Math 221: LINEAR ALGEBRA

Chapter 2. Matrix Algebra
§2-7. LU Factorization

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Emory University, 2021 Spring

(last updated on 02/08/2021)

[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

LU Factorization

Why do we need LU Factorization?

Finding the LU

Multiplier Method

LU-Algorithm

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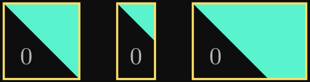
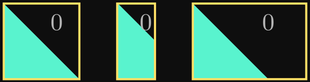
Definition

A matrix $A = [a_{ij}]$ is called upper triangular if $a_{ij} = 0$ whenever $i > j$. Thus the entries below the main diagonal equal 0.



A lower triangular matrix is defined similarly, as a matrix for which all entries above the main diagonal are equal to zero.



An LU factorization of a matrix A is written

$$A = LU$$

where L is lower triangular matrix and U is upper triangular.

We often require either L or U to have only 1's on the main diagonal.

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & 1 \end{pmatrix} \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & * \end{pmatrix}$$

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Why do we need LU Factorization?

The LU factorization often helps to quickly solve equations of the form $A\vec{x} = \vec{b}$.

Suppose we wish to find all solutions $\vec{x}$ to the system $A\vec{x} = B$. The LU factorization of A can assist in this process.

Consider the following reduction:

$$\begin{aligned} A\vec{x} &= B \\ (LU)\vec{x} &= B \\ L(U\vec{x}) &= B \\ L\vec{y} &= B \end{aligned}$$

Therefore, if we can solve $L\vec{y} = B$ for $\vec{y}$, then all that remains is to solve $U\vec{x} = \vec{y}$ for $\vec{x}$.

Example

Find all solutions to

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 3 & 10 & 5 & 1 \\ 0 & -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

Solution

Using a method of your choice, verify that the LU factorization of A gives

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Solution (continued)

Let $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and solve $L\vec{y} = \vec{b}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

The solution is $\vec{y} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$.

Now we solve $U\vec{x} = \vec{y}$.

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$$

Solution (continued)

Multiplying and solving (or finding the reduced row-echelon form), the general solution is given by

$$\vec{x} = \begin{bmatrix} -12 \\ 2 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 13 \\ -3 \\ -2 \\ 1 \end{bmatrix} t, \quad \forall t \in \mathbb{R}.$$

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Finding the LU Factorization

Condition for the existence of LU factorization: A matrix A has LU factorization provided that A can be lower reduced, namely, the row-echelon form of A can be calculated without interchanging rows.

Example

Determine if the LU factorization of A exists, and if so, find it.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

Solution

Because the row-echelon form can be obtained without interchanging rows:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix} \xrightarrow{r_2 - 2r_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 1 & 0 & 5 \end{bmatrix} \xrightarrow{r_3 - r_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & -1 & 3 \end{bmatrix} \xrightarrow{r_3 + r_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$

the LU factorization exists, or A can be lower reduced.

Solution (continued)

We proceed to finding L and U. Assign variables to the unknown entries and multiply.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} \begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} a & d & e \\ ax & dx + b & ex + f \\ ay & dy + bz & ey + fz + c \end{bmatrix}$$

Solving each entry will give us values for the unknown entries.

Solution (continued)

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix} = \begin{bmatrix} a & d & e \\ ax & dx + b & ex + f \\ ay & dy + bz & ey + fz + c \end{bmatrix}$$

We see easily that $a = 1, d = 1$, and $e = 2$. Continuing to solve the first column gives $x = 2, y = 1$. The other values are calculated as follows.

$$\begin{aligned} dx + b &= 3 \\ (1)(2) + b &= 3 \\ b &= 1 \end{aligned}$$

$$\begin{aligned} ex + f &= 0 \\ (2)(2) + f &= 0 \\ f &= -4 \end{aligned}$$

$$\begin{aligned} dy + bz &= 0 \\ (1)(1) + (1)z &= 0 \\ z &= -1 \end{aligned}$$

$$\begin{aligned} ey + fz + c &= 5 \\ (2)(1) + (-4)(-1) + c &= 5 \\ c &= -1 \end{aligned}$$

Solution (continued)

Therefore,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$

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Remark

If you want the diagonal terms of U to be all 1's:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$

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$$\underbrace{\begin{bmatrix} 1 & 0 & -0 \\ 2 & 1 & -0 \\ 1 & -1 & -1 \end{bmatrix}}_{\text{L}} \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ -0 & -0 & 1 \end{bmatrix}}_{\text{U}}$$

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Multiplier Method

The following process for finding L and U, called the multiplier method, can be more efficient.

Example

Find the LU factorization of $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$

Solution

First, write A as

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

$$||$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

Solution (continued)

To do so, we use row operations to remove the entries of A below the main diagonal. For every operation we apply to A (the matrix on the right), we apply the inverse operation to the identity matrix (on the left). This ensures the product remains the same.

The first step is to add (-2) times the first row of A to the second row. To preserve the product, add (2) times the second column to the first column, for the matrix on the left.

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

$$\|$$

$$c_1 + 2c_2 \to c_1 \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 1 & 0 & 5 \end{bmatrix} r_2 - 2r_1 \to r_2$$

Solution (continued)

We proceed in the same way.

$$c_1 + c_3 \to c_1 \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & -1 & 3 \end{bmatrix} r_3 - r_1 \to r_3$$

$$c_2 - c_3 \to c_2 \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix} r_3 + r_2 \to r_3$$

At this point we have a lower triangular matrix L on the left, and an upper triangular matrix U on the right so we are done. You can (and should!) check that this product equals A.

If you want the diagonal terms of U to be all 1's:

$$-1 \times c_3 \to c_3 \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} -1 \times r_3 \to r_3$$

Problem

Use the multiplier method to verify the LU factorization for

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 13 & 5 \\ -2 & -7 & -4 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 13 & 5 \\ -2 & -7 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = LU$$

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Theorem (LU-Algorithm)

Let A be an $m \times n$ matrix of rank r, and suppose that A can be lower reduced to a row-echelon matrix U. Then $A = LU$ where the lower triangular, invertible matrix L is constructed as follows:

1. If $A = 0$, take $L = I_m$ and $U = 0$.
2. If $A \neq 0$, write $A_1 = A$ and let $\vec{c}_1$ be the leading column of A_1. Use $\vec{c}_1$ to create the first leading 1 and make its below all zeros. When this is completed, let A_2 denote the matrix consisting of rows 2 to m of the matrix just created.
3. If $A_2 \neq 0$, let $\vec{c}_2$ be the leading column of A_2 and repeat Step 2 on A_2 to create A_3.
4. Continue in this way until U is reached, where all rows below the last leading 1 consist of zeros. This will happen after r steps.
5. Create L by placing $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_r$ at the bottom of the first r columns of I_m.

Problem

Find an LU-factorization for $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}$.

Solution

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & 5 \end{bmatrix} = L$$

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Problem

Find an LU-factorization for $A = \begin{bmatrix} 5 & -5 & 10 & 0 & 5 \\ -3 & 3 & 2 & 2 & 1 \\ -2 & 2 & 0 & -1 & 0 \\ 1 & -1 & 10 & 2 & 5 \end{bmatrix}$.

Solution

$$\begin{bmatrix} 5 & -5 & 10 & 0 & 5 \\ -3 & 3 & 2 & 2 & 1 \\ -2 & 2 & 0 & -1 & 0 \\ 1 & -1 & 10 & 2 & 5 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 8 & 2 & 4 \\ 0 & 0 & 4 & -1 & 2 \\ 0 & 0 & 8 & 2 & 4 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$$\Downarrow$$

$$L = \begin{bmatrix} 5 & 0 & 0 & 0 \\ -3 & 8 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 1 & 8 & 0 & 1 \end{bmatrix}$$

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