Math 221: LINEAR ALGEBRA

Chapter 1. Systems of Linear Equations
§1-2. Gaussian Elimination

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Row-Echelon Form

Solving Systems of Linear Equations – Gaussian Elimination

Rank

Uniqueness of the Reduced Row-Echelon Form

One Application

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Row-Echelon Matrix

Definition

A matrix is called a row-echelon matrix if

- All rows consisting entirely of zeros are at the bottom.
- The first nonzero entry in each nonzero row is a 1 (called the leading 1 for that row).
- Each leading 1 is to the right of all leading 1's in rows above it.

A matrix is said to be in the row-echelon form (REF) if it a row-echelon matrix.

Example

$$\begin{bmatrix} 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $*$ can be any number.

Definition

A matrix is called a reduced row-echelon matrix if

- Row-echelon matrix.
- Each leading 1 is the only nonzero entry in its column.

A matrix is said to be in the reduced row-echelon form (RREF) if it a reduced row-echelon matrix.

Example

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $*$ can be any number.

Examples

Which of the following matrices are in the REF?

Which ones are in the RREF?

(a) $\begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Example

Suppose that the following matrix is the augmented matrix of a system of linear equations. We see from this matrix that the system of linear equations has four equations and seven variables.

$$
\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \\
\end{array}
$$

$$
\left[\begin{array}{ccccccc|c}
1 & -3 & 4 & -2 & 5 & -7 & 0 & 4 \\
0 & 0 & 1 & 8 & 0 & 3 & -7 & 0 \\
0 & 0 & 0 & 1 & 1 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2
\end{array}\right]
$$

Note that the matrix is a row-echelon matrix.

- Each column of the matrix corresponds to a variable, and the leading variables are the variables that correspond to columns containing leading ones.
- The remaining variables are called non-leading variables.

We will use elementary row operations to transform a matrix to row-echelon (REF) or reduced row-echelon form (RREF).

Solving Systems of Linear Equations – Gaussian Elimination

Theorem

Every matrix can be brought to (reduced) row-echelon form by a sequence of elementary row operations.

Gaussian Elimination

To solve a system of linear equations proceed as follows:

1. Carry the augmented matrix to a reduced row-echelon matrix using elementary row operations.
2. If a row of the form $[0\ 0\ \cdots 0 \mid 1]$ occurs, the system is inconsistent.
3. Otherwise assign the nonleading variables (if any) parameters and use the equations corresponding to the reduced row-echelon matrix to solve for the leading variables in terms of the parameters.

Problem

Solve the system

$$\begin{cases} 2x + y + 3z = 1 \\ 2y - z + x = 0 \\ 9z + x - 4y = 2 \end{cases}$$

Solution

$$\left[\begin{array}{rrr|r} 2 & 1 & 3 & 1 \\ 1 & 2 & -1 & 0 \\ 1 & -4 & 9 & 2 \end{array}\right] \rightarrow^{r_1 \leftrightarrow r_2} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ 2 & 1 & 3 & 1 \\ 1 & -4 & 9 & 2 \end{array}\right]$$

$$\rightarrow^{-2r_1+r_2, -r_1+r_3} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ 0 & -3 & 5 & 1 \\ 0 & -6 & 10 & 2 \end{array}\right] \rightarrow^{-2r_2+r_3} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ 0 & -3 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow^{-\frac{1}{3}r_2} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ 0 & 1 & -5/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow^{-2r_2+r_1} \left[\begin{array}{rrr|r} 1 & 0 & 7/3 & 2/3 \\ 0 & 1 & -5/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution (continued)

Given the reduced row-echelon matrix

$$\left[\begin{array}{ccr|r} 1 & 0 & 7/3 & 2/3 \\ 0 & 1 & -5/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x and y are leading variables; z is a non-leading variable and so assign a parameter to z. Thus the solution to the original system is given by

$$\left.\begin{array}{rcrcl} x & = & \frac{2}{3} & - & \frac{7}{3}s \\ y & = & -\frac{1}{3} & + & \frac{5}{3}s \\ z & = & s & & \end{array}\right\} \text{ for all } s \in \mathbb{R}.$$

Problem

Solve the system

$$\left\{\begin{array}{rcrcrcr} x & + & y & + & 2z & = & -1 \\ y & + & 2x & + & 3z & = & 0 \\ z & - & 2y & & & = & 2 \end{array}\right.$$

Solution

$$\left[\begin{array}{rrr|r} 1 & 1 & 2 & -1 \\ 2 & 1 & 3 & 0 \\ 0 & -2 & 1 & 2 \end{array}\right] \rightarrow^{-2r_1+r_2} \left[\begin{array}{rrr|r} 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & 2 \\ 0 & -2 & 1 & 2 \end{array}\right]$$

$$\rightarrow^{-1\cdot r_2} \left[\begin{array}{rrr|r} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & -2 & 1 & 2 \end{array}\right] \rightarrow^{2r_2+r_3} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 3 & -2 \end{array}\right]$$

$$\rightarrow^{\frac{1}{3}r_3} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -2/3 \end{array}\right] \rightarrow^{-r_3+r_2,-r_3+r_1} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 5/3 \\ 0 & 1 & 0 & -4/3 \\ 0 & 0 & 1 & -2/3 \end{array}\right]$$

The unique solution is $x = 5/3$, $y = -4/3$, $z = -2/3$.

Check your answer!

Problem

Solve the system $\left\{ \begin{array}{rcrcrcr} -3x_1 & - & 9x_2 & + & x_3 & = & -9 \\ 2x_1 & + & 6x_2 & - & x_3 & = & 6 \\ x_1 & + & 3x_2 & - & x_3 & = & 2 \end{array} \right.$

Solution

$$\left[\begin{array}{rrr|r} 1 & 3 & -1 & 2 \\ 2 & 6 & -1 & 6 \\ -3 & -9 & 1 & -9 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 3 & -1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & -3 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The last row of the final matrix corresponds to the equation

$$0x_1 + 0x_2 + 0x_3 = 1$$

which is impossible!

Therefore, this system is inconsistent, i.e., it has no solutions.

Problem (General Patterns for Systems of Linear Equations)

Find all values of a, b and c (or conditions on a, b and c) so that the system

$$\begin{array}{rcrcrcr} 2x & + & 3y & + & az & = & b \\ & - & y & + & 2z & = & c \\ x & + & 3y & - & 2z & = & 1 \end{array}$$

has (i) a unique solution, (ii) no solutions, and (iii) infinitely many solutions. In (i) and (iii), find the solution(s).

Solution

$$\left[\begin{array}{rrr|r} 2 & 3 & a & b \\ 0 & -1 & 2 & c \\ 1 & 3 & -2 & 1 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 3 & -2 & 1 \\ 0 & -1 & 2 & c \\ 2 & 3 & a & b \end{array}\right]$$

Solution (continued)

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 1 \\ 0 & -1 & 2 & c \\ 2 & 3 & a & b \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & -2 & 1 \\ 0 & -1 & 2 & c \\ 0 & -3 & a+4 & b-2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 3 & -2 & 1 \\ 0 & 1 & -2 & -c \\ 0 & -3 & a+4 & b-2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 4 & 1+3c \\ 0 & 1 & -2 & -c \\ 0 & 0 & a-2 & b-2-3c \end{array}\right]$$

Case 1. $a - 2 \neq 0$, i.e., $a \neq 2$. In this case,

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 4 & 1+3c \\ 0 & 1 & -2 & -c \\ 0 & 0 & 1 & \frac{b-2-3c}{a-2} \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1+3c-4\left(\frac{b-2-3c}{a-2}\right) \\ 0 & 1 & 0 & -c+2\left(\frac{b-2-3c}{a-2}\right) \\ 0 & 0 & 1 & \frac{b-2-3c}{a-2} \end{array}\right]$$

Solution (continued)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1+3c-4\left(\frac{b-2-3c}{a-2}\right) \\ 0 & 1 & 0 & -c+2\left(\frac{b-2-3c}{a-2}\right) \\ 0 & 0 & 1 & \frac{b-2-3c}{a-2} \end{array}\right]$$

(i) When $a \neq 2$, the unique solution is

$$x = 1+3c-4\left(\frac{b-2-3c}{a-2}\right)$$

$$y = -c+2\left(\frac{b-2-3c}{a-2}\right)$$

$$z = \frac{b-2-3c}{a-2}$$

Solution (continued)

Case 2. If $a = 2$, then the augmented matrix becomes

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 1+3c \\ 0 & 1 & -2 & -c \\ 0 & 0 & a-2 & b-2-3c \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 4 & 1+3c \\ 0 & 1 & -2 & -c \\ 0 & 0 & 0 & b-2-3c \end{array}\right]$$

From this we see that the system has no solutions when $b - 2 - 3c \neq 0$.

(ii) When $a = 2$ and $b - 3c \neq 2$, the system has no solutions.

Solution (continued)

Finally when $a = 2$ and $b - 3c = 2$, the augmented matrix becomes

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 1+3c \\ 0 & 1 & -2 & -c \\ 0 & 0 & 0 & b-2-3c \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 4 & 1+3c \\ 0 & 1 & -2 & -c \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and the system has infinitely many solutions.

(iii) When $a = 2$ and $b - 3c = 2$, the system has infinitely many solutions:

$$\left.\begin{array}{rcrcr} x & = & 1+3c & - & 4s \\ y & = & -c & + & 2s \\ z & = & s & & \end{array}\right\} \quad \text{for all } s \in \mathbb{R}.$$

■

Row-Echelon Form

Solving Systems of Linear Equations – Gaussian Elimination

Rank

Uniqueness of the Reduced Row-Echelon Form

One Application

Rank

Definition

The rank of a matrix A, denoted $\operatorname{rank} A$, is the number of leading 1's in any row-echelon matrix obtained from A by performing elementary row operations.

Suppose A is the augmented matrix of a consistent system of m linear equations in n variables, and rank $A = r$.

$$m \left\{ \underbrace{\left[\begin{array}{cccc|c} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{array} \right]}_{n} \right. \rightarrow \underbrace{\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]}_{r \text{ leading } 1's}$$

Then the set of solutions to the system has $n - r$ parameters, so

- if $r < n$, there is at least one parameter, and the system has infinitely many solutions;
- if $r = n$, there are no parameters, and the system has a unique solution.

Problem

Find the rank of $A = \begin{bmatrix} a & b & 5 \\ 1 & -2 & 1 \end{bmatrix}$.

Solution

$$\begin{bmatrix} a & b & 5 \\ 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ a & b & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ 0 & b+2a & 5-a \end{bmatrix}$$

If $b + 2a = 0$ and $5 - a = 0$, i.e., $a = 5$ and $b = -10$, then rank $A = 1$. Otherwise, rank $A = 2$.

For any system of linear equations, exactly one of the following holds:

1. the system is inconsistent;
2. the system has a unique solution, i.e., exactly one solution;
3. the system has infinitely many solutions.

One can see what case applies by looking at the RREF matrix equivalent to the augmented matrix of the system and distinguishing three cases:

1. The last nonzero row is $[0, \cdots, 0, 1]$: no solution.
2. The last nonzero row is not $[0, \cdots, 0, 1]$ and all variables are leading: unique solution.
3. The last nonzero row is not $[0, \cdots, 0, 1]$ and there are non-leading variables: infinitely many solutions.

Problem

Solve the system

$$\begin{array}{rcrcrcrcrcr} -3x_1 & + & 6x_2 & - & 4x_3 & - & 9x_4 & + & 3x_5 & = & -1 \\ -x_1 & + & 2x_2 & - & 2x_3 & - & 4x_4 & - & 3x_5 & = & 3 \\ x_1 & - & 2x_2 & + & 2x_3 & + & 2x_4 & - & 5x_5 & = & 1 \\ x_1 & - & 2x_2 & + & x_3 & + & 3x_4 & - & x_5 & = & 1 \end{array}$$

Solution

Begin by putting the augmented matrix in reduced row-echelon form.

$$\left[\begin{array}{rrrrr|r} 1 & -2 & 2 & 2 & -5 & 1 \\ -3 & 6 & -4 & -9 & 3 & -1 \\ -1 & 2 & -2 & -4 & -3 & 3 \\ 1 & -2 & 1 & 3 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrrrr|r} 1 & -2 & 0 & 0 & -13 & 9 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system is consistent. The rank of the augmented matrix is 3.
Since the system is consistent, the set of solutions has $5 - 3 = 2$ parameters.

Solution (continued)

From the reduced row-echelon matrix

$$\left[\begin{array}{rrrrr|r} 1 & -2 & 0 & 0 & -13 & 9 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

we obtain the general solution

$$\left.\begin{array}{lcl} x_1 & = & 9 + 2r + 13s \\ x_2 & = & r \\ x_3 & = & -2 \\ x_4 & = & -2 - 4s \\ x_5 & = & s \end{array}\right\} \quad \forall r, s \in \mathbb{R}$$

The solution has two parameters (r and s) as we expected.

Uniqueness of the Reduced Row-Echelon Form

Theorem

Systems of linear equations that correspond to row equivalent augmented matrices have exactly the same solutions.

Theorem

Every matrix A is row equivalent to a unique reduced row-echelon matrix.

Problem

Solve the system

$$\begin{array}{ccccccc} 2x & + & y & + & 3z & = & 1 \\ 2y & - & z & + & x & = & 0 \\ 9z & + & x & - & 4y & = & 2 \end{array}$$

Solution

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 1 & 2 & -1 & 0 \\ 1 & -4 & 9 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 1 & 3 & 1 \\ 1 & -4 & 9 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 5 & 1 \\ 0 & -6 & 10 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -\frac{5}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -\frac{7}{3} & -\frac{2}{3} \\ 0 & 1 & -\frac{5}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution (continued)

This row-echelon matrix corresponds to the system

$$\begin{array}{rcrcrcr} x & + & 0y & + & \frac{7}{3}z & = & -\frac{2}{3} \\ & & y & - & \frac{5}{3}z & = & -\frac{1}{3} \end{array},$$

and thus

$$\begin{aligned} x &= \tfrac{2}{3} - \tfrac{7}{3}z \\ y &= -\tfrac{1}{3} + \tfrac{5}{3}z \end{aligned}$$

Setting $z = s$, where $s \in \mathbb{R}$, gives us (as before):

$$\begin{array}{rcrcr} x & = & \frac{2}{3} & - & \frac{7}{3}s \\ y & = & -\frac{1}{3} & + & \frac{5}{3}s \\ z & = & s & & \end{array}$$

Always check your answer! ■

Row-Echelon Form

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One Application

One Application

Problem

Derive the formula for $1^r + 2^r + \cdots + n^r$ for $r = 3$.

Solution

We know that $1^3 + 2^3 + \cdots + n^3$ is a polynomial in n of oder 4, namely,

$$1^3 + 2^3 + \cdots + n^3 = a_0 + a_1 n + a_2 n^2 + a_3 n^3 + a_4 n^4.$$

It is easy to see that when $n = 0$, both sides should be equal to zero. Hence, $a_0 = 0$. Now we have 4 unknowns, $a_1, \cdots, a_4$. We can let $n = 1, \cdots, 4$ to form 4 equations in order to find these unknowns:

$$\begin{array}{ccccccccccc}
1^1 a_1 & + & 1^2 a_2 & + & 1^3 a_3 & + & 1^4 a_4 & = & 1^3 & & (n = 1) \\
2^1 a_1 & + & 2^2 a_2 & + & 2^3 a_3 & + & 2^4 a_4 & = & 1^3 + 2^3 & & (n = 2) \\
3^1 a_1 & + & 3^2 a_2 & + & 3^3 a_3 & + & 3^4 a_4 & = & 1^3 + 2^3 + 3^3 & & (n = 3) \\
4^1 a_1 & + & 4^2 a_2 & + & 4^3 a_3 & + & 4^4 a_4 & = & 1^3 + 2^3 + 3^3 + 4^3 & & (n = 4)
\end{array}$$

Solution (continued)

Hence, we have the following augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & 4 & 8 & 16 & 9 \\ 3 & 9 & 27 & 81 & 36 \\ 4 & 16 & 64 & 256 & 100 \end{array}\right)$$

You can use Octave or Matlab to compute the reduced echelon form:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/4 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 1/4 \end{array}\right)$$

Therefore, we have that

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2}{4} + \frac{n^3}{2} + \frac{n^4}{4} = \frac{1}{4}n^2(n+1)^2.$$

■