and f'' exist. Show that f_1, f_2, and f_3 in V are linearly independent provided that their **wronskian** $w(x)$ is nonzero for some x, where

$$w(x) = \det \begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{bmatrix}$$

Exercise 6.2 Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis of $\mathbb{R}^n$ (written as columns), and let A be an $n \times n$ matrix.

a. If A is invertible, show that $\{A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_n\}$ is a basis of $\mathbb{R}^n$.

b. If $\{A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_n\}$ is a basis of $\mathbb{R}^n$, show that A is invertible.

Exercise 6.3 If A is an $m \times n$ matrix, show that A has rank m if and only if col A contains every column of I_m.

Exercise 6.4 Show that null A = null (A^TA) for any real matrix A.

Exercise 6.5 Let A be an $m \times n$ matrix of rank r. Show that $\dim(\text{null } A) = n - r$ (Theorem 5.4.3) as follows. Choose a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ of null A and extend it to a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{z}_1, \dots, \mathbf{z}_m\}$ of $\mathbb{R}^n$. Show that $\{A\mathbf{z}_1, \dots, A\mathbf{z}_m\}$ is a basis of col A.