

# Math 221: LINEAR ALGEBRA

## Chapter 3. Determinants and Diagonalization

### §3-1. The Cofactor Expansion

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Emory University, 2021 Spring

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<sup>1</sup>Slides are adapted from those by Karen Seyffarth from University of Calgary.

Determinant of Small Matrices

The Cofactor Expansion

Elementary Row Operations and Determinants

Determinant and Scalar Multiple

Determinant of Triangular Matrices

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Some More Exercises

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# Determinant of Small Matrices

## Determinant of Small Matrices

Recall that if  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , then the **determinant** of  $A$  is defined as

$$\det A = ad - bc,$$

and that  $A$  is invertible if and only if  $\det A \neq 0$ .

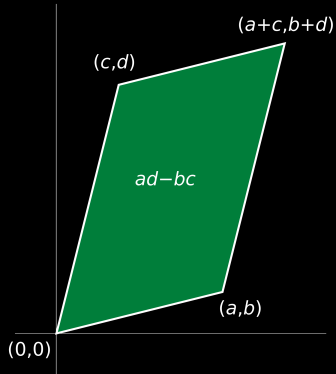
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and that  $A$  is invertible if and only if  $\det A \neq 0$ .

Notation: For  $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , we often write  $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ , i.e., use **vertical bars** instead of **square brackets**.



$$\det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \text{signed area of parallelogram}$$

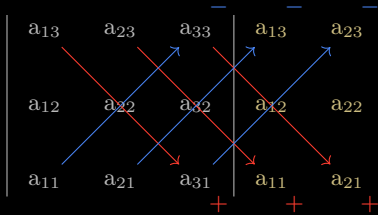
## Problem

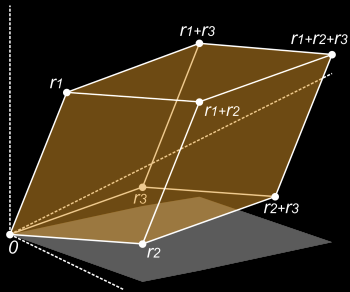
How to define determinant for a general  $n \times n$  matrix?





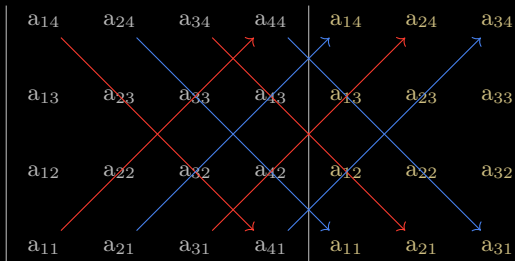
$3 \times 3$





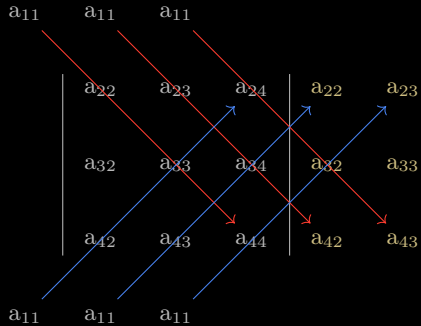
$$\det (\vec{r}_1 \quad \vec{r}_2 \quad \vec{r}_3) = \text{signed volume of the parallelepiped}$$

$4 \times 4$

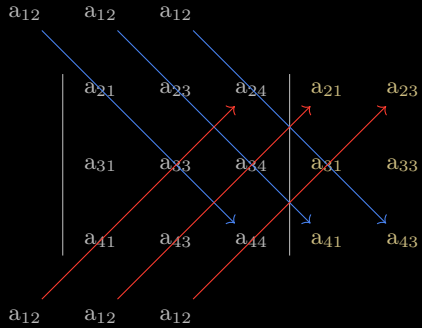


Only partially right... still missing many terms...

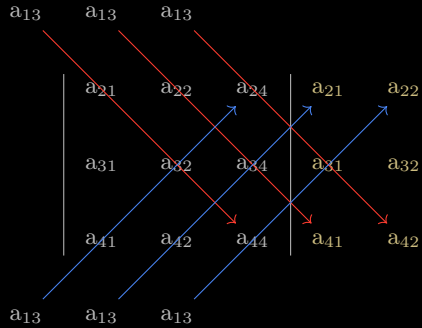
$4 \times 4$  part I:



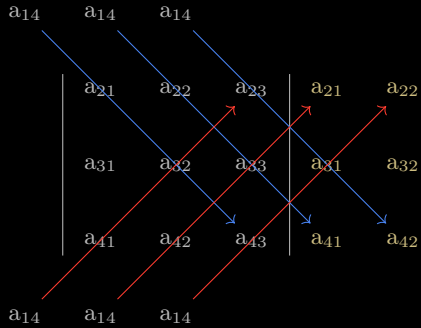
$4 \times 4$  part II:



### $4 \times 4$ part III:



$4 \times 4$  part IV:







$5 \times 5?$  ...

The determinant of an  $n \times n$  matrix is more effectively defined through recursion...

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# Cofactor and cofactor expansion

## Definition

Let  $A = [a_{ij}]$  be an  $n \times n$  matrix.

- The **sign** of the  $(i, j)$  position is  $(-1)^{i+j}$ . (Thus the sign is 1 if  $(i + j)$  is even, and  $-1$  if  $(i + j)$  is odd.)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \Rightarrow \begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

## Definition (continued)

- Let  $A_{ij}$  denote the  $(n - 1) \times (n - 1)$  matrix obtained from  $A$  by deleting row  $i$  and column  $j$ . The  $(i, j)$ -cofactor of  $A$  is

$$c_{ij}(A) = (-1)^{i+j} \det(A_{ij}).$$

- The determinant of  $A$  is defined as

$$\det A = a_{11}c_{11}(A) + a_{12}c_{12}(A) + a_{13}c_{13}(A) + \cdots + a_{1n}c_{1n}(A)$$

and is called the cofactor expansion of  $\det A$  along row 1.

### Example

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}. \text{ Find } \det A.$$

Using cofactor expansion along row 1,

$$\begin{aligned} \det A &= 1c_{11}(A) + 2c_{12}(A) + 3c_{13}(A) \\ &= 1(-1)^2 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3(-1)^4 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ &= (45 - 48) - 2(36 - 42) + 3(32 - 35) \\ &= -3 - 2(-6) + 3(-3) \\ &= -3 + 12 - 9 \\ &= 0 \end{aligned}$$

Example (continued)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Now try cofactor expansion along column 2.



### Example (continued)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Now try cofactor expansion along column 2.

$$\begin{aligned} \det A &= 2c_{12}(A) + 5c_{22}(A) + 8c_{32}(A) \\ &= 2(-1)^3 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5(-1)^4 \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} + 8(-1)^5 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \\ &= -2(36 - 42) + 5(9 - 21) - 8(6 - 12) \\ &= -2(-6) + 5(-12) - 8(-6) \\ &= 12 - 60 + 48 \\ &= 0. \end{aligned}$$

Example (continued)

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We get the same answer!

## Theorem (Cofactor Expansion Theorem)

The determinant of an  $n \times n$  matrix  $A$  can be computed using the cofactor expansion along **any row or column** of  $A$ .

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### Example

$$\text{Let } A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}. \text{ Cofactor expansion along row 1 yields}$$

$$\begin{aligned} \det A &= 0c_{11}(A) + 1c_{12}(A) + 2c_{13}(A) + 1c_{14}(A) \\ &= 1c_{12}(A) + 2c_{13}(A) + c_{14}(A), \end{aligned}$$

whereas cofactor expansion along, row 3 yields

$$\begin{aligned} \det A &= 0c_{31}(A) + 1c_{32}(A) + (-1)c_{33}(A) + 0c_{34}(A) \\ &= 1c_{32}(A) + (-1)c_{33}(A), \end{aligned}$$

i.e., in the first case we have to compute three cofactors, but in the second we only have to compute two.

### Example (continued)

We can save ourselves some work by using cofactor expansion along row 3 rather than row 1.

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned} \det A &= 1c_{32}(A) + (-1)c_{33}(A) \\ &= 1(-1)^5 \begin{vmatrix} 0 & 2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} + (-1)(-1)^6 \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \\ &= (-1)2(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} + (-1)1(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} \\ &= 2(10 - 21) + 1(10 - 21) \\ &= 2(-11) + (-11) \\ &= -33. \end{aligned}$$

### Example (continued)

Try computing  $\det \begin{bmatrix} 0 & 1 & 2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$  using cofactor expansion along other

rows and columns, for instance column 2 or row 4. You will still get  $\det A = -33$ . ■

## Problem

Find  $\det A$  for  $A = \begin{bmatrix} -8 & 1 & 0 & -4 \\ 5 & 7 & 0 & -7 \\ 12 & -3 & 0 & 8 \\ -3 & 11 & 0 & 2 \end{bmatrix}$ .

## Problem

Find  $\det A$  for  $A = \begin{bmatrix} -8 & 1 & 0 & -4 \\ 5 & 7 & 0 & -7 \\ 12 & -3 & 0 & 8 \\ -3 & 11 & 0 & 2 \end{bmatrix}$ .

## Solution

Using cofactor expansion along column 3,  $\det A = 0$ . ■



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Find  $\det A$  for  $A = \begin{bmatrix} -8 & 1 & 0 & -4 \\ 5 & 7 & 0 & -7 \\ 12 & -3 & 0 & 8 \\ -3 & 11 & 0 & 2 \end{bmatrix}$ .

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Using cofactor expansion along column 3,  $\det A = 0$ . ■

### Remark

If  $A$  is an  $n \times n$  matrix with a row or column of zeros, then  $\det A = 0$ .

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# Elementary Row Operations and Determinants

## Elementary Row Operations and Determinants

### Example

Let  $A = \begin{bmatrix} 2 & 0 & -3 \\ 0 & 4 & 0 \\ 1 & 0 & -2 \end{bmatrix}$ . Then

$$\det A = 4(-1)^4 \begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} = 4(-1) = -4.$$

# Elementary Row Operations and Determinants

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Let  $B_1, B_2,$  and  $B_3$  be obtained from  $A$  by performing a type 1, 2 and 3 elementary row operation, respectively, i.e.,

$$B_1 = \begin{array}{c} r_2 \leftrightarrow r_3 \\ \left[ \begin{array}{ccc} 2 & 0 & -3 \\ 1 & 0 & -2 \\ 0 & 4 & 0 \end{array} \right] \end{array} \quad \left| \quad \begin{array}{c} -3 \times r_3 \rightarrow r_3 \\ \left[ \begin{array}{ccc} 2 & 0 & -3 \\ 0 & 4 & 0 \\ -3 & 0 & 6 \end{array} \right] \end{array} \quad \left| \quad \begin{array}{c} 2 \times r_1 + r_3 \rightarrow r_3 \\ \left[ \begin{array}{ccc} 2 & 0 & -3 \\ 0 & 4 & 0 \\ 5 & 0 & -8 \end{array} \right] \end{array} \right.$$

Example (continued)

$$\det B_1 = 4(-1)^5 \begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} = (-4)(-1) = 4 = (-1) \det A.$$

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$$\det B_3 = 4(-1)^4 \begin{vmatrix} 2 & -3 \\ 5 & -8 \end{vmatrix} = 4(-16 + 15) = 4(-1) = -4 = \det A.$$



## Theorem (Determinant and Elementary Row Operations)

Let  $A$  be an  $n \times n$  matrix.

1. If  $B$  is obtained from  $A$  by exchanging two different rows (or columns) of  $A$ , then  $\det B = -\det A$ .

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4. If  $A$  has a row or column of zeros, then  $\det A = 0$ .
5. If two different rows (or columns) of  $A$  are identical, then  $\det A = 0$ .

## Example

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix} = \begin{vmatrix} -3 & -6 \\ -6 & -12 \end{vmatrix} = 36 - 36 = 0.$$



## Example

$$\begin{aligned} \det \begin{bmatrix} 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & -1 & 5 & 5 \\ 1 & 1 & 2 & -1 \end{bmatrix} &= \begin{vmatrix} 0 & -8 & 26 & 4 \\ -1 & -3 & 8 & 0 \\ 0 & -4 & 13 & 5 \\ 0 & -2 & 10 & -1 \end{vmatrix} \\ &= (-1)(-1)^3 \begin{vmatrix} -8 & 26 & 4 \\ -4 & 13 & 5 \\ -2 & 10 & -1 \end{vmatrix} \\ &= \begin{vmatrix} 0 & -14 & 8 \\ 0 & -7 & 7 \\ -2 & 10 & -1 \end{vmatrix} \\ &= (-2)(-1)^4 \begin{vmatrix} -14 & 8 \\ -7 & 7 \end{vmatrix} \\ &= -2 \begin{vmatrix} 0 & -6 \\ -7 & 7 \end{vmatrix} \\ &= (-2)(-42) = 84. \end{aligned}$$



### Problem

$$\text{If } \det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = -1, \text{ find } \det \begin{bmatrix} -x & -y & -z \\ 3p + a & 3q + b & 3r + c \\ 2p & 2q & 2r \end{bmatrix}.$$



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### Solution

$$\begin{vmatrix} -x & -y & -z \\ 3p + a & 3q + b & 3r + c \\ 2p & 2q & 2r \end{vmatrix} = (-1)(2) \begin{vmatrix} x & y & z \\ 3p + a & 3q + b & 3r + c \\ p & q & r \end{vmatrix}$$

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## Example

$$\begin{aligned}\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix} &= 1 \det \begin{bmatrix} 5 & 6 \\ 0 & 9 \end{bmatrix} \\ &= (1)(5) \det [ 9 ] \\ &= (1)(5)(9) \\ &= 45.\end{aligned}$$



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### Problem

Suppose  $A$  is a  $3 \times 3$  matrix with  $\det A = 7$ . What is  $\det(-3A)$ ?



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$$\text{Write } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \text{ Then } -3A = \begin{bmatrix} -3a_{11} & -3a_{12} & -3a_{13} \\ -3a_{21} & -3a_{22} & -3a_{23} \\ -3a_{31} & -3a_{32} & -3a_{33} \end{bmatrix}.$$

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$$\text{Write } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \text{ Then } -3A = \begin{bmatrix} -3a_{11} & -3a_{12} & -3a_{13} \\ -3a_{21} & -3a_{22} & -3a_{23} \\ -3a_{31} & -3a_{32} & -3a_{33} \end{bmatrix}.$$

$$\det(-3A) = \begin{vmatrix} -3a_{11} & -3a_{12} & -3a_{13} \\ -3a_{21} & -3a_{22} & -3a_{23} \\ -3a_{31} & -3a_{32} & -3a_{33} \end{vmatrix} = (-3) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ -3a_{21} & -3a_{22} & -3a_{23} \\ -3a_{31} & -3a_{32} & -3a_{33} \end{vmatrix}$$

$$= (-3)(-3) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ -3a_{31} & -3a_{32} & -3a_{33} \end{vmatrix} = (-3)(-3)(-3) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= (-3)^3 \det A = (-27) \times 7 = -189. \quad \blacksquare$$

### Theorem (Determinant of Scalar Multiple of Matrices)

If  $A$  is an  $n \times n$  matrix and  $k \in \mathbb{R}$  is a scalar, then

$$\det(kA) = k^n \det A.$$

## Problem

Let

$$A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2a + p & 2b + q & 2c + r \\ 2p + x & 2q + y & 2r + z \\ 2x + a & 2y + b & 2z + c \end{bmatrix}$$

Show that  $\det B = 9 \det A$ .

## Solution

$$\det B = \begin{vmatrix} 2a + p & 2b + q & 2c + r \\ 2p + x & 2q + y & 2r + z \\ 2x + a & 2y + b & 2z + c \end{vmatrix} = \begin{vmatrix} p - 4x & q - 4y & r - 4z \\ 2p + x & 2q + y & 2r + z \\ 2x + a & 2y + b & 2z + c \end{vmatrix}$$

## Solution

$$\begin{aligned} \det B &= \begin{vmatrix} 2a + p & 2b + q & 2c + r \\ 2p + x & 2q + y & 2r + z \\ 2x + a & 2y + b & 2z + c \end{vmatrix} = \begin{vmatrix} p - 4x & q - 4y & r - 4z \\ 2p + x & 2q + y & 2r + z \\ 2x + a & 2y + b & 2z + c \end{vmatrix} \\ &= \begin{vmatrix} p - 4x & q - 4y & r - 4z \\ 9x & 9y & 9z \\ 2x + a & 2y + b & 2z + c \end{vmatrix} = 9 \begin{vmatrix} p - 4x & q - 4y & r - 4z \\ x & y & z \\ 2x + a & 2y + b & 2z + c \end{vmatrix} \end{aligned}$$

## Solution

$$\begin{aligned}\det B &= \begin{vmatrix} 2a + p & 2b + q & 2c + r \\ 2p + x & 2q + y & 2r + z \\ 2x + a & 2y + b & 2z + c \end{vmatrix} = \begin{vmatrix} p - 4x & q - 4y & r - 4z \\ 2p + x & 2q + y & 2r + z \\ 2x + a & 2y + b & 2z + c \end{vmatrix} \\ &= \begin{vmatrix} p - 4x & q - 4y & r - 4z \\ 9x & 9y & 9z \\ 2x + a & 2y + b & 2z + c \end{vmatrix} = 9 \begin{vmatrix} p - 4x & q - 4y & r - 4z \\ x & y & z \\ 2x + a & 2y + b & 2z + c \end{vmatrix} \\ &= 9 \begin{vmatrix} p & q & r \\ x & y & z \\ 2x + a & 2y + b & 2z + c \end{vmatrix} = 9 \begin{vmatrix} p & q & r \\ x & y & z \\ a & b & c \end{vmatrix} = -9 \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}\end{aligned}$$



## Solution

$$\begin{aligned}\det B &= \begin{vmatrix} 2a + p & 2b + q & 2c + r \\ 2p + x & 2q + y & 2r + z \\ 2x + a & 2y + b & 2z + c \end{vmatrix} = \begin{vmatrix} p - 4x & q - 4y & r - 4z \\ 2p + x & 2q + y & 2r + z \\ 2x + a & 2y + b & 2z + c \end{vmatrix} \\ &= \begin{vmatrix} p - 4x & q - 4y & r - 4z \\ 9x & 9y & 9z \\ 2x + a & 2y + b & 2z + c \end{vmatrix} = 9 \begin{vmatrix} p - 4x & q - 4y & r - 4z \\ x & y & z \\ 2x + a & 2y + b & 2z + c \end{vmatrix} \\ &= 9 \begin{vmatrix} p & q & r \\ x & y & z \\ 2x + a & 2y + b & 2z + c \end{vmatrix} = 9 \begin{vmatrix} p & q & r \\ x & y & z \\ a & b & c \end{vmatrix} = -9 \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} \\ &= 9 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = 9 \det A.\end{aligned}$$



Determinant of Small Matrices

The Cofactor Expansion

Elementary Row Operations and Determinants

Determinant and Scalar Multiple

**Determinant of Triangular Matrices**

Determinant of Block Matrices

Some More Exercises



# Determinant of Triangular Matrices

## Theorem

If  $A = [a_{ij}]$  is an  $n \times n$  (square, upper or lower) triangular matrix, then

$$\det A = a_{11}a_{22}a_{33} \cdots a_{nn},$$

i.e.,  $\det A$  is the product of the entries of the main diagonal of  $A$ .



## Determinants of Upper Triangular Matrices

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

⇓

$$\det(U) = u_{11}u_{22} \cdots u_{nn}$$

## Determinants of **lower** Triangular Matrices

$$\mathbf{L} = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix}$$

⇓

$$\det(\mathbf{L}) = l_{11}l_{22} \cdots l_{nn}$$

## Determinants of **diagonal** Matrices

$$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}$$

⇓

$$\det(D) = d_{11}d_{22} \cdots d_{nn}$$

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# Determinant of Block Matrices

# Determinant of Block Matrices

## Theorem

Consider the matrices

$$\begin{bmatrix} A & X \\ 0 & B \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & 0 \\ Y & B \end{bmatrix}$$

where  $A$  and  $B$  are square matrices. Then

$$\det \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} = \det A \det B \quad \text{and} \quad \det \begin{bmatrix} A & 0 \\ Y & B \end{bmatrix} = \det A \det B.$$

## Example

$$\det \begin{bmatrix} 1 & -1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 4 & 1 \\ 1 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} =$$

## Example

$$\det \begin{bmatrix} 1 & -1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 4 & 1 \\ 1 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} = \det \left[ \begin{array}{ccc|cc} 1 & -1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 4 & 1 \\ 1 & 1 & 5 & 0 & 0 \\ \hline 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{array} \right]$$

### Example

$$\det \begin{bmatrix} 1 & -1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 4 & 1 \\ 1 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} = \det \left[ \begin{array}{ccc|cc} 1 & -1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 4 & 1 \\ 1 & 1 & 5 & 0 & 0 \\ \hline 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{array} \right]$$
$$= \det \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 5 \end{bmatrix} \det \begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix}$$

## Example

$$\begin{aligned} \det \begin{bmatrix} 1 & -1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 4 & 1 \\ 1 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} &= \det \left[ \begin{array}{ccc|cc} 1 & -1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 4 & 1 \\ 1 & 1 & 5 & 0 & 0 \\ \hline 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{array} \right] \\ &= \det \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 5 \end{bmatrix} \det \begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix} \\ &= \det \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix} \det \begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix} \end{aligned}$$

### Example

$$\begin{aligned} \det \begin{bmatrix} 1 & -1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 4 & 1 \\ 1 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} &= \det \left[ \begin{array}{ccc|cc} 1 & -1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 4 & 1 \\ 1 & 1 & 5 & 0 & 0 \\ \hline 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{array} \right] \\ &= \det \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 5 \end{bmatrix} \det \begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix} \\ &= \det \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix} \det \begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix} \\ &= 3 \times (-2) = -6. \end{aligned}$$

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## Some More Exercises

## Some More Exercises

Example (From Exercise)

Evaluate by inspection.

$$\det \begin{bmatrix} a & b & c \\ a+1 & b+1 & c+1 \\ a-1 & b-1 & c-1 \end{bmatrix} = ?$$

## Some More Exercises

### Example (From Exercise)

Evaluate by inspection.

$$\det \begin{bmatrix} a & b & c \\ a+1 & b+1 & c+1 \\ a-1 & b-1 & c-1 \end{bmatrix} = ?$$

$$\text{row2} + \text{row3} - 2(\text{row1}) = [ 0 \quad 0 \quad 0 ]$$

Example (From Exercise)

(a) Find  $\det A$  if  $A$  is  $3 \times 3$  and  $\det(2A) = 6$ .

### Example (From Exercise)

(a) Find  $\det A$  if  $A$  is  $3 \times 3$  and  $\det(2A) = 6$ .

(b) Let  $A$  be an  $n \times n$  matrix. Under what conditions is  $\det(-A) = \det A$ ?

### Example (From Exercise)

In each case, prove the statement is true or give a counterexample showing that the statement is false.

(a)  $\det(A + B) = \det A + \det B$ .

### Example (From Exercise)

In each case, prove the statement is true or give a counterexample showing that the statement is false.

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(c) If  $A$  is  $2 \times 2$ , then  $\det(A^T) = \det A$ .

### Example (From Exercise)

In each case, prove the statement is true or give a counterexample showing that the statement is false.

- (a)  $\det(A + B) = \det A + \det B$ .
- (c) If  $A$  is  $2 \times 2$ , then  $\det(A^T) = \det A$ .
- (e) If  $A$  is  $2 \times 2$ , then  $\det(7A) = 49 \det A$ .



### Example (From Exercise)

In each case, prove the statement is true or give a counterexample showing that the statement is false.

- (a)  $\det(A + B) = \det A + \det B$ .
- (c) If  $A$  is  $2 \times 2$ , then  $\det(A^T) = \det A$ .
- (e) If  $A$  is  $2 \times 2$ , then  $\det(7A) = 49 \det A$ .
- (g)  $\det(-A) = -\det A$ .