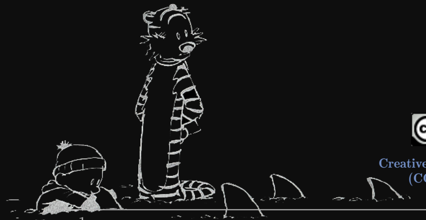
Math 221: LINEAR ALGEBRA

Chapter 2. Matrix Algebra
§2-3. Matrix Multiplication

Le Chen[1]

Emory University, 2021 Spring

(last updated on 01/31/2021)



[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Matrix Multiplication

Properties of Matrix Multiplication

Matrix Multiplication

Properties of Matrix Multiplication

Matrix Multiplication

Definition (Product of two matrices)

Let A be an $m \times n$ matrix and let $B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_p \end{bmatrix}$ be an $n \times p$ matrix, whose columns are $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p$. The product of A and B is the matrix

$$AB = A \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_p \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_p \end{bmatrix}$$

i.e., the first column of AB is $A\vec{b}_1$, the second column of AB is $A\vec{b}_2$, etc. Note that AB has size $m \times p$.

Problem

Find the product AB of matrices

$$\mathrm{A} = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathrm{B} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}.$$

Solution

AB has columns

$$\mathrm{A}\vec{b}_1 = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix},$$

$$\mathrm{A}\vec{b}_2 = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix},$$

$$\mathrm{A}\vec{b}_3 = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}.$$

Thus, $\mathrm{AB} = \begin{bmatrix} 4 & -1 & -2 \\ -1 & 4 & 0 \end{bmatrix}$.

Definition

Let A and B be matrices, and suppose that A is $m \times n$.

- In order for the product AB to exist, the number of rows in B must be equal to the number of columns in A, implying that B is an $n \times p$ matrix for some p.
- When defined, AB is an $m \times p$ matrix.

If the product is defined, then A and B are said to be compatible for (matrix) multiplication.

Example

As we saw in the previous problem

$$\overset{2\times3}{\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}} \overset{3\times3}{\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}} = \overset{2\times3}{\begin{bmatrix} 4 & -1 & -2 \\ -1 & 4 & 0 \end{bmatrix}}$$

Note that the product

$$\overset{3\times3}{\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}} \overset{2\times3}{\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}}$$

does not exist.

Example (Multiplication by the zero matrix)

Compute the product AO for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and the 2×2 zero matrix given by $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad AO = O.$$

■

Definition (The (i, j)-entry of a product)

Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be an $n \times p$ matrix. Then the (i, j)-entry of AB is given by the dot product of row i of A and column j of B:

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

$$\underset{m \times n}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}} \underset{n \times s}{\begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1s} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2s} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{ns} \end{bmatrix}} = \underset{m \times s}{\begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1s} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & \boxed{c_{ij}} & \cdots & c_{is} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{ms} \end{bmatrix}}$$

Example

Using the above definition, the $(2,3)$-entry of the product

$$\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}$$

is computed by the dot product of the second row of the first matrix and the third column of the second matrix:

$$2 \times 2 + (-1) \times 4 + 1 \times 0 = 4 - 4 + 0 = 0.$$

Matrix Multiplication

Properties of Matrix Multiplication

Properties of Matrix Multiplication

Given matrices A and B, is $AB = BA$?

Problem

Let

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 0 \\ 1 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & -2 & 1 & -3 \end{bmatrix}$$

- Does AB exist? If so, compute it.
- Does BA exist? If so, compute it.

Solution

$$AB = \begin{bmatrix} 7 & -5 & 4 & -6 \\ -3 & 3 & -6 & 0 \\ -11 & 7 & -2 & 12 \end{bmatrix}$$

BA does not exist!

Problem

Let

$$G = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- Does GH exist? If so, compute it.
- Does HG exist? If so, compute it.

Solution

$$GH = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$HG = \begin{bmatrix} 1 \end{bmatrix}$$

Remark

In this example, GH and HG both exist, but they are not equal. They aren't even the same size!

Problem

Let

$$P = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix}$$

- Does PQ exist? If so, compute it.
- Does QP exist? If so, compute it.

Solution

$$PQ = \begin{bmatrix} -1 & 1 \\ -2 & -1 \end{bmatrix}$$

$$QP = \begin{bmatrix} 1 & -1 \\ 6 & -3 \end{bmatrix}$$

■

Remark

In this example, PQ and QP both exist and are the same size, but $PQ \neq QP$.

Fact

The three preceding problems illustrate an important property of matrix multiplication.

> In general, matrix multiplication is not commutative, i.e., the order of the matrices in the product is important.

In other words, in general

$$AB \neq BA.$$

Multiplying from left or right, it MATTERS!

Problem

Let

$$U = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

- Does UV exist? If so, compute it.
- Does VU exist? If so, compute it.

Solution

$$UV = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

$$VU = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

■

Remark

In this particular example, the matrices commute, i.e., $UV = VU$.

Theorem (Properties of Matrix Multiplication)

Let A, B, and C be matrices of the appropriate sizes, and let $r \in \mathbb{R}$ be a scalar. Then the following properties hold.

1. $IA = A$ and $AI = A$.
2. $A(B + C) = AB + AC$.
   (matrix multiplication distributes over matrix addition).
3. $(B + C)A = BA + CA$.
   (matrix multiplication distributes over matrix addition).
4. $A\left(BC\right) = \left(AB\right)C$. (matrix multiplication is associative).
5. $r(AB) = (rA)B = A(rB)$.
6. $\left(AB\right)^T = B^T A^T$.

Remark

This applies to matrix-vector multiplication as well, since a vector is a row matrix or a column matrix.

Problem

Let $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$ be three $n \times n$ matrices. For $1 \leq i, j \leq n$ write down a formula for the (i, j)-entry of each of the following matrices.

1. AB
2. BA
3. $A+C$
4. $C(A+B)$
5. $A(BC)$
6. $(AB)C$

Problem

Let A and B be $m \times n$ matrices, and let C be an $n \times p$ matrix. Prove that if A and B commute with C, then $A + B$ commutes with C.

Proof.

We are given that $AC = CA$ and $BC = CB$. Consider $(A + B)C$.

$$\begin{aligned}(A+B)C &= AC + BC \\ &= CA + CB \\ &= C(A+B)\end{aligned}$$

Since $(A + B)C = C(A + B)$, $A + B$ commutes with C. ∎

Problem

Let A, B and C be $n \times n$ matrices, and suppose that both A and B commute with C, i.e., $AC = CA$ and $BC = CB$. Show that AB commutes with C.

Proof.

We must show that $(AB)C = C(AB)$ given that $AC = CA$ and $BC = CB$.

$$
\begin{aligned}
(AB)C &= A(BC) && \text{(matrix multiplication is associative)} \\
&= A(CB) && \text{(B commutes with C)} \\
&= (AC)B && \text{(matrix multiplication is associative)} \\
&= (CA)B && \text{(A commutes with C)} \\
&= C(AB) && \text{(matrix multiplication is associative)}
\end{aligned}
$$

Therefore, AB commutes with C. ■

Partitioned matrix and block multiplication

Observation

We can partition matrix into blocks so that each entry of the partitioned matrix is again a matrix.

Example

$$A = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 2 & -1 & 4 & 2 & 1 \\ 3 & 1 & -1 & 7 & 5 \end{array}\right] = \begin{bmatrix} I_2 & 0_{23} \\ P & Q \end{bmatrix} \quad \text{and} \quad B = \left[\begin{array}{cc} 4 & -2 \\ 5 & 6 \\ \hline 7 & 3 \\ -1 & 0 \\ 1 & 6 \end{array}\right] = \begin{bmatrix} X \\ Y \end{bmatrix}$$

Example

Let A and B be $\mathrm{m} \times \mathrm{n}$ and $\mathrm{n} \times \mathrm{k}$ matrices, respectively. We can partition then into either column vectors or row vectors: When viewed as partitioned matrices, AB can be equivalently written in one of the following four ways:

$$\mathrm{A_{mn}} = (\vec{\mathrm{a}}_1, \cdots, \vec{\mathrm{a}}_\mathrm{n}) = \begin{pmatrix} \vec{\alpha}_1^\mathrm{T} \\ \vdots \\ \vec{\alpha}_\mathrm{m}^\mathrm{T} \end{pmatrix} \quad \text{and} \quad \mathrm{B_{nk}} = \left(\vec{\mathrm{b}}_1, \cdots, \vec{\mathrm{b}}_\mathrm{k}\right) = \begin{pmatrix} \vec{\beta}_1^\mathrm{T} \\ \vdots \\ \vec{\beta}_\mathrm{n}^\mathrm{T} \end{pmatrix}$$

1. $$\mathrm{AB} = \mathrm{A}\left(\vec{\mathrm{b}}_1, \cdots, \vec{\mathrm{b}}_\mathrm{k}\right) = \left(\mathrm{A}\vec{\mathrm{b}}_1, \cdots, \mathrm{A}\vec{\mathrm{b}}_\mathrm{k}\right)$$

2. $$\mathrm{AB} = \begin{pmatrix} \vec{\alpha}_1^\mathrm{T} \\ \vdots \\ \vec{\alpha}_\mathrm{m}^\mathrm{T} \end{pmatrix} \mathrm{B} = \begin{pmatrix} \vec{\alpha}_1^\mathrm{T}\mathrm{B} \\ \vdots \\ \vec{\alpha}_\mathrm{m}^\mathrm{T}\mathrm{B} \end{pmatrix}$$

Example (continued)

3

$$\mathrm{AB} = (\vec{\mathrm{a}}_1, \cdots, \vec{\mathrm{a}}_\mathrm{n}) \begin{pmatrix} \vec{\beta}_1^\mathrm{T} \\ \vdots \\ \vec{\beta}_\mathrm{n}^\mathrm{T} \end{pmatrix} = \vec{\mathrm{a}}_1 \vec{\beta}_1^\mathrm{T} + \vec{\mathrm{a}}_2 \vec{\beta}_2^\mathrm{T} + \cdots \vec{\mathrm{a}}_\mathrm{n} \vec{\beta}_\mathrm{n}^\mathrm{T}$$

4

$$\mathrm{AB} = \begin{pmatrix} \vec{\alpha}_1^\mathrm{T} \\ \vdots \\ \vec{\alpha}_\mathrm{m}^\mathrm{T} \end{pmatrix} \left(\vec{\mathrm{b}}_1, \cdots, \vec{\mathrm{b}}_\mathrm{k}\right) = \begin{pmatrix} \vec{\alpha}_1^\mathrm{T} \mathrm{b}_1 & \vec{\alpha}_1^\mathrm{T} \mathrm{b}_2 & \cdots & \vec{\alpha}_1^\mathrm{T} \mathrm{b}_\mathrm{k} \\ \vec{\alpha}_2^\mathrm{T} \mathrm{b}_1 & \vec{\alpha}_2^\mathrm{T} \mathrm{b}_2 & \cdots & \vec{\alpha}_2^\mathrm{T} \mathrm{b}_\mathrm{k} \\ \vdots & \vdots & \ddots & \vdots \\ \vec{\alpha}_\mathrm{m}^\mathrm{T} \mathrm{b}_1 & \vec{\alpha}_\mathrm{m}^\mathrm{T} \mathrm{b}_\mathrm{m} & \cdots & \vec{\alpha}_\mathrm{m}^\mathrm{T} \mathrm{b}_\mathrm{k} \end{pmatrix}$$

Example (continued)

One can also partition A and B as follows:

$$\mathrm{A} = \begin{pmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} \\ \mathrm{A}_{21} & \mathrm{A}_{22} \end{pmatrix} \quad \text{and} \quad \mathrm{B} = \begin{pmatrix} \mathrm{B}_{11} & \mathrm{B}_{12} \\ \mathrm{B}_{21} & \mathrm{B}_{22} \end{pmatrix}$$

in a way that dimensions match. Then

$$\mathrm{AB} = \begin{pmatrix} \mathrm{A}_{11}\mathrm{B}_{11} + \mathrm{A}_{12}\mathrm{B}_{21} & \mathrm{A}_{11}\mathrm{B}_{12} + \mathrm{A}_{12}\mathrm{B}_{22} \\ \mathrm{A}_{21}\mathrm{B}_{11} + \mathrm{A}_{22}\mathrm{B}_{21} & \mathrm{A}_{21}\mathrm{B}_{12} + \mathrm{A}_{22}\mathrm{B}_{22} \end{pmatrix}$$

Problem

Let A be a square matrix. Compute A^k where $A = \begin{pmatrix} I & X \\ O & O \end{pmatrix}$.

Solution

$$A^2 = \cdots = A.$$

Hence, $A^k = A$ for all $k \geq 2$.