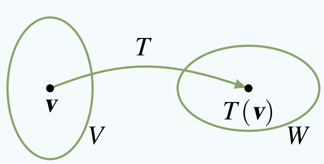
7. Linear Transformations

If V and W are vector spaces, a function $T : V \to W$ is a rule that assigns to each vector $\mathbf{v}$ in V a uniquely determined vector $T(\mathbf{v})$ in W. As mentioned in Section 2.2, two functions $S : V \to W$ and $T : V \to W$ are equal if $S(\mathbf{v}) = T(\mathbf{v})$ for every $\mathbf{v}$ in V. A function $T : V \to W$ is called a *linear transformation* if $T(\mathbf{v}+\mathbf{v}_1) = T(\mathbf{v}) + T(\mathbf{v}_1)$ for all $\mathbf{v}$, $\mathbf{v}_1$ in V and $T(r\mathbf{v}) = rT(\mathbf{v})$ for all $\mathbf{v}$ in V and all scalars r. $T(\mathbf{v})$ is called the *image* of $\mathbf{v}$ under T. We have already studied linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ and shown (in Section 2.6) that they are all given by multiplication by a uniquely determined $m \times n$ matrix A; that is $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. In the case of linear operators $\mathbb{R}^2 \to \mathbb{R}^2$, this yields an important way to describe geometric functions such as rotations about the origin and reflections in a line through the origin.

In the present chapter we will describe linear transformations in general, introduce the *kernel* and *image* of a linear transformation, and prove a useful result (called the *dimension theorem*) that relates the dimensions of the kernel and image, and unifies and extends several earlier results. Finally we study the notion of *isomorphic* vector spaces, that is, spaces that are identical except for notation, and relate this to composition of transformations that was introduced in Section 2.3.

7.1 Examples and Elementary Properties

Definition 7.1 Linear Transformations of Vector Spaces



If V and W are two vector spaces, a function $T : V \to W$ is called a ***linear transformation*** *if it satisfies the following axioms.*

T1.	$T(\mathbf{v}+\mathbf{v}_1) = T(\mathbf{v}) + T(\mathbf{v}_1)$	*for all $\mathbf{v}$ and $\mathbf{v}_1$ in V.*
T2.	$T(r\mathbf{v}) = rT(\mathbf{v})$	*for all $\mathbf{v}$ in V and r in $\mathbb{R}$.*

A linear transformation $T : V \to V$ is called a ***linear operator*** *on V. The situation can be visualized as in the diagram.*

Axiom T1 is just the requirement that T *preserves* vector addition. It asserts that the result $T(\mathbf{v}+\mathbf{v}_1)$ of adding $\mathbf{v}$ and $\mathbf{v}_1$ first and then applying T is the same as applying T first to get $T(\mathbf{v})$ and $T(\mathbf{v}_1)$ and then adding. Similarly, axiom T2 means that T *preserves* scalar multiplication. Note that, even though the additions in axiom T1 are both denoted by the same symbol $+$, the addition on the left forming $\mathbf{v}+\mathbf{v}_1$ is carried out in V, whereas the addition $T(\mathbf{v}) + T(\mathbf{v}_1)$ is done in W. Similarly, the scalar multiplications $r\mathbf{v}$ and $rT(\mathbf{v})$ in axiom T2 refer to the spaces V and W, respectively.

We have already seen many examples of linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$. In fact, writing vectors in $\mathbb{R}^n$ as columns, Theorem 2.6.2 shows that, for each such T, there is an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x}$ in $\mathbb{R}^n$. Moreover, the matrix A is given by $A = \left[\begin{array}{cccc} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{array}\right]$ where $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$. We denote this transformation by $T_A : \mathbb{R}^n \to \mathbb{R}^m$, defined by

$$T_A(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

Example 7.1.1 lists three important linear transformations that will be referred to later. The verification of axioms T1 and T2 is left to the reader.

Example 7.1.1

If V and W are vector spaces, the following are linear transformations:

Identity operator $V \to V$	$1_V : V \to V$	where $1_V(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v}$ in V
Zero transformation $V \to W$	$0 : V \to W$	where $0(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v}$ in V
Scalar operator $V \to V$	$a : V \to V$	where $a(\mathbf{v}) = a\mathbf{v}$ for all $\mathbf{v}$ in V (Here a is any real number.)

The symbol 0 will be used to denote the zero transformation from V to W for *any* spaces V and W. It was also used earlier to denote the zero function $[a, b] \to \mathbb{R}$.

The next example gives two important transformations of matrices. Recall that the trace $\text{tr} A$ of an $n \times n$ matrix A is the sum of the entries on the main diagonal.

Example 7.1.2

Show that the transposition and trace are linear transformations. More precisely,

$$\begin{aligned} R : \mathbf{M}_{mn} \to \mathbf{M}_{nm} \qquad & \text{where } R(A) = A^T \text{ for all } A \text{ in } \mathbf{M}_{mn} \\ S : \mathbf{M}_{mn} \to \mathbb{R} \qquad & \text{where } S(A) = \text{tr } A \text{ for all } A \text{ in } \mathbf{M}_{nn} \end{aligned}$$

are both linear transformations.

Solution. Axioms T1 and T2 for transposition are $(A+B)^T = A^T + B^T$ and $(rA)^T = r(A^T)$, respectively (using Theorem 2.1.2). The verifications for the trace are left to the reader.

Example 7.1.3

If a is a scalar, define $E_a : \mathbf{P}_n \to \mathbb{R}$ by $E_a(p) = p(a)$ for each polynomial p in $\mathbf{P}_n$. Show that E_a is a linear transformation (called **evaluation** at a).

Solution. If p and q are polynomials and r is in $\mathbb{R}$, we use the fact that the sum $p+q$ and scalar product rp are defined as for functions:

$$(p+q)(x) = p(x) + q(x) \quad \text{and} \quad (rp)(x) = rp(x)$$

for all x. Hence, for all p and q in $\mathbf{P}_n$ and all r in $\mathbb{R}$:

$$\begin{aligned} E_a(p+q) &= (p+q)(a) = p(a) + q(a) = E_a(p) + E_a(q), \quad \text{and} \\ E_a(rp) &= (rp)(a) = rp(a) = rE_a(p). \end{aligned}$$

Hence E_a is a linear transformation.

The next example involves some calculus.

Example 7.1.4

Show that the differentiation and integration operations on $\mathbf{P}_n$ are linear transformations. More precisely,

$$D : \mathbf{P}_n \to \mathbf{P}_{n-1} \quad \text{where } D[p(x)] = p'(x) \text{ for all } p(x) \text{ in } \mathbf{P}_n$$
$$I : \mathbf{P}_n \to \mathbf{P}_{n+1} \quad \text{where } I[p(x)] = \int_0^x p(t)dt \text{ for all } p(x) \text{ in } \mathbf{P}_n$$

are linear transformations.

Solution. These restate the following fundamental properties of differentiation and integration.

$$[p(x) + q(x)]' = p'(x) + q'(x) \quad \text{and} \quad [rp(x)]' = (rp)'(x)$$

$$\textstyle\int_0^x [p(t) + q(t)]dt = \int_0^x p(t)dt + \int_0^x q(t)dt \quad \text{and} \quad \int_0^x rp(t)dt = r\int_0^x p(t)dt$$

The next theorem collects three useful properties of *all* linear transformations. They can be described by saying that, in addition to preserving addition and scalar multiplication (these are the axioms), linear transformations preserve the zero vector, negatives, and linear combinations.

Theorem 7.1.1

Let $T : V \to W$ be a linear transformation.

1. $T(\mathbf{0}) = \mathbf{0}$.
2. $T(-\mathbf{v}) = -T(\mathbf{v})$ *for all* $\mathbf{v}$ *in* V.
3. $T(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k) = r_1T(\mathbf{v}_1) + r_2T(\mathbf{v}_2) + \cdots + r_kT(\mathbf{v}_k)$ *for all* $\mathbf{v}_i$ *in* V *and all* r_i *in* $\mathbb{R}$.

Proof.

1. $T(\mathbf{0}) = T(0\mathbf{v}) = 0T(\mathbf{v}) = \mathbf{0}$ for any $\mathbf{v}$ in V.
2. $T(-\mathbf{v}) = T[(-1)\mathbf{v}] = (-1)T(\mathbf{v}) = -T(\mathbf{v})$ for any $\mathbf{v}$ in V.
3. The proof of Theorem 2.6.1 goes through. □

The ability to use the last part of Theorem 7.1.1 effectively is vital to obtaining the benefits of linear transformations. Example 7.1.5 and Theorem 7.1.2 provide illustrations.

Example 7.1.5

Let $T : V \to W$ be a linear transformation. If $T(\mathbf{v} - 3\mathbf{v}_1) = \mathbf{w}$ and $T(2\mathbf{v} - \mathbf{v}_1) = \mathbf{w}_1$, find $T(\mathbf{v})$ and $T(\mathbf{v}_1)$ in terms of $\mathbf{w}$ and $\mathbf{w}_1$.

Solution. The given relations imply that

$$\begin{aligned} T(\mathbf{v}) - 3T(\mathbf{v}_1) &= \mathbf{w} \\ 2T(\mathbf{v}) - T(\mathbf{v}_1) &= \mathbf{w}_1 \end{aligned}$$

by Theorem 7.1.1. Subtracting twice the first from the second gives $T(\mathbf{v}_1) = \frac{1}{5}(\mathbf{w}_1 - 2\mathbf{w})$. Then substitution gives $T(\mathbf{v}) = \frac{1}{5}(3\mathbf{w}_1 - \mathbf{w})$.

The full effect of property (3) in Theorem 7.1.1 is this: If $T : V \to W$ is a linear transformation and $T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)$ are known, then $T(\mathbf{v})$ can be computed for *every* vector $\mathbf{v}$ in span $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. In particular, if $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ spans V, then $T(\mathbf{v})$ is determined for all $\mathbf{v}$ in V by the choice of $T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)$. The next theorem states this somewhat differently. As for functions in general, two linear transformations $T : V \to W$ and $S : V \to W$ are called **equal** (written $T = S$) if they have the same **action**; that is, if $T(\mathbf{v}) = S(\mathbf{v})$ for all $\mathbf{v}$ in V.

Theorem 7.1.2

Let $T : V \to W$ and $S : V \to W$ be two linear transformations. Suppose that $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. If $T(\mathbf{v}_i) = S(\mathbf{v}_i)$ for each i, then $T = S$.

Proof. If $\mathbf{v}$ is any vector in $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, write $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$ where each a_i is in $\mathbb{R}$. Since $T(\mathbf{v}_i) = S(\mathbf{v}_i)$ for each i, Theorem 7.1.1 gives

$$\begin{aligned} T(\mathbf{v}) &= T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) \\ &= a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2) + \cdots + a_nT(\mathbf{v}_n) \\ &= a_1S(\mathbf{v}_1) + a_2S(\mathbf{v}_2) + \cdots + a_nS(\mathbf{v}_n) \\ &= S(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) \\ &= S(\mathbf{v}) \end{aligned}$$

Since $\mathbf{v}$ was arbitrary in V, this shows that $T = S$. □

Example 7.1.6

Let $V = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. Let $T : V \to W$ be a linear transformation. If $T(\mathbf{v}_1) = \cdots = T(\mathbf{v}_n) = \mathbf{0}$, show that $T = 0$, the zero transformation from V to W.

Solution. The zero transformation $0 : V \to W$ is defined by $0(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v}$ in V (Example 7.1.1), so $T(\mathbf{v}_i) = 0(\mathbf{v}_i)$ holds for each i. Hence $T = 0$ by Theorem 7.1.2.

Theorem 7.1.2 can be expressed as follows: If we know what a linear transformation $T : V \to W$ does to each vector in a spanning set for V, then we know what T does to *every* vector in V. If the spanning set is a basis, we can say much more.

Theorem 7.1.3

Let V and W be vector spaces and let $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ be a basis of V. Given any vectors $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$ in W (they need not be distinct), there exists a unique linear transformation $T : V \to W$ satisfying $T(\mathbf{b}_i) = \mathbf{w}_i$ for each $i = 1, 2, \dots, n$. In fact, the action of T is as follows: Given $\mathbf{v} = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n$ in V, v_i in $\mathbb{R}$, then

$$T(\mathbf{v}) = T(v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n) = v_1\mathbf{w}_1 + v_2\mathbf{w}_2 + \cdots + v_n\mathbf{w}_n.$$

Proof. If a transformation T *does* exist with $T(\mathbf{b}_i) = \mathbf{w}_i$ for each i, and if S is any other such transformation, then $T(\mathbf{b}_i) = \mathbf{w}_i = S(\mathbf{b}_i)$ holds for each i, so $S = T$ by Theorem 7.1.2. Hence T is unique if it exists, and it remains to show that there really is such a linear transformation. Given $\mathbf{v}$ in V, we must specify $T(\mathbf{v})$ in W. Because $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis of V, we have $\mathbf{v} = v_1\mathbf{b}_1 + \cdots + v_n\mathbf{b}_n$, where $v_1, \dots, v_n$ are *uniquely* determined by $\mathbf{v}$ (this is Theorem 6.3.1). Hence we may define $T : V \to W$ by

$$T(\mathbf{v}) = T(v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n) = v_1\mathbf{w}_1 + v_2\mathbf{w}_2 + \cdots + v_n\mathbf{w}_n$$

for all $\mathbf{v} = v_1\mathbf{b}_1 + \cdots + v_n\mathbf{b}_n$ in V. This satisfies $T(\mathbf{b}_i) = \mathbf{w}_i$ for each i; the verification that T is linear is left to the reader. □

This theorem shows that linear transformations can be defined almost at will: Simply specify where the basis vectors go, and the rest of the action is dictated by the linearity. Moreover, Theorem 7.1.2 shows that deciding whether two linear transformations are equal comes down to determining whether they have the same effect on the basis vectors. So, given a basis $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ of a vector space V, there is a different linear transformation $V \to W$ for every ordered selection $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$ of vectors in W (not necessarily distinct).

Example 7.1.7

Find a linear transformation $T : \mathbf{P}_2 \to \mathbf{M}_{22}$ such that

$$T(1+x) = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right], \quad T(x+x^2) = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right], \quad \text{and} \quad T(1+x^2) = \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right].$$

Solution. The set $\{1+x, x+x^2, 1+x^2\}$ is a basis of $\mathbf{P}_2$, so every vector $p = a + bx + cx^2$ in $\mathbf{P}_2$ is a linear combination of these vectors. In fact

$$p(x) = \frac{1}{2}(a+b-c)(1+x) + \frac{1}{2}(-a+b+c)(x+x^2) + \frac{1}{2}(a-b+c)(1+x^2)$$

Hence Theorem 7.1.3 gives

$$\begin{aligned} T[p(x)] &= \frac{1}{2}(a+b-c)\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] + \frac{1}{2}(-a+b+c)\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] + \frac{1}{2}(a-b+c)\left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right] \\ &= \frac{1}{2}\left[\begin{array}{cc} a+b-c & -a+b+c \\ -a+b+c & a-b+c \end{array}\right] \end{aligned}$$

Exercises for 7.1

Exercise 7.1.1 Show that each of the following functions is a linear transformation.

a. $T : \mathbb{R}^2 \to \mathbb{R}^2$; $T(x,\ y) = (x,\ -y)$ (reflection in the x axis)

b. $T : \mathbb{R}^3 \to \mathbb{R}^3$; $T(x,\ y,\ z) = (x,\ y,\ -z)$ (reflection in the x-y plane)

c. $T : \mathbb{C} \to \mathbb{C}$; $T(z) = \overline{z}$ (conjugation)

d. $T : \mathbf{M}_{mn} \to \mathbf{M}_{kl}$; $T(A) = PAQ$, P a $k \times m$ matrix, Q an $n \times l$ matrix, both fixed

e. $T : \mathbf{M}_{nn} \to \mathbf{M}_{nn}$; $T(A) = A^T + A$

f. $T : \mathbf{P}_n \to \mathbb{R}$; $T[p(x)] = p(0)$

g. $T : \mathbf{P}_n \to \mathbb{R}$; $T(r_0 + r_1x + \cdots + r_nx^n) = r_n$

h. $T : \mathbb{R}^n \to \mathbb{R}$; $T(\mathbf{x}) = \mathbf{x} \cdot \mathbf{z}$, $\mathbf{z}$ a fixed vector in $\mathbb{R}^n$

i. $T : \mathbf{P}_n \to \mathbf{P}_n$; $T[p(x)] = p(x+1)$

j. $T : \mathbb{R}^n \to V$; $T(r_1,\ \cdots,\ r_n) = r_1\mathbf{e}_1 + \cdots + r_n\mathbf{e}_n$ where $\{\mathbf{e}_1,\ \dots,\ \mathbf{e}_n\}$ is a fixed basis of V

k. $T : V \to \mathbb{R}$; $T(r_1\mathbf{e}_1 + \cdots + r_n\mathbf{e}_n) = r_1$, where $\{\mathbf{e}_1,\ \dots,\ \mathbf{e}_n\}$ is a fixed basis of V

Exercise 7.1.2 In each case, show that T is *not* a linear transformation.

a. $T : \mathbf{M}_{nn} \to \mathbb{R}$; $T(A) = \det A$

b. $T : \mathbf{M}_{nm} \to \mathbb{R}$; $T(A) = \operatorname{rank} A$

c. $T : \mathbb{R} \to \mathbb{R}$; $T(x) = x^2$

d. $T : V \to V$; $T(\mathbf{v}) = \mathbf{v} + \mathbf{u}$ where $\mathbf{u} \neq \mathbf{0}$ is a fixed vector in V (T is called the **translation** by $\mathbf{u}$)

Exercise 7.1.3 In each case, assume that T is a linear transformation.

a. If $T : V \to \mathbb{R}$ and $T(\mathbf{v}_1) = 1$, $T(\mathbf{v}_2) = -1$, find $T(3\mathbf{v}_1 - 5\mathbf{v}_2)$.

b. If $T : V \to \mathbb{R}$ and $T(\mathbf{v}_1) = 2$, $T(\mathbf{v}_2) = -3$, find $T(3\mathbf{v}_1 + 2\mathbf{v}_2)$.

c. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $T\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, find $T\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

d. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $T\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, find $T\begin{bmatrix} 1 \\ -7 \end{bmatrix}$.

e. If $T : \mathbf{P}_2 \to \mathbf{P}_2$ and $T(x+1) = x$, $T(x-1) = 1$, $T(x^2) = 0$, find $T(2 + 3x - x^2)$.

f. If $T : \mathbf{P}_2 \to \mathbb{R}$ and $T(x+2) = 1$, $T(1) = 5$, $T(x^2 + x) = 0$, find $T(2 - x + 3x^2)$.

Exercise 7.1.4 In each case, find a linear transformation with the given properties and compute $T(\mathbf{v})$.

a. $T : \mathbb{R}^2 \to \mathbb{R}^3$; $T(1,\ 2) = (1,\ 0,\ 1)$, $T(-1,\ 0) = (0,\ 1,\ 1)$; $\mathbf{v} = (2,\ 1)$

b. $T : \mathbb{R}^2 \to \mathbb{R}^3$; $T(2,\ -1) = (1,\ -1,\ 1)$, $T(1,\ 1) = (0,\ 1,\ 0)$; $\mathbf{v} = (-1,\ 2)$

c. $T : \mathbf{P}_2 \to \mathbf{P}_3$; $T(x^2) = x^3$, $T(x+1) = 0$, $T(x-1) = x$; $\mathbf{v} = x^2 + x + 1$

d. $T : \mathbf{M}_{22} \to \mathbb{R}$; $T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 3$, $T\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$, $T\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = 0 = T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$; $\mathbf{v} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Exercise 7.1.5 If $T : V \to V$ is a linear transformation, find $T(\mathbf{v})$ and $T(\mathbf{w})$ if:

a. $T(\mathbf{v} + \mathbf{w}) = \mathbf{v} - 2\mathbf{w}$ and $T(2\mathbf{v} - \mathbf{w}) = 2\mathbf{v}$

b. $T(\mathbf{v} + 2\mathbf{w}) = 3\mathbf{v} - \mathbf{w}$ and $T(\mathbf{v} - \mathbf{w}) = 2\mathbf{v} - 4\mathbf{w}$

Exercise 7.1.6 If $T : V \to W$ is a linear transformation, show that $T(\mathbf{v} - \mathbf{v}_1) = T(\mathbf{v}) - T(\mathbf{v}_1)$ for all $\mathbf{v}$ and $\mathbf{v}_1$ in V.

Exercise 7.1.7 Let $\{\mathbf{e}_1,\ \mathbf{e}_2\}$ be the standard basis of $\mathbb{R}^2$. Is it possible to have a linear transformation T such that $T(\mathbf{e}_1)$ lies in $\mathbb{R}$ while $T(\mathbf{e}_2)$ lies in $\mathbb{R}^2$? Explain your answer.

Exercise 7.1.8 Let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis of V and let $T : V \to V$ be a linear transformation.

a. If $T(\mathbf{v}_i) = \mathbf{v}_i$ for each i, show that $T = 1_V$.

b. If $T(\mathbf{v}_i) = -\mathbf{v}_i$ for each i, show that $T = -1$ is the scalar operator (see Example 7.1.1).

Exercise 7.1.9 If A is an $m \times n$ matrix, let $C_k(A)$ denote column k of A. Show that $C_k : \mathbf{M}_{mn} \to \mathbb{R}^m$ is a linear transformation for each $k = 1, \dots, n$.

Exercise 7.1.10 Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis of $\mathbb{R}^n$. Given k, $1 \leq k \leq n$, define $P_k : \mathbb{R}^n \to \mathbb{R}^n$ by $P_k(r_1\mathbf{e}_1 + \cdots + r_n\mathbf{e}_n) = r_k\mathbf{e}_k$. Show that P_k a linear transformation for each k.

Exercise 7.1.11 Let $S : V \to W$ and $T : V \to W$ be linear transformations. Given a in $\mathbb{R}$, define functions $(S+T) : V \to W$ and $(aT) : V \to W$ by $(S+T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v})$ and $(aT)(\mathbf{v}) = aT(\mathbf{v})$ for all $\mathbf{v}$ in V. Show that $S+T$ and aT are linear transformations.

Exercise 7.1.12 Describe all linear transformations $T : \mathbb{R} \to V$.

Exercise 7.1.13 Let V and W be vector spaces, let V be finite dimensional, and let $\mathbf{v} \neq \mathbf{0}$ in V. Given any $\mathbf{w}$ in W, show that there exists a linear transformation $T : V \to W$ with $T(\mathbf{v}) = \mathbf{w}$. [*Hint*: Theorem 6.4.1 and Theorem 7.1.3.]

Exercise 7.1.14 Given $\mathbf{y}$ in $\mathbb{R}^n$, define $S_\mathbf{y} : \mathbb{R}^n \to \mathbb{R}$ by $S_\mathbf{y}(\mathbf{x}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}$ in $\mathbb{R}^n$ (where $\cdot$ is the dot product introduced in Section 5.3).

a. Show that $S_\mathbf{y} : \mathbb{R}^n \to \mathbb{R}$ is a linear transformation for any $\mathbf{y}$ in $\mathbb{R}^n$.

b. Show that every linear transformation $T : \mathbb{R}^n \to \mathbb{R}$ arises in this way; that is, $T = S_\mathbf{y}$ for some $\mathbf{y}$ in $\mathbb{R}^n$. [*Hint*: If $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$, write $S_\mathbf{y}(\mathbf{e}_i) = y_i$ for each i. Use Theorem 7.1.1.]

Exercise 7.1.15 Let $T : V \to W$ be a linear transformation.

a. If U is a subspace of V, show that $T(U) = \{T(\mathbf{u}) \mid \mathbf{u} \text{ in } U\}$ is a subspace of W (called the **image** of U under T).

b. If P is a subspace of W, show that $\{\mathbf{v} \text{ in } V \mid T(\mathbf{v}) \text{ in } P\}$ is a subspace of V (called the **preimage** of P under T).

Exercise 7.1.16 Show that differentiation is the only linear transformation $\mathbf{P}_n \to \mathbf{P}_n$ that satisfies $T(x^k) = kx^{k-1}$ for each $k = 0, 1, 2, \dots, n$.

Exercise 7.1.17 Let $T : V \to W$ be a linear transformation and let $\mathbf{v}_1, \dots, \mathbf{v}_n$ denote vectors in V.

a. If $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ is linearly independent, show that $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is also independent.

b. Find $T : \mathbb{R}^2 \to \mathbb{R}^2$ for which the converse of part (a) is false.

Exercise 7.1.18 Suppose $T : V \to V$ is a linear operator with the property that $T[T(\mathbf{v})] = \mathbf{v}$ for all $\mathbf{v}$ in V. (For example, transposition in $\mathbf{M}_{nn}$ or conjugation in $\mathbb{C}$.) If $\mathbf{v} \neq \mathbf{0}$ in V, show that $\{\mathbf{v}, T(\mathbf{v})\}$ is linearly independent if and only if $T(\mathbf{v}) \neq \mathbf{v}$ and $T(\mathbf{v}) \neq -\mathbf{v}$.

Exercise 7.1.19 If a and b are real numbers, define $T_{a,\,b} : \mathbb{C} \to \mathbb{C}$ by $T_{a,\,b}(r+si) = ra + sbi$ for all $r+si$ in $\mathbb{C}$.

a. Show that $T_{a,\,b}$ is linear and $T_{a,\,b}(\overline{z}) = \overline{T_{a,\,b}(z)}$ for all z in $\mathbb{C}$. (Here $\overline{z}$ denotes the conjugate of z.)

b. If $T : \mathbb{C} \to \mathbb{C}$ is linear and $T(\overline{z}) = \overline{T(z)}$ for all z in $\mathbb{C}$, show that $T = T_{a,\,b}$ for some real a and b.

Exercise 7.1.20 Show that the following conditions are equivalent for a linear transformation $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$.

1. $\text{tr}\,[T(A)] = \text{tr}\,A$ for all A in $\mathbf{M}_{22}$.

2. $T\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = r_{11}B_{11} + r_{12}B_{12} + r_{21}B_{21} + r_{22}B_{22}$ for matrices B_{ij} such that $\text{tr}\,B_{11} = 1 = \text{tr}\,B_{22}$ and $\text{tr}\,B_{12} = 0 = \text{tr}\,B_{21}$.

Exercise 7.1.21 Given a in $\mathbb{R}$, consider the **evaluation** map $E_a : \mathbf{P}_n \to \mathbb{R}$ defined in Example 7.1.3.

a. Show that E_a is a linear transformation satisfying the additional condition that $E_a(x^k) = [E_a(x)]^k$ holds for all $k = 0, 1, 2, \dots$. [*Note*: $x^0 = 1$.]

b. If $T : \mathbf{P}_n \to \mathbb{R}$ is a linear transformation satisfying $T(x^k) = [T(x)]^k$ for all $k = 0, 1, 2, \dots$, show that $T = E_a$ for some a in R.

Exercise 7.1.22 If $T : \mathbf{M}_{nn} \to \mathbb{R}$ is any linear transformation satisfying $T(AB) = T(BA)$ for all A and B in $\mathbf{M}_{nn}$, show that there exists a number k such that $T(A) = k \operatorname{tr} A$ for all A. (See Lemma 5.5.1.) [*Hint*: Let E_{ij} denote the $n \times n$ matrix with 1 in the (i, j) position and zeros elsewhere.

Show that $E_{ik}E_{lj} = \begin{cases} 0 & \text{if } k \neq l \\ E_{ij} & \text{if } k = l \end{cases}$. Use this to show that $T(E_{ij}) = 0$ if $i \neq j$ and $T(E_{11}) = T(E_{22}) = \cdots = T(E_{nn})$. Put $k = T(E_{11})$ and use the fact that $\{E_{ij} \mid 1 \leq i, j \leq n\}$ is a basis of $\mathbf{M}_{nn}$.]

Exercise 7.1.23 Let $T : \mathbb{C} \to \mathbb{C}$ be a linear transformation of the real vector space $\mathbb{C}$ and assume that $T(a) = a$ for every real number a. Show that the following are equivalent:

a. $T(zw) = T(z)T(w)$ for all z and w in $\mathbb{C}$.

b. Either $T = 1_{\mathbb{C}}$ or $T(z) = \overline{z}$ for each z in $\mathbb{C}$ (where $\overline{z}$ denotes the conjugate).

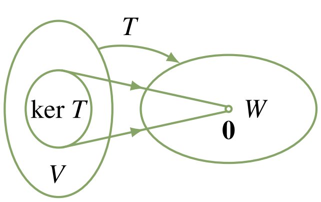
7.2 Kernel and Image of a Linear Transformation

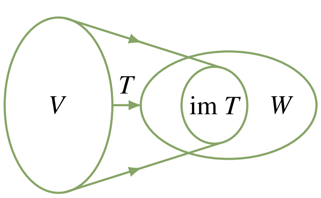
This section is devoted to two important subspaces associated with a linear transformation $T : V \to W$.

Definition 7.2 Kernel and Image of a Linear Transformation

The ***kernel*** *of* T *(denoted* $\ker T$*) and the* ***image*** *of* T *(denoted* $\operatorname{im} T$ *or* $T(V)$*) are defined by*

$$\ker T = \{\mathbf{v} \text{ in } V \mid T(\mathbf{v}) = \mathbf{0}\}$$
$$\operatorname{im} T = \{T(\mathbf{v}) \mid \mathbf{v} \text{ in } V\} = T(V)$$





The kernel of T is often called the **nullspace** of T because it consists of all vectors $\mathbf{v}$ in V satisfying the *condition* that $T(\mathbf{v}) = \mathbf{0}$. The image of T is often called the **range** of T and consists of all vectors $\mathbf{w}$ in W of the *form* $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v}$ in V. These subspaces are depicted in the diagrams.

Example 7.2.1

Let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation induced by the $m \times n$ matrix A, that is $T_A(\mathbf{x}) = A\mathbf{x}$ for all columns $\mathbf{x}$ in $\mathbb{R}^n$. Then

$$\ker T_A = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\} = \operatorname{null} A \quad \text{and}$$
$$\operatorname{im} T_A = \{A\mathbf{x} \mid \mathbf{x} \text{ in } \mathbb{R}^n\} = \operatorname{im} A$$

Hence the following theorem extends Example 5.1.2.