Math 221: LINEAR ALGEBRA

Chapter 2. Matrix Algebra
§2-6. Linear Transformations

Le Chen[1]

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Transformations

Finding the Matrix of a Linear Transformation

Composition of Linear Transformations

Rotations and Reflections in $\mathbb{R}^2$

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Definition

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if it satisfies the following two properties for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and all (scalars) $a \in \mathbb{R}$.

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1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ (preservation of addition)
2. $T(a\vec{x}) = aT(\vec{x})$ (preservation of scalar multiplication)

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\vec{x} \in \mathbb{R}^n$.

Let $\mathrm{T} : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}^{\mathrm{m}}$ be a linear transformation, and let $\vec{\mathrm{x}} \in \mathbb{R}^{\mathrm{n}}$. Since T preserves scalar multiplication, Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\vec{x} \in \mathbb{R}^n$. Since T preserves scalar multiplication,

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1. $T(0\vec{x}) = 0T(\vec{x})$ implying $T(0) = 0$, so T preserves the zero vector.
2. $T((-1)\vec{x}) = (-1)T(\vec{x})$, implying $T(-\vec{x}) = -T(\vec{x})$, so T preserves the negative of a vector.

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Suppose $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ are vectors in $\mathbb{R}^n$ and for some $a_1, a_2, \ldots, a_k \in \mathbb{R}$.

$$\vec{y} = a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_k\vec{x}_k.$$

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\vec{x} \in \mathbb{R}^n$. Since T preserves scalar multiplication,

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$$\vec{y} = a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_k\vec{x}_k.$$

$$\Downarrow$$

3.
$$\begin{aligned} T(\vec{y}) &= T(a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_k\vec{x}_k) \\ &= a_1T(\vec{x}_1) + a_2T(\vec{x}_2) + \cdots + a_kT(\vec{x}_k), \end{aligned}$$

i.e., T preserves linear combinations.

Problem

Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation such that

$$T\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \end{bmatrix} \quad \text{and} \quad T\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -1 \\ 5 \end{bmatrix}.$$

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Solution

The only way it is possible to solve this problem is if

$$\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} \text{ is a linear combination of } \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix},$$

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i.e., if there exist $a, b \in \mathbb{R}$ so that

$$\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = a\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + b\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}.$$

Solution (continued)

To find a and b, solve the system of three equations in two variables:

$$\left[\begin{array}{cc|c} 1 & 4 & -7 \\ 3 & 0 & 3 \\ 1 & 5 & -9 \end{array}\right] \to \cdots \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

Thus $a = 1$, $b = -2$, and

$$\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}.$$

Solution (continued)

We now use that fact that linear transformations preserve linear combinations, implying that

$$T\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = T\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}\right)$$

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Therefore, $T\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 2 \\ -12 \end{bmatrix}$. ■

Problem

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation such that

$$T\begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad \text{and} \quad T\begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}. \text{ Find } T\begin{bmatrix} 1 \\ 3 \\ -2 \\ -4 \end{bmatrix}.$$

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Solution (Final Answer)

$$T\begin{bmatrix} 1 \\ 3 \\ -2 \\ -4 \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \\ -3 \end{bmatrix}.$$

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proving that T preserves addition.

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proving that T preserves scalar multiplication.

Since T preserves addition and scalar multiplication T is a linear transformation. ∎

Example (The Zero Transformation)

If A is the $m \times n$ matrix of zeros, then the transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ induced by A is called the zero transformation because for every vector $\vec{x}$ in $\mathbb{R}^n$

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The transformation of $\mathbb{R}^n$ induced by I_n, the $n \times n$ identity matrix, is called the identity transformation because for every vector $\vec{x}$ in $\mathbb{R}^n$,

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The identity transformation on $\mathbb{R}^n$ is usually written as $1_{\mathbb{R}^n}$.

Problem (Revisited)

Is the following $T : \mathbb{R}^3 \to \mathbb{R}^4$ a matrix transformation?

$$T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + b \\ b + c \\ a - c \\ c - b \end{bmatrix}$$

Problem (Revisited)

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Solution

$$A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+c \\ a-c \\ c-b \end{bmatrix} = T \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

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Yes, T is a matrix transformation. ■

Problem (Not all transformations are matrix transformations)

Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\vec{x}) = \vec{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ for all } \vec{x} \in \mathbb{R}^2.$$

Show that T NOT a matrix transformation.

Solution

We have $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\vec{x}) = \vec{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ for all } \vec{x} \in \mathbb{R}^2.$$

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Since every matrix transformation is a linear transformation,

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We have $\mathrm{T} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mathrm{T}(\vec{\mathrm{x}}) = \vec{\mathrm{x}} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ for all } \vec{\mathrm{x}} \in \mathbb{R}^2.$$

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$$T \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solution

We have $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\vec{x}) = \vec{x} + \left[\begin{array}{r} 1 \\ -1 \end{array} \right] \text{ for all } \vec{x} \in \mathbb{R}^2.$$

Since every matrix transformation is a linear transformation, we consider $T(0)$, where 0 is the zero vector of $\mathbb{R}^2$.

$$T \left[\begin{array}{c} 0 \\ 0 \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \end{array} \right] + \left[\begin{array}{r} 1 \\ -1 \end{array} \right]$$

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We have $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

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Since every matrix transformation is a linear transformation, we consider $\mathrm{T}(0)$, where 0 is the zero vector of $\mathbb{R}^2$.

$$\mathrm{T}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

violating one of the properties of a linear transformation.

Solution

We have $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\vec{x}) = \vec{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ for all } \vec{x} \in \mathbb{R}^2.$$

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violating one of the properties of a linear transformation.

Therefore, T is not a linear transformation, and hence is not a matrix transformation. ■

Remark

Recall that a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if it satisfies the following two properties for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and all (scalars) $a \in \mathbb{R}$.

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Recall that a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if it satisfies the following two properties for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and all (scalars) $a \in \mathbb{R}$.

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Theorem (Every Linear Transformation is a Matrix Transformation)

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then we can find an $n \times m$ matrix A such that

$$T(\vec{x}) = A\vec{x}$$

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In this case, we say that T is induced, or determined, by A and we write

$$T_A(\vec{x}) = A\vec{x}$$

Problem

The transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+c \\ a-c \\ c-b \end{bmatrix}$$

for each $\vec{x} \in \mathbb{R}^3$ is another matrix transformation, that is, $T(\vec{x}) = A\vec{x}$ for some matrix A. Can you find a matrix A that works?

Problem

The transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+c \\ a-c \\ c-b \end{bmatrix}$$

for each $\vec{x} \in \mathbb{R}^3$ is another matrix transformation, that is, $T(\vec{x}) = A\vec{x}$ for some matrix A. Can you find a matrix A that works?

Solution

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and try to fill in the values of the matrix.

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and try to fill in the values of the matrix.

$$\vdots$$

We can deduce from the product that T is induced by the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

■

Linear Transformations

Finding the Matrix of a Linear Transformation

Composition of Linear Transformations

Rotations and Reflections in $\mathbb{R}^2$

Finding the Matrix of a Linear Transformation

Finding the Matrix of a Linear Transformation

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Definition

The set of columns $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ of I_n is called the standard basis of $\mathbb{R}^n$.

Theorem (Matrix of a Linear Transformation)

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

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Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is a matrix transformation. Furthermore, T is induced by the unique matrix

$$A = \left[\begin{array}{cccc} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{array}\right],$$

where $\vec{e}_j$ is the j-th column of I_n, and $T(\vec{e}_j)$ is the j-th column of A.

Theorem (Matrix of a Linear Transformation)

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is a matrix transformation. Furthermore, T is induced by the unique matrix

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where $\vec{e}_j$ is the j-th column of I_n, and $T(\vec{e}_j)$ is the j-th column of A.

Corollary

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if it is a matrix transformation.

"linear" = "matrix"

Problem

Let $\mathrm{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by

$$\mathrm{T}\begin{bmatrix} \mathrm{x} \\ \mathrm{y} \end{bmatrix} = \begin{bmatrix} \mathrm{x} + 2\mathrm{y} \\ \mathrm{x} - \mathrm{y} \end{bmatrix}$$

for each $\vec{\mathrm{x}} \in \mathbb{R}^2$. Find the matrix, A, of T.

Problem

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ x - y \end{bmatrix}$$

for each $\vec{x} \in \mathbb{R}^2$. Find the matrix, A, of T.

Solution

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 + 2(0) \\ 1 - 0 \end{bmatrix}$$

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Solution

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 + 2(0) \\ 1 - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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Problem

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$$\Downarrow$$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

■

Sometimes, T is defined through its actions several concrete vectors.

Problem

Find the matrix A of T where T is given as

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad T\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Solution (continued)

We need to write $\vec{e}_1$ and $\vec{e}_2$ as a linear combination of the vectors provided. First, find x and y such that

$$\left[\begin{array}{c} 1 \\ 0 \end{array}\right] = x \left[\begin{array}{c} 1 \\ 1 \end{array}\right] + y \left[\begin{array}{r} 0 \\ -1 \end{array}\right]$$

Solution (continued)

We need to write $\vec{e}_1$ and $\vec{e}_2$ as a linear combination of the vectors provided. First, find x and y such that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Once we find x and y we can compute

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = xT \begin{bmatrix} 1 \\ 1 \end{bmatrix} + yT \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$= x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Solution (continued)

Finding x and y involves solving the following system of equations.

$$
\begin{aligned}
x &= 1 \\
x - y &= 0
\end{aligned}
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The solution is $x = 1, y = 1$.

Solution (continued)

Finding x and y involves solving the following system of equations.

$$\begin{aligned} x &= 1 \\ x - y &= 0 \end{aligned}$$

The solution is $x = 1, y = 1$. Hence, we can find $T(\vec{e}_1)$ as follows.

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

As for $T(\vec{e}_2)$,

$$T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -T \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}.$$

$$\Downarrow$$

$$A = \begin{bmatrix} 4 & -3 \\ 4 & -2 \end{bmatrix}$$

■

Problem

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a transformation defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \\ -x + 2y \end{bmatrix}$.

Is T a linear transformation?

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If T were a linear transformation, then T would be induced by the matrix

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It remains to verify the matrix transform induced by A indeed coincides with T:

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \\ -x+2y \end{bmatrix} = T\begin{bmatrix} x \\ y \end{bmatrix}.$$

Therefore, T is a matrix transformation induced by A above. ■

Problem

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xy \\ x + y \end{bmatrix}$. Is T a linear transformation?

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If T were a linear transformation, then T would be induced by the matrix

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Problem

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xy \\ x+y \end{bmatrix}$. Is T a linear transformation?

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If T were a linear transformation, then T would be induced by the matrix

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However, the matrix transform induced by A doesn't pass the verification:

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ x+y \end{bmatrix} \neq \begin{bmatrix} xy \\ x+y \end{bmatrix} = T\begin{bmatrix} x \\ y \end{bmatrix}$$

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If T were a linear transformation, then T would be induced by the matrix

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Therefore, T in NOT a linear transformation. ■

Linear Transformations

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Composition of Linear Transformations

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Definition

Suppose $T : \mathbb{R}^k \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations.

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is defined by

$$(S \circ T)(\vec{x}) = S(T(\vec{x})) \text{ for all } \vec{x} \in \mathbb{R}^k.$$

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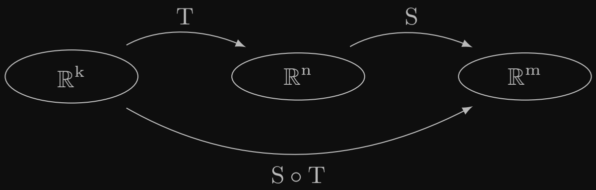

Composition of Linear Transformations

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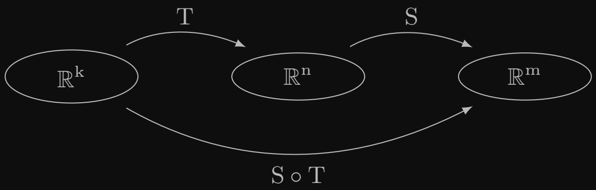

Remark (Convention on the order)

$S \circ T$ means that the transformation T is applied first, followed by the transformation S.

Theorem

Let $\mathbb{R}^k \xrightarrow{T} \mathbb{R}^n \xrightarrow{S} \mathbb{R}^m$ be linear transformations, and suppose that S is induced by matrix A, and T is induced by matrix B. Then $S \circ T$ is a linear transformation, and $S \circ T$ is induced by the matrix AB.

Theorem

Let $\mathbb{R}^k \xrightarrow{T} \mathbb{R}^n \xrightarrow{S} \mathbb{R}^m$ be linear transformations, and suppose that S is induced by matrix A, and T is induced by matrix B. Then $S \circ T$ is a linear transformation, and $S \circ T$ is induced by the matrix AB.

Problem

Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations defined by

$$S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} \quad \text{and} \quad T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} \text{ for all } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2.$$

Find $S \circ T$.

Solution

Then S and T are induced by matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

respectively.

Solution

Then S and T are induced by matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

respectively. The composite of S and T is the transformation
$S \circ T : \mathbb{R}^2 \to \mathbb{R}^2$

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Then S and T are induced by matrices

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respectively. The composite of S and T is the transformation $S \circ T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$(S \circ T) \begin{bmatrix} x \\ y \end{bmatrix} = S\left(T \begin{bmatrix} x \\ y \end{bmatrix}\right),$$

Solution

Then S and T are induced by matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

respectively. The composite of S and T is the transformation $S \circ T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$(S \circ T) \begin{bmatrix} x \\ y \end{bmatrix} = S\left(T \begin{bmatrix} x \\ y \end{bmatrix}\right),$$

and has matrix (or is induced by the matrix)

$$AB = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Example (continued)

Therefore the composite of S and T is the linear transformation

$$(S \circ T)\begin{bmatrix} x \\ y \end{bmatrix} = AB\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix},$$

for all $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$. ■

Example (continued)

Therefore the composite of S and T is the linear transformation

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for all $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$. ■

Remark

Compare this with the composite of T and S which is the linear transformation

$$(T \circ S)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix},$$

for all $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$.

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Definition

The transformation

$$\mathrm{R}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$

denotes counterclockwise rotation about the origin through an angle of θ.

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Definition

The transformation

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denotes counterclockwise rotation about the origin through an angle of θ.

Rotation through an angle of θ preserves scalar multiplication.

Rotation through an angle of θ preserves vector addition.

R_θ is a linear transformation

Since R_θ preserves addition and scalar multiplication, R_θ is a linear transformation, and hence a matrix transformation.

The matrix that induces R_θ can be found by computing $R_\theta(\vec{e}_1)$ and $R_\theta(\vec{e}_2)$, where

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

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The Matrix for R_θ

The rotation $\mathrm{R}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, and is induced by the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Example (Rotation through π)

We denote by

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counterclockwise rotation about the origin through an angle of π.

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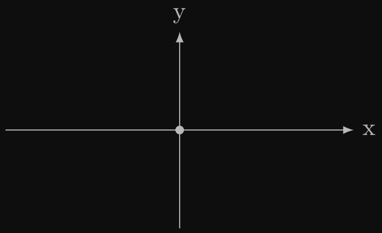

We see that $R_\pi \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ -b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$, so R_π is a matrix transformation.

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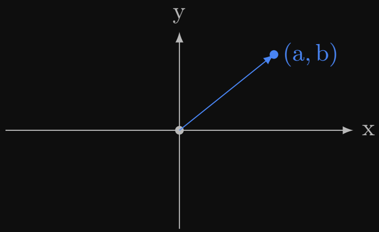

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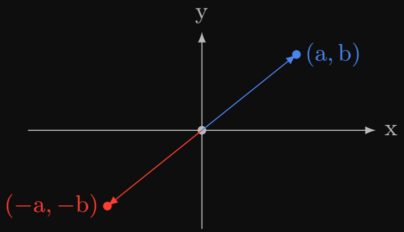

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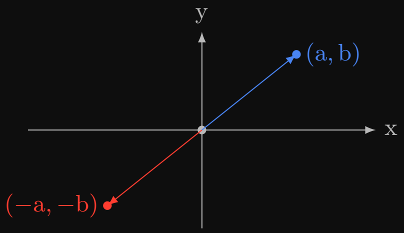

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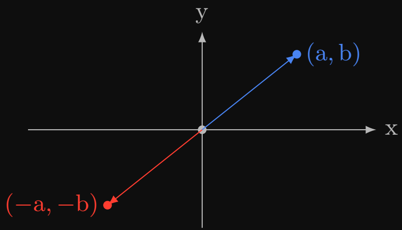

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The transformation $\mathrm{R}_{\frac{\pi}{2}} : \mathbb{R}^2 \to \mathbb{R}^2$ denotes a counterclockwise rotation about the origin through an angle of $\frac{\pi}{2}$ radians. Find the matrix of $\mathrm{R}_{\frac{\pi}{2}}$.

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$$\mathrm{R}_{\frac{\pi}{2}} \begin{bmatrix} \mathrm{a} \\ \mathrm{b} \end{bmatrix} = \begin{bmatrix} -\mathrm{b} \\ \mathrm{a} \end{bmatrix}$$

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Solution

First,

$$R_{\frac{\pi}{2}} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}$$

Furthermore $R_{\frac{\pi}{2}}$ is a matrix transformation, and the matrix it is induced by is

$$\begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

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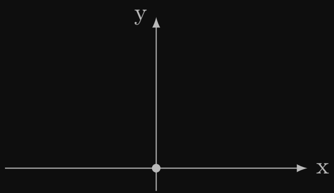

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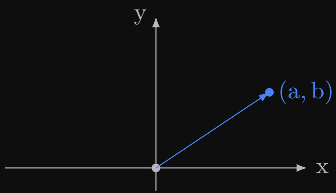

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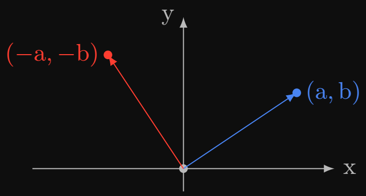

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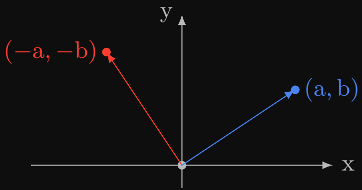

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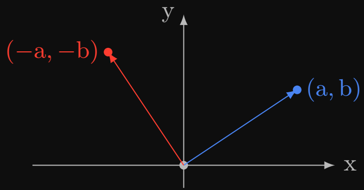

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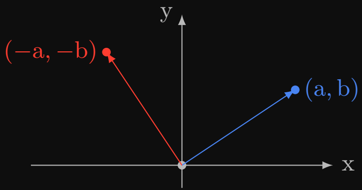

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Reflection in $\mathbb{R}^2$

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Example

In $\mathbb{R}^2$, reflection in the x-axis, which transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} a \\ -b \end{bmatrix}$, is a matrix transformation because

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Reflection in the line $y = x$ transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} b \\ a \end{bmatrix}$.

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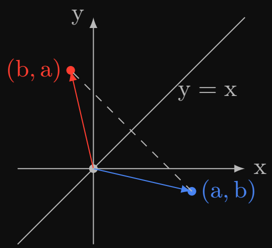

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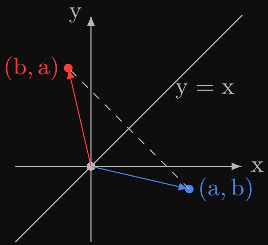

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Reflection in the line

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Example (Reflection in $y = mx$ preserves scalar multiplication)

Let $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection in the line $y = mx$, and let $\vec{u} \in \mathbb{R}^2$.

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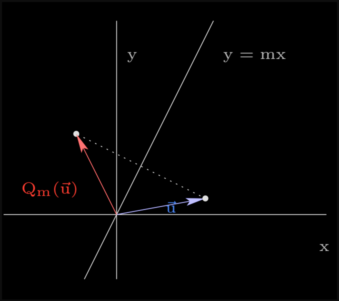

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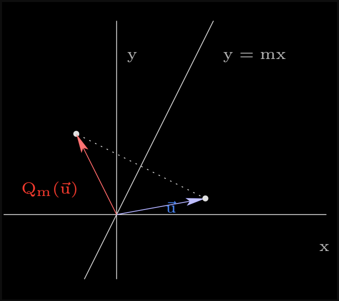

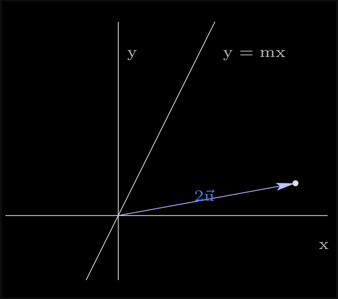

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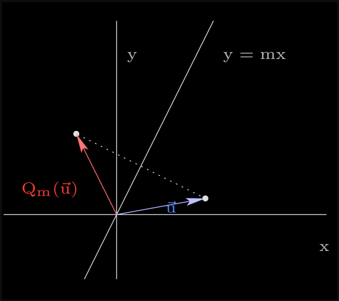

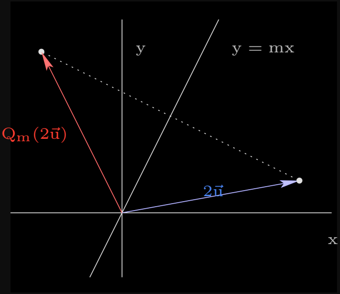

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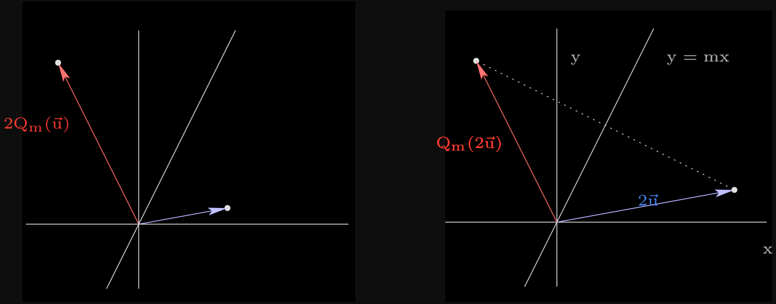

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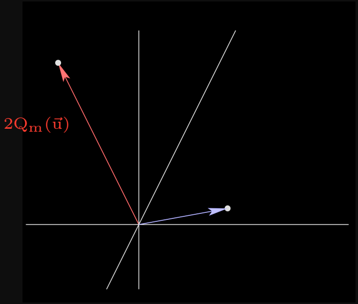

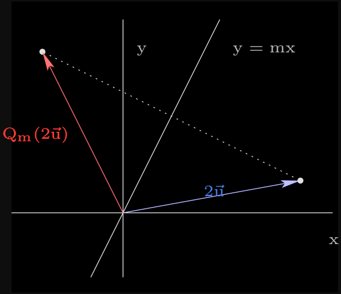

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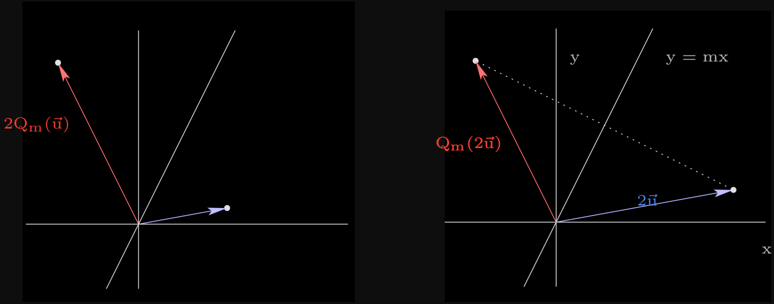

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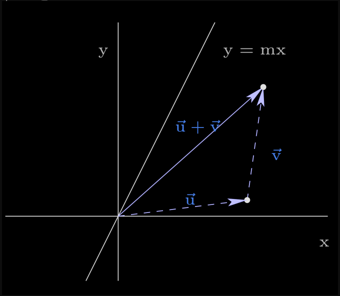

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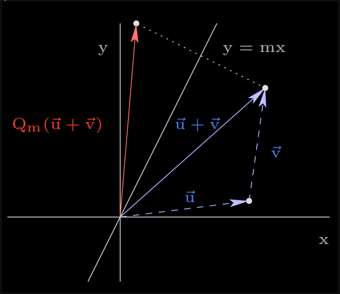

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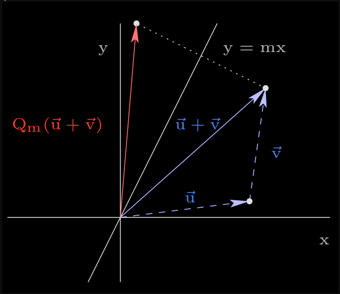

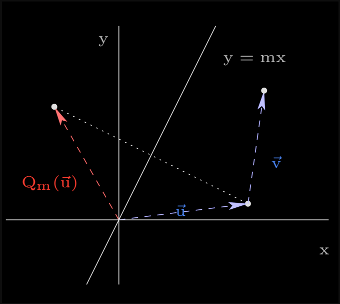

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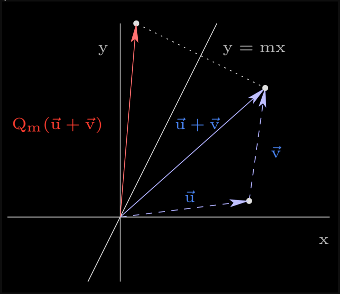

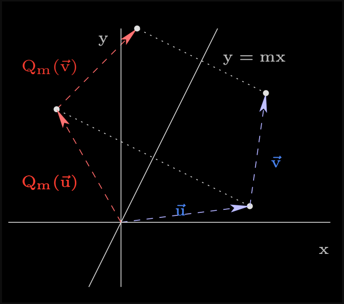

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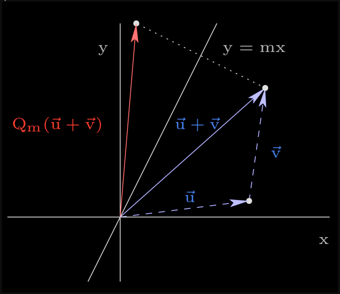

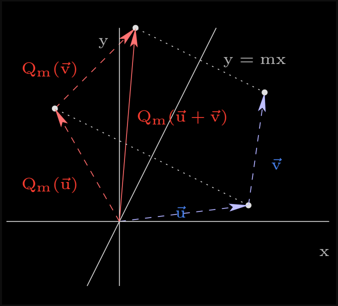

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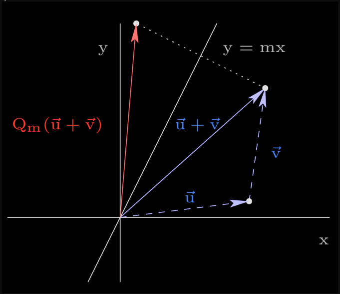

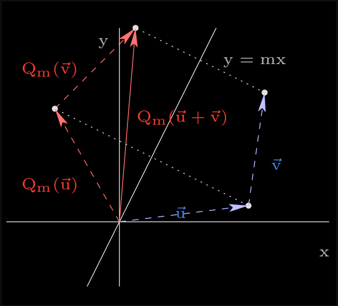

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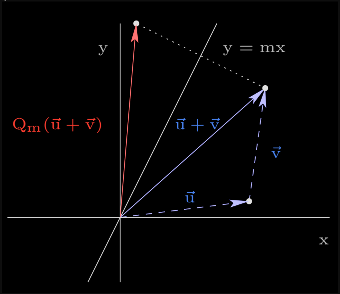

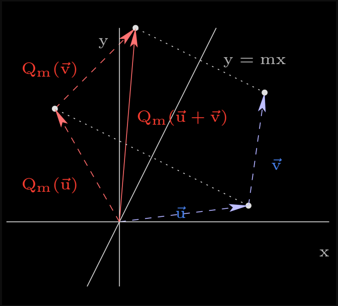

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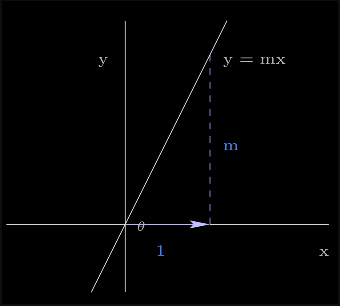

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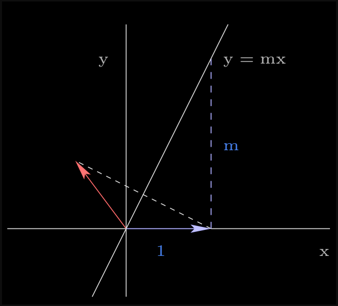

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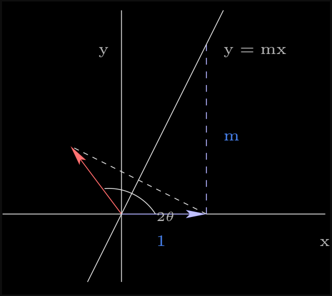

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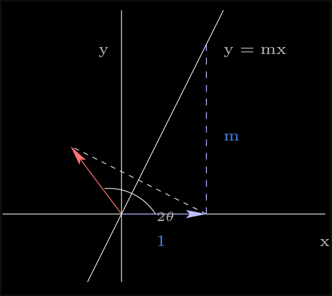

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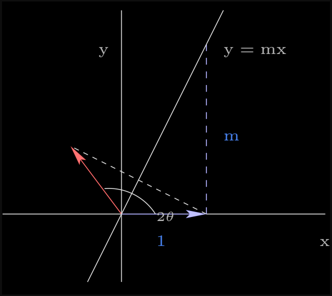

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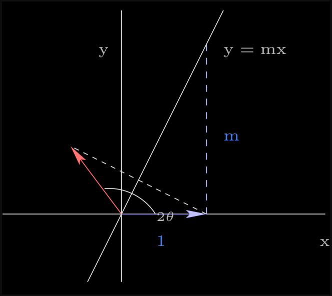

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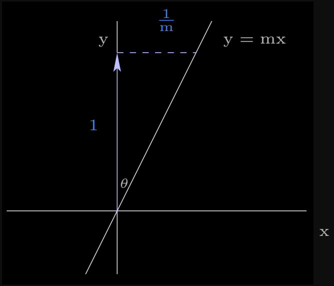

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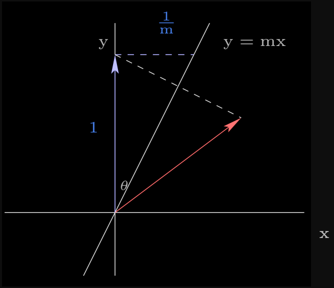

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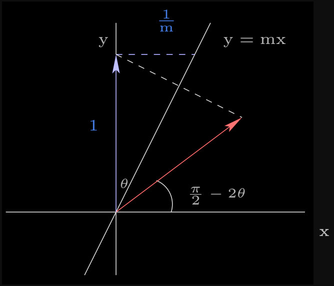

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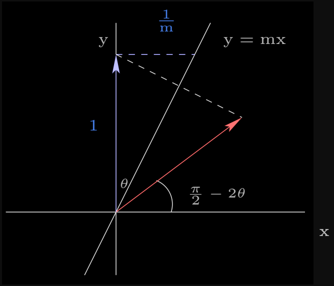

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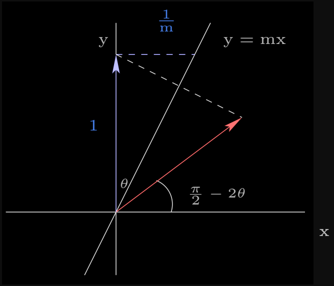

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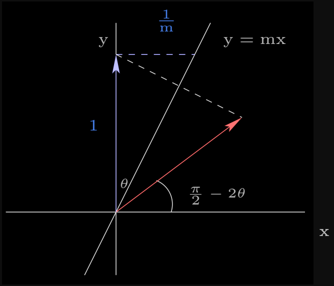

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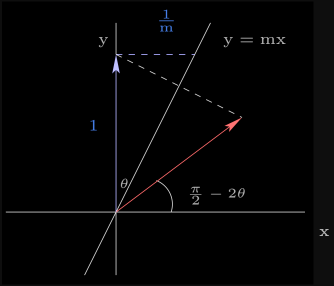

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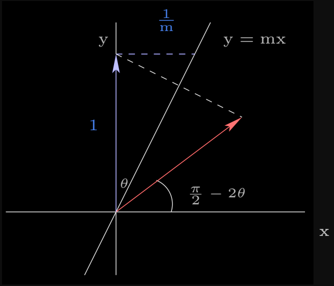

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Then multiply these three matrices ...

The Matrix for Reflection in $y = mx$

The transformation $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$, reflection in the line $y = mx$, is a linear transformation and is induced by the matrix

$$\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

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Q_0 is induced by $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $R_{\frac{\pi}{2}}$ is induced by

$$B = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

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How do we know this?

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Therefore,

$$R_{\frac{\pi}{2}} \circ Q_0 = Q_1,$$

reflection in the line $y = x$. ∎

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Therefore, $Q_Y \circ Q_{-1}$ is induced by BA.

Solution (continued)

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What transformation does BA induce?

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Rotation through an angle θ such that

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Therefore, $\mathrm{Q_Y} \circ \mathrm{Q}_{-1} = \mathrm{R}_{-\frac{\pi}{2}} = \mathrm{R}_{\frac{3\pi}{2}}$. ∎

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- The composite of a reflection and a rotation is a reflection.

$$R_\theta \circ Q_n = Q_m \circ Q_n \circ Q_n = Q_m$$