

# Math 221: LINEAR ALGEBRA

## Chapter 1. Systems of Linear Equations §1-1. Solutions and Elementary Operations

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Emory University, 2021 Spring

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<sup>1</sup>Slides are adapted from those by Karen Seyffarth from University of Calgary.

Solutions of Linear Equations

Elementary Operations

The Augmented Matrix

Solving a System using Back Substitution

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## Objective:

Can we do the same for linear equations in more variables?

## Definition

A **linear equation** is an expression

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where  $n \geq 1$ ,  $a_1, \dots, a_n$  are real numbers, **not all of them equal to zero**, and  $b$  is a real number.

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**Solve a system** means ‘find **all** solutions to the system’.

## Example

A system of linear equations:

$$\begin{array}{rcccccc} x_1 & - & 2x_2 & - & 7x_3 & = & -1 \\ -x_1 & + & 3x_2 & + & 6x_3 & = & 0 \end{array}$$

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► constant terms:

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$x_1 = -3$ ,  $x_2 = -1$ ,  $x_3 = 0$  is a **solution** to the system

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The system above is **consistent**, meaning that the system has at least one solution.

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Why are there no solutions?

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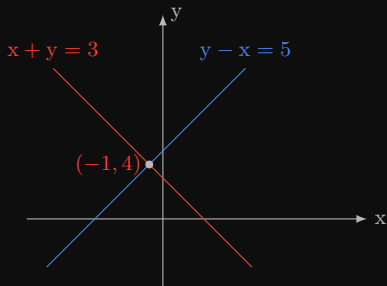
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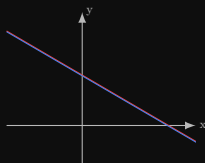
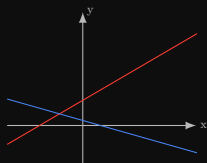
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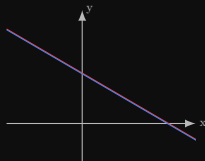
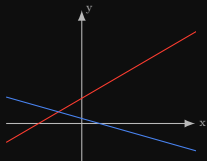


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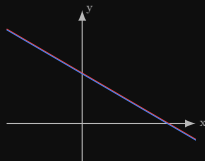
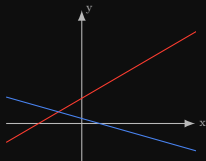
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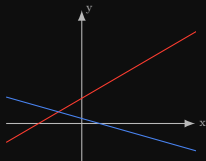


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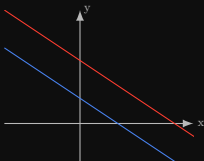
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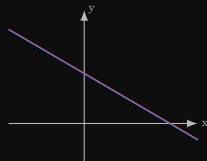
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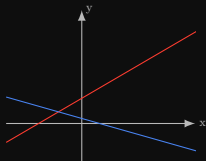
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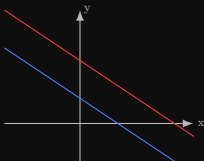
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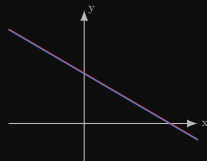
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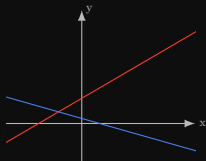
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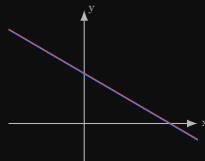
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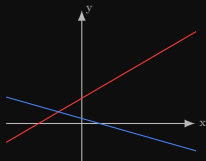
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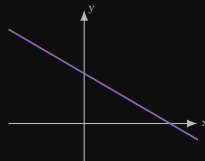
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## Remark

We will see in what follows that this generalizes to systems of linear equations in more than two variables.

## Example

The system of linear equations in three variables that we saw earlier

$$\begin{array}{rcccccc} x_1 & - & 2x_2 & - & 7x_3 & = & -1 \\ -x_1 & + & 3x_2 & + & 6x_3 & = & 0, \end{array}$$

has solutions  $x_1 = -3 + 9s$ ,  $x_2 = -1 + s$ ,  $x_3 = s$  where  $s$  is any real number (written  $s \in \mathbb{R}$ ).

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$s$  is called a **parameter**, and the expression

$$x_1 = -3 + 9s, \quad x_2 = -1 + s, \quad x_3 = s, \quad \text{where } s \in \mathbb{R}$$

is called the **general solution** in parametric form.

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The two systems of linear equations

$$\begin{array}{rcl} 2x & + & y = 2 \\ 3x & & = 3 \end{array} \quad \text{and} \quad \begin{array}{rcl} x & + & y = 1 \\ & & y = 0 \end{array}$$

are equivalent because both systems have the unique solution  $x = 1, y = 0$ .



Solutions of Linear Equations

**Elementary Operations**

The Augmented Matrix

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- Type II: Multiply an equation by a nonzero number,  $-2r_1$ .
- Type III: Add a multiple of one equation to a different equation,  $3r_3 + r_2$ .

## Example

Consider the system of linear eq's:

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3. Add 3 time the second equation to the first equation (Type III):

$$\begin{array}{rccccrcr} & & & & 7x_2 & + & 11x_3 & = & 2 \\ 3r_2 + r_1 & -x_1 & + & 3x_2 & + & 6x_3 & = & 1 \\ 2x_1 & & & & - & x_3 & = & 3 \end{array}$$

## Theorem (Elementary Operations and Solutions)

Suppose that a sequence of elementary operations is performed on a system of linear equations. Then the resulting system has the same set of solutions as the original, so the two systems are equivalent.

## Theorem (Elementary Operations and Solutions)

Suppose that a sequence of elementary operations is performed on a system of linear equations. Then the resulting system has the same set of solutions as the original, so the two systems are equivalent.

As a consequence, performing a sequence of elementary operations on a system of linear equations results in an equivalent system of linear equations, with the exact same solutions.

Solutions of Linear Equations

Elementary Operations

**The Augmented Matrix**

Solving a System using Back Substitution



## The Augmented Matrix

Represent a system of linear equations with its augmented matrix.

### Example

The system of linear equations

$$\begin{array}{rcccccc} x_1 & - & 2x_2 & - & 7x_3 & = & -1 \\ -x_1 & + & 3x_2 & + & 6x_3 & = & 0 \end{array}$$

is represented by the **augmented matrix**

$$\left[ \begin{array}{ccc|c} 1 & -2 & -7 & -1 \\ -1 & 3 & 6 & 0 \end{array} \right]$$

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## Remark

Two other **matrices** associated with a system of linear equations are the **coefficient matrix** and the **constant matrix**:

$$\left[ \begin{array}{ccc} 1 & -2 & -7 \\ -1 & 3 & 6 \end{array} \right], \quad \left[ \begin{array}{c} -1 \\ 0 \end{array} \right].$$



For convenience, instead of performing elementary operations on a system of linear equations, perform corresponding **elementary row operations** on the corresponding augmented matrix.

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Type I: Interchange two rows.

### Example

Interchange rows 1 and 3.

$$\left[ \begin{array}{cccc|c} 2 & -1 & 0 & 5 & -3 \\ -2 & 0 & 3 & 3 & -1 \\ 0 & 5 & -6 & 1 & 0 \\ 1 & -4 & 2 & 2 & 2 \end{array} \right] \xrightarrow[r_1 \leftrightarrow r_3]{} \left[ \begin{array}{cccc|c} 0 & 5 & -6 & 1 & 0 \\ -2 & 0 & 3 & 3 & -1 \\ 2 & -1 & 0 & 5 & -3 \\ 1 & -4 & 2 & 2 & 2 \end{array} \right]$$

Type II: Multiply a row by a nonzero number.

### Example

Multiply row 4 by 2.

$$\left[ \begin{array}{cccc|c} 2 & -1 & 0 & 5 & -3 \\ -2 & 0 & 3 & 3 & -1 \\ 0 & 5 & -6 & 1 & 0 \\ 1 & -4 & 2 & 2 & 2 \end{array} \right] \xrightarrow{2r_4} \left[ \begin{array}{cccc|c} 2 & -1 & 0 & 5 & -3 \\ -2 & 0 & 3 & 3 & -1 \\ 0 & 5 & -6 & 1 & 0 \\ 2 & -8 & 4 & 4 & 4 \end{array} \right]$$

Type III: Add a multiple of one row to a different row.

### Example

Add 2 times row 4 to row 2.

$$\left[ \begin{array}{cccc|c} 2 & -1 & 0 & 5 & -3 \\ -2 & 0 & 3 & 3 & -1 \\ 0 & 5 & -6 & 1 & 0 \\ 1 & -4 & 2 & 2 & 2 \end{array} \right] \xrightarrow{2r_4+r_2} \left[ \begin{array}{cccc|c} 2 & -1 & 0 & 5 & -3 \\ 0 & -8 & 7 & 7 & 3 \\ 0 & 5 & -6 & 1 & 0 \\ 1 & -4 & 2 & 2 & 2 \end{array} \right]$$

## Definition

Two matrices  $A$  and  $B$  are **row equivalent** (or simply equivalent) if one can be obtained from the other by a sequence of **elementary row operations**.

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## Problem

Prove that  $A$  can be obtained from  $B$  by a sequence of elementary row operations if and only if  $B$  can be obtained from  $A$  by a sequence of elementary row operations.

Prove that row equivalence is an equivalence relation.

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# Solving a System using Back Substitution

## Problem

Solve the system using back substitution

$$2x + y = 4$$

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Add  $(-2)$  times the second equation to the first equation.

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The result is an equivalent system

$$7y = 2$$

$$x - 3y = 1$$



## Solution (continued)

The first equation of the system,

$$7y = 2$$

can be rearranged to give us

$$y = \frac{2}{7}.$$

Substituting  $y = \frac{2}{7}$  into second equation:

$$x - 3y = x - 3\left(\frac{2}{7}\right) = 1,$$

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We shall describe an **algorithm** for solving any given system of linear equations.