Math 221: LINEAR ALGEBRA

Chapter 3. Determinants and Diagonalization
§3-3. Diagonalization and Eigenvalues

Le Chen[1]

Emory University, 2021 Spring

(last updated on 02/22/2021)

[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Why Diagonalization?

Eigenvalues and Eigenvectors

Geometric Interpretation of Eigenvalues and Eigenvectors

Diagonalization

Linear Dynamical Systems

Why Diagonalization?

Eigenvalues and Eigenvectors

Geometric Interpretation of Eigenvalues and Eigenvectors

Diagonalization

Linear Dynamical Systems

Why Diagonalization?

Why Diagonalization?

Example

Let $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$. Find A^{100}.

How can we do this efficiently?

Why Diagonalization?

Example

Let $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$. Find A^{100}.

How can we do this efficiently?

Consider the matrix $P = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$. Observe that P is invertible (why?), and that

$$P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}.$$

Why Diagonalization?

Example

Let $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$. Find A^{100}.

How can we do this efficiently?

Consider the matrix $P = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$. Observe that P is invertible (why?), and that

$$P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}.$$

Furthermore,

$$P^{-1}AP = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} = D,$$

where D is a diagonal matrix.

Example (continued)

This is significant, because

$$\begin{aligned}
P^{-1}AP &= D \\
P(P^{-1}AP)P^{-1} &= PDP^{-1} \\
(PP^{-1})A(PP^{-1}) &= PDP^{-1} \\
IAI &= PDP^{-1} \\
A &= PDP^{-1},
\end{aligned}$$

Example (continued)

This is significant, because

$$\begin{aligned} P^{-1}AP &= D \\ P(P^{-1}AP)P^{-1} &= PDP^{-1} \\ (PP^{-1})A(PP^{-1}) &= PDP^{-1} \\ IAI &= PDP^{-1} \\ A &= PDP^{-1}, \end{aligned}$$

and so

$$\begin{aligned} A^{100} &= (PDP^{-1})^{100} \\ &= (PDP^{-1})(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1}) \\ &= PD(P^{-1}P)D(P^{-1}P)D(P^{-1}\cdots P)DP^{-1} \\ &= PDIDIDI\cdots IDP^{-1} \\ &= PD^{100}P^{-1}. \end{aligned}$$

Example (continued)

Now,

$$D^{100} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}^{100} = \begin{bmatrix} 2^{100} & 0 \\ 0 & 5^{100} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} A^{100} &= PD^{100}P^{-1} \\ &= \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^{100} & 0 \\ 0 & 5^{100} \end{bmatrix} \left(\frac{1}{3}\right) \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2^{100} + 2 \cdot 5^{100} & 2^{100} - 2 \cdot 5^{100} \\ 2^{100} - 5^{100} & 2 \cdot 2^{100} + 5^{100} \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2^{100} + 2 \cdot 5^{100} & 2^{100} - 2 \cdot 5^{100} \\ 2^{100} - 5^{100} & 2^{101} + 5^{100} \end{bmatrix}. \end{aligned}$$

■

Theorem (Diagonalization and Matrix Powers)

If $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$ for each $k = 1, 2, 3, \ldots$

Theorem (Diagonalization and Matrix Powers)

If $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$ for each $k = 1, 2, 3, \ldots$

The process of finding an invertible matrix P and a diagonal matrix D so that $A = PDP^{-1}$ is referred to as diagonalizing the matrix A, and P is called the diagonalizing matrix for A.

Theorem (Diagonalization and Matrix Powers)

If $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$ for each $k = 1, 2, 3, \ldots$

The process of finding an invertible matrix P and a diagonal matrix D so that $A = PDP^{-1}$ is referred to as diagonalizing the matrix A, and P is called the diagonalizing matrix for A.

Problem

- When is it possible to diagonalize a matrix?
- How do we find a diagonalizing matrix?

Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors

Definition

Let A be an $n \times n$ matrix, λ a real number, and $\vec{x} \neq \vec{0}$ an n-vector. If $A\vec{x} = \lambda\vec{x}$, then λ is an eigenvalue of A, and $\vec{x}$ is an eigenvector of A corresponding to λ, or a λ-eigenvector.

Eigenvalues and Eigenvectors

Definition

Let A be an $n \times n$ matrix, λ a real number, and $\vec{x} \neq \vec{0}$ an n-vector. If $A\vec{x} = \lambda\vec{x}$, then λ is an eigenvalue of A, and $\vec{x}$ is an eigenvector of A corresponding to λ, or a λ-eigenvector.

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then

$$A\vec{x} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\vec{x}.$$

This means that 3 is an eigenvalue of A, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to 3 (or a 3-eigenvector of A).

Suppose that A is an $n \times n$ matrix, $\vec{x} \neq 0$ an n-vector, $\lambda \in \mathbb{R}$, and that $A\vec{x} = \lambda\vec{x}$.

Suppose that A is an n $\times$ n matrix, $\vec{x} \neq 0$ an n-vector, $\lambda \in \mathbb{R}$, and that $A\vec{x} = \lambda\vec{x}$.

Then

$$\begin{aligned} \lambda\vec{x} - A\vec{x} &= \vec{0} \\ \lambda I\vec{x} - A\vec{x} &= \vec{0} \\ (\lambda I - A)\vec{x} &= \vec{0} \end{aligned}$$

Suppose that A is an n × n matrix, $\vec{x} \neq 0$ an n-vector, $\lambda \in \mathbb{R}$, and that $A\vec{x} = \lambda\vec{x}$.

Then

$$\begin{aligned} \lambda\vec{x} - A\vec{x} &= \vec{0} \\ \lambda I\vec{x} - A\vec{x} &= \vec{0} \\ (\lambda I - A)\vec{x} &= \vec{0} \end{aligned}$$

Since $\vec{x} \neq \vec{0}$, the matrix $\lambda I - A$ has no inverse, and thus

$$\det(\lambda I - A) = 0.$$

Definition

The characteristic polynomial of an $n \times n$ matrix A is

$$c_A(x) = \det(xI - A).$$

Definition

The characteristic polynomial of an $n \times n$ matrix A is

$$c_A(x) = \det(xI - A).$$

Example

The characteristic polynomial of $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$ is

$$\begin{aligned} c_A(x) &= \det\left(\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}\right) \\ &= \det\begin{bmatrix} x-4 & 2 \\ 1 & x-3 \end{bmatrix} \\ &= (x-4)(x-3) - 2 \\ &= x^2 - 7x + 10 \end{aligned}$$

Theorem (Eigenvalues and Eigenvectors of a Matrix)

Let A be an $n \times n$ matrix.

1. The eigenvalues of A are the roots of $c_A(x)$.
2. The λ-eigenvectors $\vec{x}$ are the nontrivial solutions to $(\lambda I - A)\vec{x} = \vec{0}$.

Theorem (Eigenvalues and Eigenvectors of a Matrix)

Let A be an $n \times n$ matrix.

1. The eigenvalues of A are the roots of $c_A(x)$.
2. The λ-eigenvectors $\vec{x}$ are the nontrivial solutions to $(\lambda I - A)\vec{x} = \vec{0}$.

Example (continued)

For $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$, we have

$$c_A(x) = x^2 - 7x + 10 = (x-2)(x-5),$$

so A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$.

Theorem (Eigenvalues and Eigenvectors of a Matrix)

Let A be an $n \times n$ matrix.

1. The eigenvalues of A are the roots of $c_A(x)$.
2. The λ-eigenvectors $\vec{x}$ are the nontrivial solutions to $(\lambda I - A)\vec{x} = \vec{0}$.

Example (continued)

For $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$, we have

$$c_A(x) = x^2 - 7x + 10 = (x-2)(x-5),$$

so A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$.

To find the 2-eigenvectors of A, solve $(2I - A)\vec{x} = \vec{0}$:

$$\left[\begin{array}{rr|r} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & -1 & 0 \\ -2 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Example (continued)

The general solution, in parametric form, is

$$\vec{x} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ where } t \in \mathbb{R}.$$

Example (continued)

The general solution, in parametric form, is

$$\vec{x} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ where } t \in \mathbb{R}.$$

To find the 5-eigenvectors of A, solve $(5I - A)\vec{x} = \vec{0}$:

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Example (continued)

The general solution, in parametric form, is

$$\vec{x} = \left[\begin{array}{c} t \\ t \end{array} \right] = t \left[\begin{array}{c} 1 \\ 1 \end{array} \right] \text{ where } t \in \mathbb{R}.$$

To find the 5-eigenvectors of A, solve $(5I - A)\vec{x} = \vec{0}$:

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

The general solution, in parametric form, is

$$\vec{x} = \left[\begin{array}{r} -2s \\ s \end{array} \right] = s \left[\begin{array}{r} -2 \\ 1 \end{array} \right] \text{ where } s \in \mathbb{R}.$$

Definition

A basic eigenvector of an $n \times n$ matrix A is any nonzero multiple of a basic solution to $(\lambda I - A)\vec{x} = \vec{0}$, where λ is an eigenvalue of A.

Definition

A basic eigenvector of an $n \times n$ matrix A is any nonzero multiple of a basic solution to $(\lambda I - A)\vec{x} = \vec{0}$, where λ is an eigenvalue of A.

Example (continued)

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ are basic eigenvectors of the matrix

$$A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$$

corresponding to eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$, respectively.

Problem

For $A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$, find $c_A(x)$, the eigenvalues of A, and the corresponding basic eigenvectors.

Problem

For $A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$, find $c_A(x)$, the eigenvalues of A, and the corresponding basic eigenvectors.

Solution

$$\det(xI - A) = \begin{vmatrix} x-3 & 4 & -2 \\ -1 & x+2 & -2 \\ -1 & 5 & x-5 \end{vmatrix} = \begin{vmatrix} x-3 & 4 & -2 \\ 0 & x-3 & -x+3 \\ -1 & 5 & x-5 \end{vmatrix}$$

$$= \begin{vmatrix} x-3 & 4 & 2 \\ 0 & x-3 & 0 \\ -1 & 5 & x \end{vmatrix} = (x-3)\begin{vmatrix} x-3 & 2 \\ -1 & x \end{vmatrix}$$

$$= (x-3)(x^2 - 3x + 2) = (x-3)(x-2)(x-1) = c_A(x).$$

Solution (continued)

Therefore, the eigenvalues of A are $\lambda_1 = 3, \lambda_2 = 2$, and $\lambda_3 = 1$.

Solution (continued)

Therefore, the eigenvalues of A are $\lambda_1 = 3, \lambda_2 = 2$, and $\lambda_3 = 1$.

Basic eigenvectors corresponding to $\lambda_1 = 3$: solve $(3\mathrm{I} - \mathrm{A})\vec{\mathrm{x}} = \vec{0}$.

$$\left[\begin{array}{rrr|r} 0 & 4 & -2 & 0 \\ -1 & 5 & -2 & 0 \\ -1 & 5 & -2 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution (continued)

Therefore, the eigenvalues of A are $\lambda_1 = 3, \lambda_2 = 2$, and $\lambda_3 = 1$.

Basic eigenvectors corresponding to $\lambda_1 = 3$: solve $(3I - A)\vec{x} = \vec{0}$.

$$\left[\begin{array}{rrr|r} 0 & 4 & -2 & 0 \\ -1 & 5 & -2 & 0 \\ -1 & 5 & -2 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $\vec{x} = \left[\begin{array}{r} \frac{1}{2}t \\ \frac{1}{2}t \\ t \end{array}\right] = t \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{array}\right], t \in \mathbb{R}$.

Solution (continued)

Therefore, the eigenvalues of A are $\lambda_1 = 3, \lambda_2 = 2$, and $\lambda_3 = 1$.

Basic eigenvectors corresponding to $\lambda_1 = 3$: solve $(3I - A)\vec{x} = \vec{0}$.

$$\left[\begin{array}{rrr|r} 0 & 4 & -2 & 0 \\ -1 & 5 & -2 & 0 \\ -1 & 5 & -2 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $\vec{x} = \left[\begin{array}{r} \frac{1}{2}t \\ \frac{1}{2}t \\ t \end{array}\right] = t \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{array}\right]$, $t \in \mathbb{R}$.

Choosing $t = 2$ gives us $\vec{x}_1 = \left[\begin{array}{c} 1 \\ 1 \\ 2 \end{array}\right]$ as a basic eigenvector corresponding to $\lambda_1 = 3$.

Solution (continued)

Basic eigenvectors corresponding to $\lambda_2 = 2$: solve $(2\mathrm{I} - \mathrm{A})\vec{\mathrm{x}} = \vec{0}$.

$$\left[\begin{array}{rrr|r} -1 & 4 & -2 & 0 \\ -1 & 4 & -2 & 0 \\ -1 & 5 & -3 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution (continued)

Basic eigenvectors corresponding to $\lambda_2 = 2$: solve $(2I - A)\vec{x} = \vec{0}$.

$$\left[\begin{array}{rrr|r} -1 & 4 & -2 & 0 \\ -1 & 4 & -2 & 0 \\ -1 & 5 & -3 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $\vec{x} = \left[\begin{array}{r} 2s \\ s \\ s \end{array}\right] = s\left[\begin{array}{r} 2 \\ 1 \\ 1 \end{array}\right], s \in \mathbb{R}$.

Solution (continued)

Basic eigenvectors corresponding to $\lambda_2 = 2$: solve $(2\mathrm{I} - \mathrm{A})\vec{x} = \vec{0}$.

$$\left[\begin{array}{rrr|r} -1 & 4 & -2 & 0 \\ -1 & 4 & -2 & 0 \\ -1 & 5 & -3 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $\vec{x} = \left[\begin{array}{r} 2s \\ s \\ s \end{array}\right] = s\left[\begin{array}{r} 2 \\ 1 \\ 1 \end{array}\right], s \in \mathbb{R}$.

Choosing $s = 1$ gives us $\vec{x}_2 = \left[\begin{array}{r} 2 \\ 1 \\ 1 \end{array}\right]$ as a basic eigenvector corresponding to $\lambda_2 = 2$.

Solution (continued)

Basic eigenvectors corresponding to $\lambda_3 = 1$: solve $(\mathrm{I} - \mathrm{A})\vec{\mathrm{x}} = \vec{0}$.

$$\left[\begin{array}{rrr|r} -2 & 4 & -2 & 0 \\ -1 & 3 & -2 & 0 \\ -1 & 5 & -4 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution (continued)

Basic eigenvectors corresponding to $\lambda_3 = 1$: solve $(I - A)\vec{x} = \vec{0}$.

$$\left[\begin{array}{rrr|r} -2 & 4 & -2 & 0 \\ -1 & 3 & -2 & 0 \\ -1 & 5 & -4 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $\vec{x} = \left[\begin{array}{c} r \\ r \\ r \end{array}\right] = r\left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right], r \in \mathbb{R}.$

Solution (continued)

Basic eigenvectors corresponding to $\lambda_3 = 1$: solve $(I - A)\vec{x} = \vec{0}$.

$$\left[\begin{array}{rrr|r} -2 & 4 & -2 & 0 \\ -1 & 3 & -2 & 0 \\ -1 & 5 & -4 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $\vec{x} = \left[\begin{array}{c} r \\ r \\ r \end{array}\right] = r \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$, $r \in \mathbb{R}$.

Choosing $r = 1$ gives us $\vec{x}_3 = \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$ as a basic eigenvector corresponding to $\lambda_3 = 1$. ∎

Geometric Interpretation of Eigenvalues and Eigenvectors

Geometric Interpretation of Eigenvalues and Eigenvectors

Let A be a 2×2 matrix. Then A can be interpreted as a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Geometric Interpretation of Eigenvalues and Eigenvectors

Let A be a 2×2 matrix. Then A can be interpreted as a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Problem

How does the linear transformation affect the eigenvectors of the matrix?

Geometric Interpretation of Eigenvalues and Eigenvectors

Let A be a 2×2 matrix. Then A can be interpreted as a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Problem

How does the linear transformation affect the eigenvectors of the matrix?

Definition

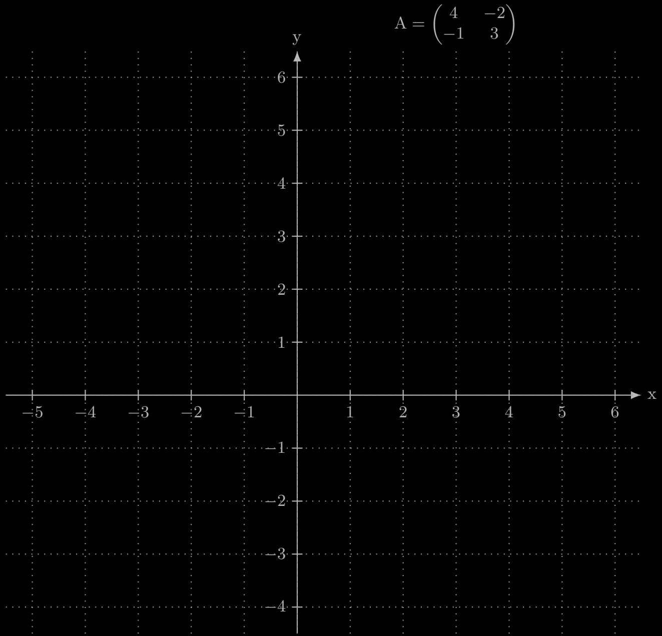
Let $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ be a nonzero vector in $\mathbb{R}^2$. Then $L_{\vec{v}}$ is the set of all scalar multiples of $\vec{v}$, i.e.,

$$L_{\vec{v}} = \mathbb{R}\vec{v} = \{t\vec{v} \mid t \in \mathbb{R}\}.$$

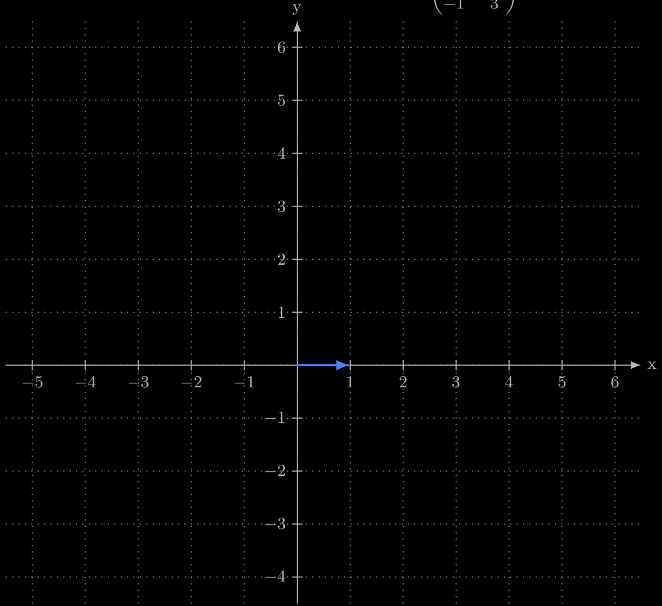
Example (revisited)

$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$ has two eigenvalues: $\lambda_1 = 2$ and $\lambda_2 = 5$ with corresponding eigenvectors

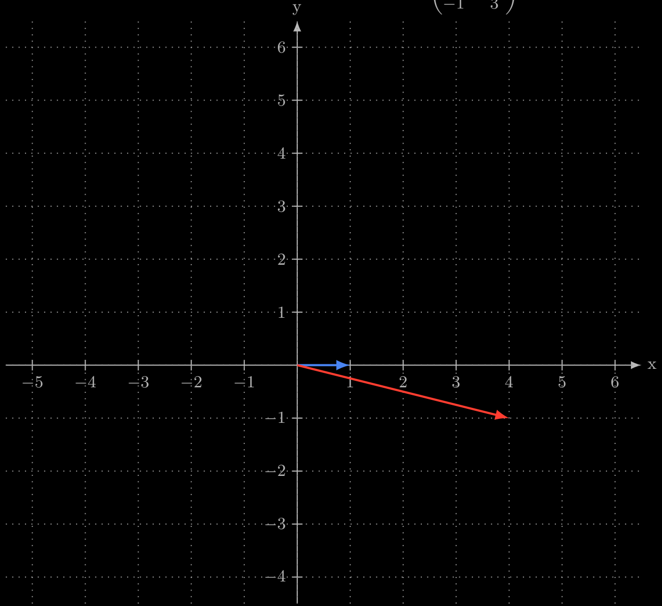
$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1/2 \end{pmatrix}$$


A = (4 −2; −1 3)
y
x
6
5
4
3
2
1
−1
−2
−3
−4
−5
−4
−3
−2
−1
1
2
3
4
5
6

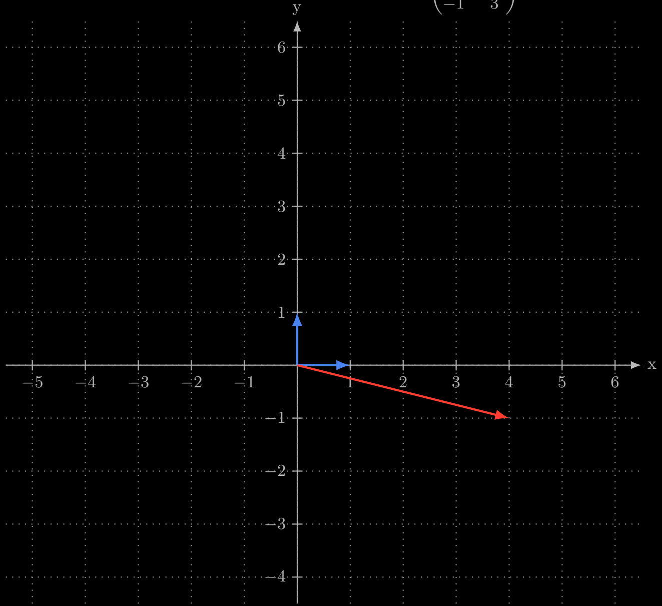
$$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$$



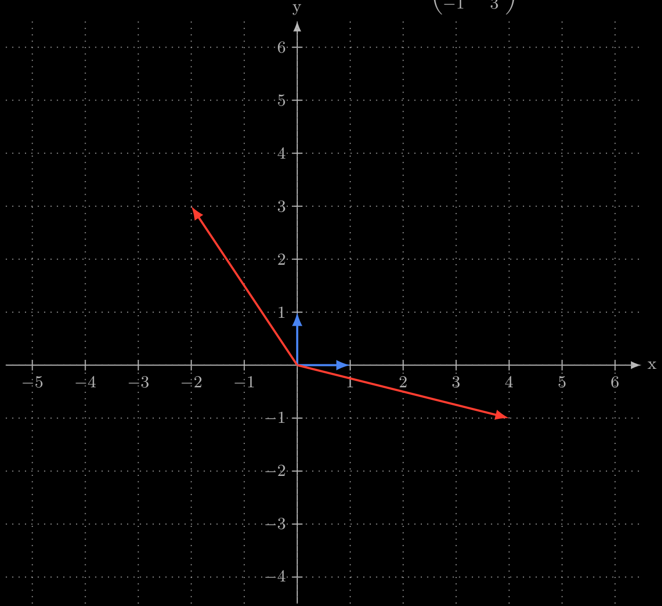
$$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$$

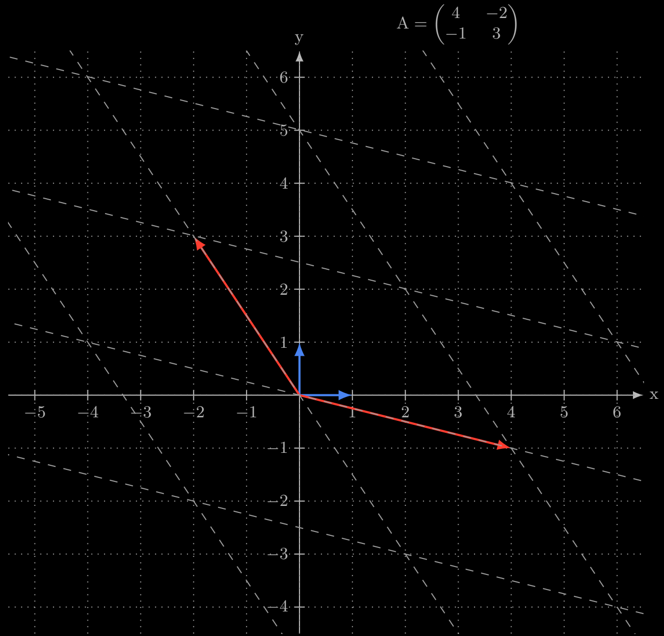


$$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$$

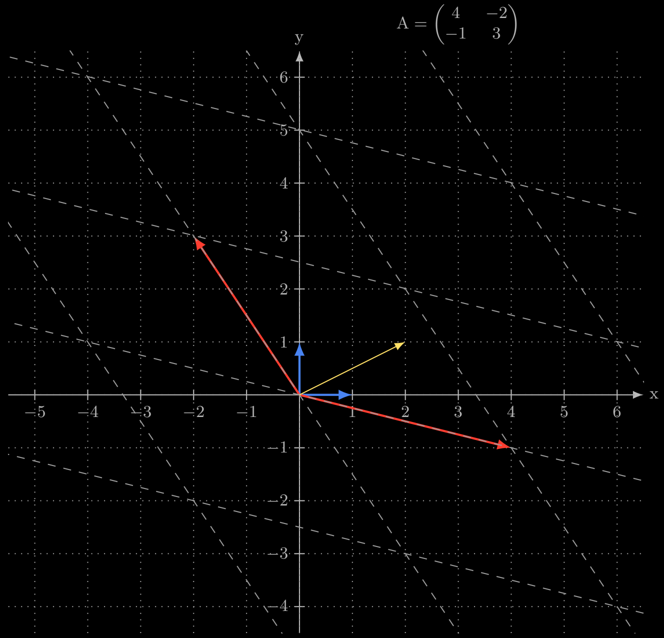


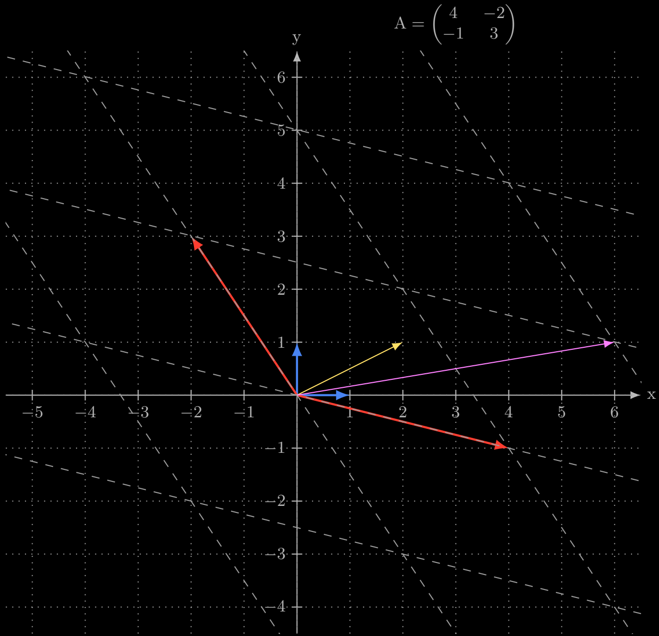
$$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$$



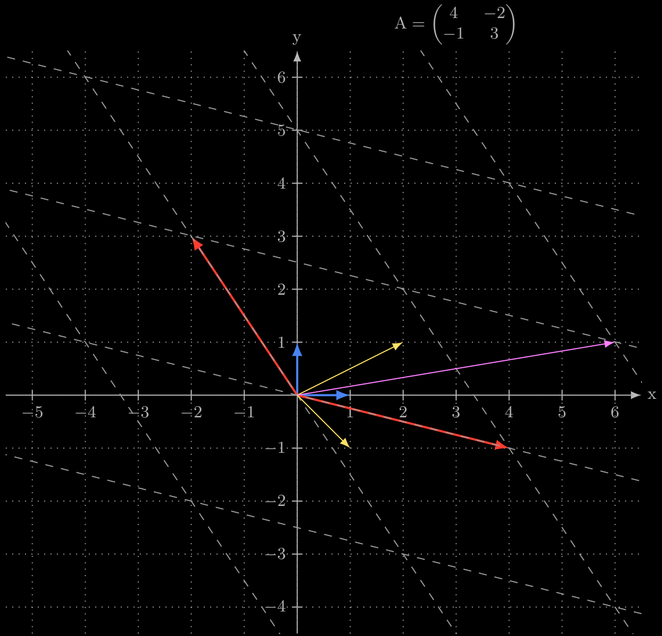


$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$
y
x
−5
−4
−3
−2
−1
1
2
3
4
5
6
−4
−3
−2
−1
1
2
3
4
5
6

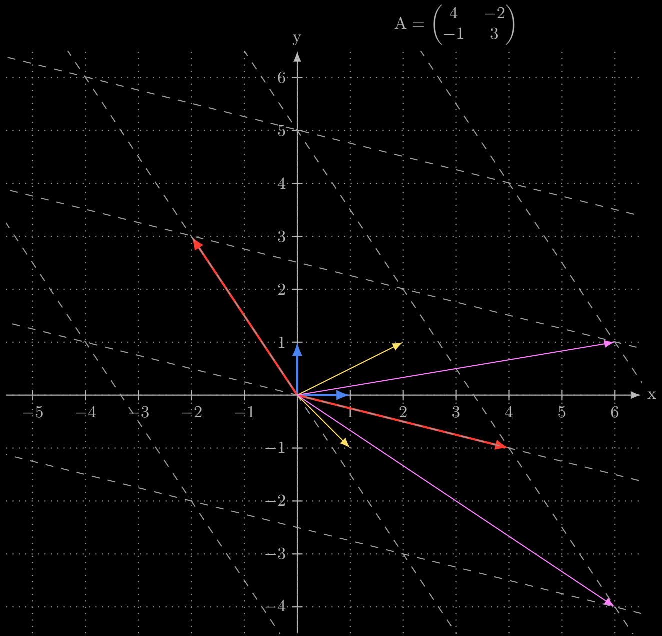

$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$
y
x
−5
−4
−3
−2
−1
1
2
3
4
5
6
6
5
4
3
2
1
−1
−2
−3
−4



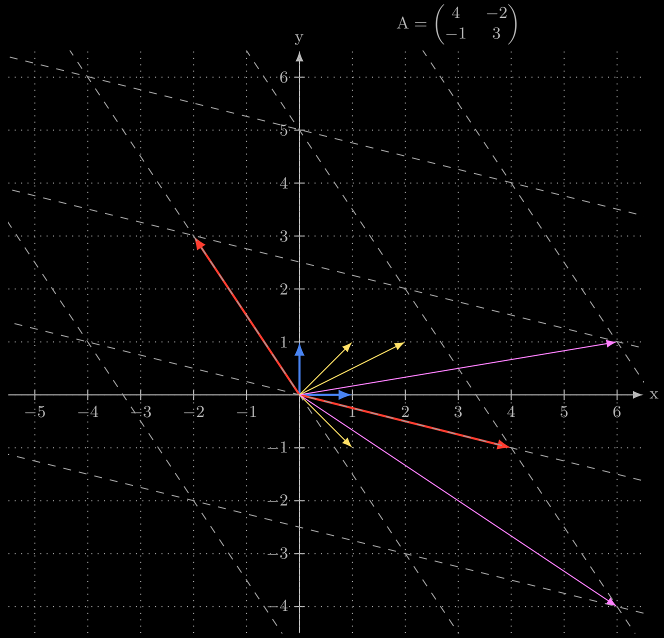
$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$
y
x
−5 −4 −3 −2 −1 1 2 3 4 5 6
6 5 4 3 2 1 −1 −2 −3 −4



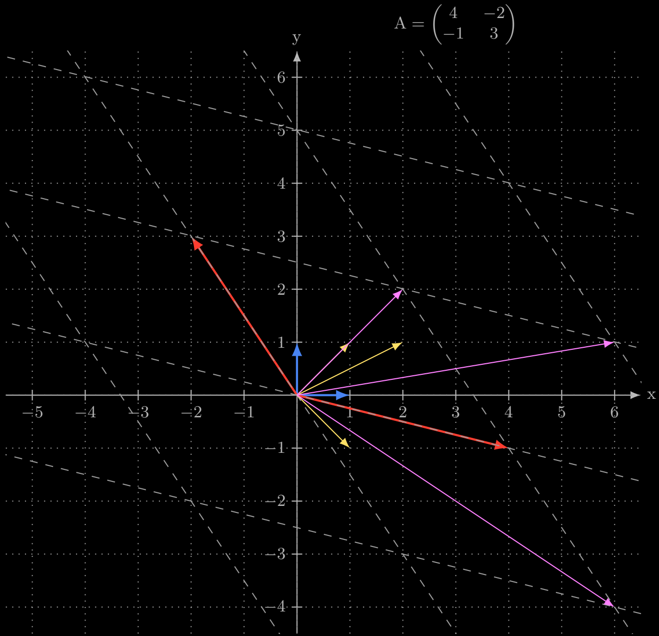
$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$
y
x
6
5
4
3
2
1
−1
−2
−3
−4
−5
−4
−3
−2
−1
1
2
3
4
5
6



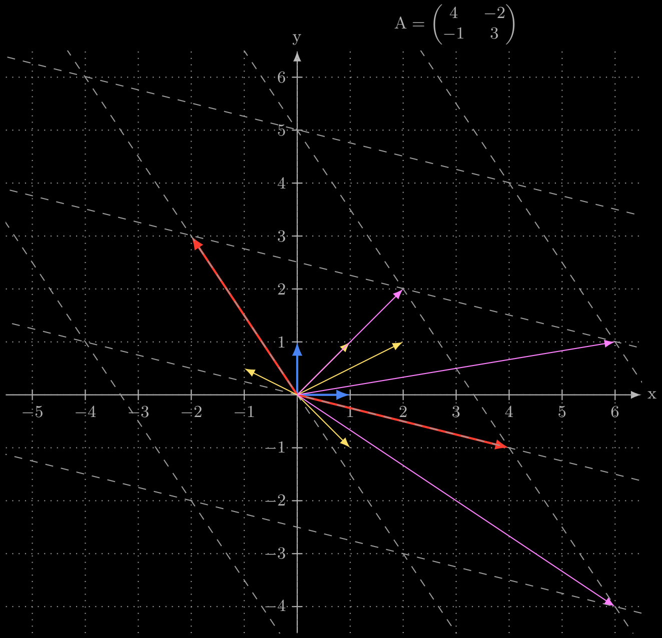
$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$
y
x
−5 −4 −3 −2 −1 1 2 3 4 5 6
6 5 4 3 2 1 −1 −2 −3 −4



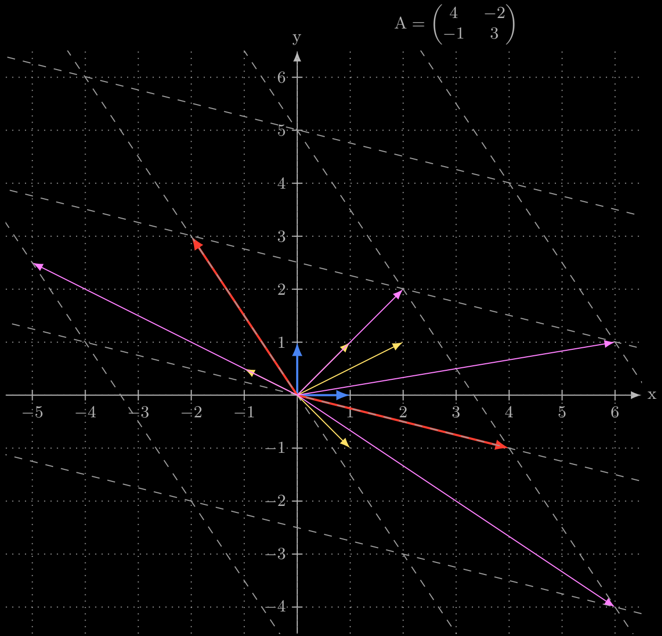
$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$
y
x
6
5
4
3
2
1
−1
−2
−3
−4
−5
−4
−3
−2
−1
1
2
3
4
5
6



$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$
x
y



A = (4 −2 / −1 3)
y
x
−5 −4 −3 −2 −1 1 2 3 4 5 6
6 5 4 3 2 1 −1 −2 −3 −4



$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$
y
x
−5
−4
−3
−2
−1
1
2
3
4
5
6
−1
−2
−3
−4

Definition

Let A be a 2×2 matrix and L a line in $\mathbb{R}^2$ through the origin. Then L is said to be A-invariant if the vector $A\vec{x}$ lies in L whenever $\vec{x}$ lies in L,

Definition

Let A be a 2×2 matrix and L a line in $\mathbb{R}^2$ through the origin. Then L is said to be A-invariant if the vector $A\vec{x}$ lies in L whenever $\vec{x}$ lies in L, i.e., $A\vec{x}$ is a scalar multiple of $\vec{x}$,

Definition

Let A be a 2×2 matrix and L a line in $\mathbb{R}^2$ through the origin. Then L is said to be A-invariant if the vector $A\vec{x}$ lies in L whenever $\vec{x}$ lies in L,
i.e., $A\vec{x}$ is a scalar multiple of $\vec{x}$,
i.e., $A\vec{x} = \lambda\vec{x}$ for some scalar $\lambda \in \mathbb{R}$,

Definition

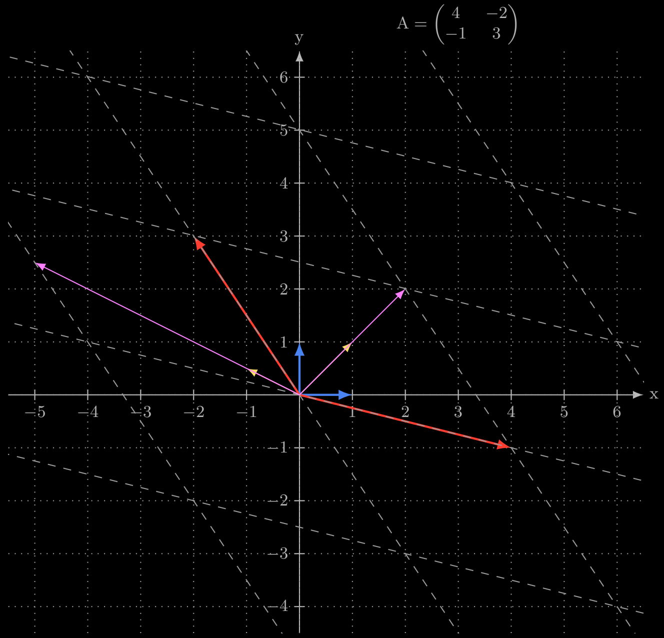
Let A be a 2×2 matrix and L a line in $\mathbb{R}^2$ through the origin. Then L is said to be A-invariant if the vector $A\vec{x}$ lies in L whenever $\vec{x}$ lies in L,
i.e., $A\vec{x}$ is a scalar multiple of $\vec{x}$,
i.e., $A\vec{x} = \lambda\vec{x}$ for some scalar $\lambda \in \mathbb{R}$,
i.e., $\vec{x}$ is an eigenvector of A.

Definition

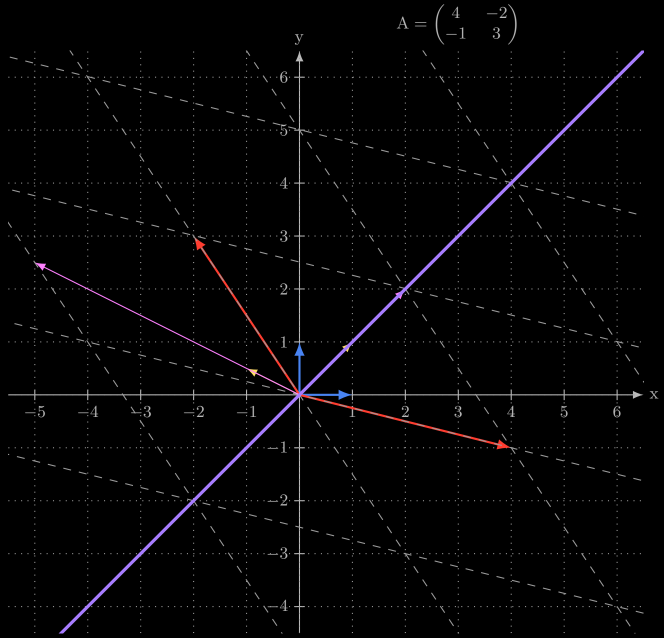
Let A be a 2×2 matrix and L a line in $\mathbb{R}^2$ through the origin. Then L is said to be A-invariant if the vector $A\vec{x}$ lies in L whenever $\vec{x}$ lies in L,
i.e., $A\vec{x}$ is a scalar multiple of $\vec{x}$,
i.e., $A\vec{x} = \lambda\vec{x}$ for some scalar $\lambda \in \mathbb{R}$,
i.e., $\vec{x}$ is an eigenvector of A.

Theorem (A-Invariance)

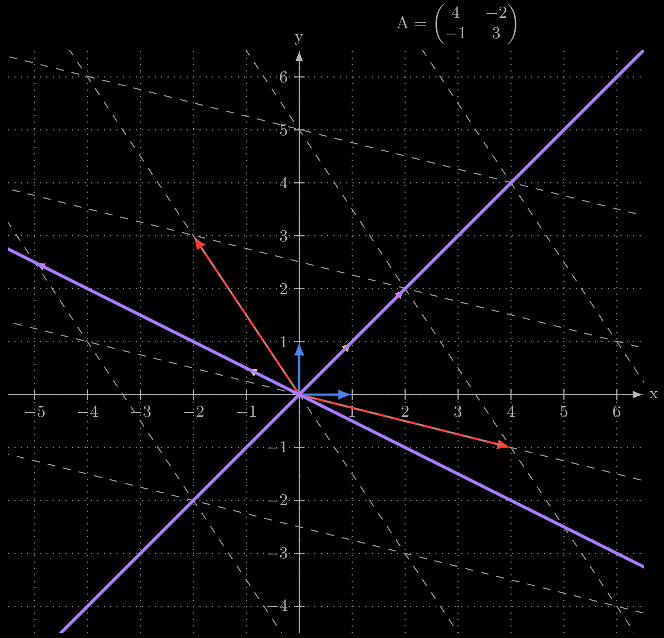
Let A be a 2×2 matrix and let $\vec{v} \neq 0$ be a vector in $\mathbb{R}^2$. Then $L_{\vec{v}}$ is A-invariant if and only if $\vec{v}$ is an eigenvector of A.



$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$
y
x
−5
−4
−3
−2
−1
1
2
3
4
5
6
−1
−2
−3
−4



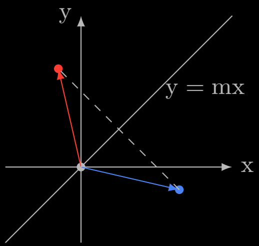
$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$
y
x
6
5
4
3
2
1
−1
−2
−3
−4
−5
−4
−3
−2
−1
1
2
3
4
5
6



$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$
y
x
6
5
4
3
2
1
−1
−2
−3
−4
−5
−4
−3
−2
−1
1
2
3
4
5
6

Problem

Let $m \in \mathbb{R}$ and consider the linear transformation $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$, i.e., reflection in the line $y = mx$.

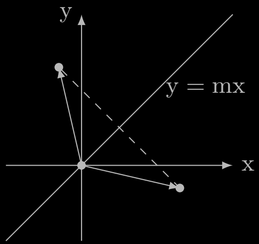


Recall that this is a matrix transformation induced by

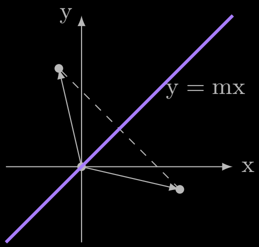
$$A = \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

Find the lines that pass through origin and are A-invariant. Determine corresponding eigenvalues.

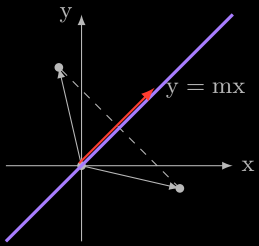
Solution



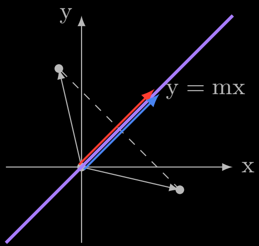
Solution



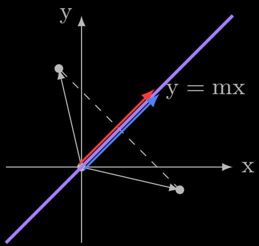
Solution



Solution



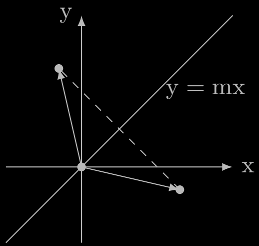
Solution



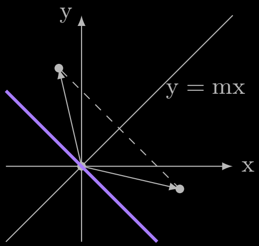
Let $\vec{x}_1 = \begin{bmatrix} 1 \\ m \end{bmatrix}$. Then $L_{\vec{x}_1}$ is A-invariant, that is, $\vec{x}_1$ is an eigenvector. Since the vector won't change, its eigenvalue should be 1. Indeed, one can verify that

$$A\vec{x}_1 = \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}\begin{pmatrix} 1 \\ m \end{pmatrix} = \ldots = \begin{pmatrix} 1 \\ m \end{pmatrix} = \vec{x}_1 .$$

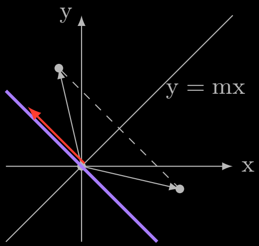
Solution (continued)



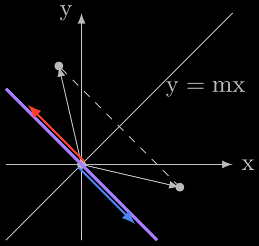
Solution (continued)



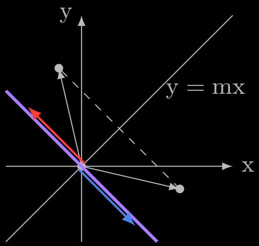
Solution (continued)



Solution (continued)



Solution (continued)



Let $\vec{x}_2 = \begin{bmatrix} -m \\ 1 \end{bmatrix}$. Then $L_{\vec{x}_2}$ is A-invariant, that is, $\vec{x}_2$ is an eigenvector. Since the vector won't change the size, only flip the direction, its eigenvalue should be -1. Indeed, one can verify that

$$A\vec{x}_2 = \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}\begin{pmatrix} -m \\ 1 \end{pmatrix} = \cdots = \begin{pmatrix} m \\ -1 \end{pmatrix} = -\vec{x}_2.$$

■

Example

Let θ be a real number, and $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ rotation through an angle of θ, induced by the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Example

Let θ be a real number, and $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ rotation through an angle of θ, induced by the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Claim: A has no real eigenvalues unless θ is an integer multiple of π, i.e., $\pm\pi, \pm 2\pi, \pm 3\pi$, etc.

Example

Let θ be a real number, and $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ rotation through an angle of θ, induced by the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Claim: A has no real eigenvalues unless θ is an integer multiple of π, i.e., $\pm\pi, \pm 2\pi, \pm 3\pi$, etc.

Consequence: a line L in $\mathbb{R}^2$ is A invariant if and only if θ is an integer multiple of π.

Diagonalization

Diagonalization

Denote an $n \times n$ diagonal matrix by

$$\mathrm{diag}(a_1, a_2, \ldots, a_n) = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & a_n \end{bmatrix}$$

Diagonalization

Denote an $n \times n$ diagonal matrix by

$$\mathrm{diag}(a_1, a_2, \ldots, a_n) = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & a_n \end{bmatrix}$$

Recall that if A is an $n \times n$ matrix and P is an invertible $n \times n$ matrix so that $P^{-1}AP$ is diagonal, then P is called a diagonalizing matrix of A, and A is diagonalizable.

- Suppose we have n eigenvalue-eigenvector pairs:

$$A\vec{x}_j = \lambda_j \vec{x}_j\,, \quad j = 1, 2, \ldots, n$$

- Suppose we have n eigenvalue-eigenvector pairs:

$$A\vec{x}_j = \lambda_j \vec{x}_j\,, \quad j = 1, 2, \ldots, n$$

- Pack the above n columns vectors into a matrix:

$$\left[\; A\vec{x}_1 \;\middle|\; A\vec{x}_2 \;\middle|\; \cdots \;\middle|\; A\vec{x}_n \;\right] = \left[\; \lambda_1\vec{x}_1 \;\middle|\; \lambda_2\vec{x}_2 \;\middle|\; \cdots \;\middle|\; \lambda_n\vec{x}_n \;\right]$$

$$||$$

$$A\left[\; \vec{x}_1 \;\middle|\; \vec{x}_2 \;\middle|\; \cdots \;\middle|\; \vec{x}_n \;\right]$$

$$||$$

$$\left[\; \vec{x}_1 \;\middle|\; \vec{x}_2 \;\middle|\; \cdots \;\middle|\; \vec{x}_n \;\right] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

- By denoting:

$$P = \left[\ \vec{x}_1 \mid \vec{x}_2 \mid \cdots \mid \vec{x}_n\ \right] \quad \text{and} \quad D = \mathrm{diag}\left(\lambda_1, \cdots, \lambda_n\right)$$

we see that

$$AP = PD$$

- By denoting:

$$P = \left[\ \vec{x}_1 \mid \vec{x}_2 \mid \cdots \mid \vec{x}_n\ \right] \quad \text{and} \quad D = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$$

  we see that

$$AP = PD$$

- Hence, provided P is invertible, we have

$$A = PDP^{-1} \quad \text{or equivalently} \quad D = P^{-1}AP$$

- By denoting:

$$P = \left[\ \vec{x}_1 \mid \vec{x}_2 \mid \cdots \mid \vec{x}_n\ \right] \quad \text{and} \quad D = \operatorname{diag}\left(\lambda_1, \cdots, \lambda_n\right)$$

we see that

$$AP = PD$$

- Hence, provided P is invertible, we have

$$A = PDP^{-1} \quad \text{or equivalently} \quad D = P^{-1}AP$$

that is, A is diagonalizable.

Theorem (Matrix Diagonalization)

Let A be an $n \times n$ matrix.

1. A is diagonalizable if and only if it has eigenvectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ so that

$$P = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix}$$

is invertible.

Theorem (Matrix Diagonalization)

Let A be an $n \times n$ matrix.

1. A is diagonalizable if and only if it has eigenvectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ so that

$$P = \left[\begin{array}{cccc} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{array} \right]$$

is invertible.

2. If P is invertible, then

$$P^{-1}AP = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

where λ_i is the eigenvalue of A corresponding to the eigenvector $\vec{x}_i$, i.e., $A\vec{x}_i = \lambda_i \vec{x}_i$.

Example

$A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$ has eigenvalues and corresponding basic eigenvectors

$$\lambda_1 = 3 \quad \text{and} \quad \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix};$$

$$\lambda_2 = 2 \quad \text{and} \quad \vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix};$$

$$\lambda_3 = 1 \quad \text{and} \quad \vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Example (continued)

Let $P = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$.

Example (continued)

Let $P = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$. Then P is invertible, so by the above Theorem,

$$P^{-1}AP = \text{diag}(3, 2, 1) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

■

Remark

It is not always possible to find n eigenvectors so that P is invertible.

Remark

It is not always possible to find n eigenvectors so that P is invertible.

Example

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{bmatrix}$.

Remark

It is not always possible to find n eigenvectors so that P is invertible.

Example

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{bmatrix}$.

Then

$$c_A(x) = \begin{vmatrix} x-1 & 2 & -3 \\ -2 & x-6 & 6 \\ -1 & -2 & x+1 \end{vmatrix} = \cdots = (x-2)^3.$$

Remark

It is not always possible to find n eigenvectors so that P is invertible.

Example

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{bmatrix}$.

Then

$$c_A(x) = \begin{vmatrix} x-1 & 2 & -3 \\ -2 & x-6 & 6 \\ -1 & -2 & x+1 \end{vmatrix} = \cdots = (x-2)^3.$$

A has only one eigenvalue, $\lambda_1 = 2$, with multiplicity three. Sometimes, one writes

$$\lambda_1 = \lambda_2 = \lambda_3 = 2.$$

Example (continued)

To find the 2-eigenvectors of A, solve the system $(2I - A)\vec{x} = \vec{0}$.

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 0 \\ -2 & -4 & 6 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example (continued)

To find the 2-eigenvectors of A, solve the system $(2I - A)\vec{x} = \vec{0}$.

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 0 \\ -2 & -4 & 6 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution in parametric form is

$$\vec{x} = \left[\begin{array}{c} -2s + 3t \\ s \\ t \end{array}\right] = s\left[\begin{array}{r} -2 \\ 1 \\ 0 \end{array}\right] + t\left[\begin{array}{r} 3 \\ 0 \\ 1 \end{array}\right], \quad s, t \in \mathbb{R}.$$

Example (continued)

To find the 2-eigenvectors of A, solve the system $(2I - A)\vec{x} = \vec{0}$.

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 0 \\ -2 & -4 & 6 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution in parametric form is

$$\vec{x} = \left[\begin{array}{c} -2s + 3t \\ s \\ t \end{array}\right] = s\left[\begin{array}{r} -2 \\ 1 \\ 0 \end{array}\right] + t\left[\begin{array}{r} 3 \\ 0 \\ 1 \end{array}\right], \quad s, t \in \mathbb{R}.$$

Since the system has only two basic solutions, there are only two basic eigenvectors, implying that the matrix A is not diagonalizable. ■

Problem

Diagonalize, if possible, the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

Problem

Diagonalize, if possible, the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

Solution

$$c_A(x) = \det(xI - A) = \begin{vmatrix} x-1 & 0 & -1 \\ 0 & x-1 & 0 \\ 0 & 0 & x+3 \end{vmatrix} = (x-1)^2(x+3).$$

Problem

Diagonalize, if possible, the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

Solution

$$c_A(x) = \det(xI - A) = \begin{vmatrix} x-1 & 0 & -1 \\ 0 & x-1 & 0 \\ 0 & 0 & x+3 \end{vmatrix} = (x-1)^2(x+3).$$

A has eigenvalues $\lambda_1 = 1$ of multiplicity two; $\lambda_2 = -3$ of multiplicity one.

Solution (continued)

Eigenvectors for $\lambda_1 = 1$: solve $(\mathrm{I} - \mathrm{A})\vec{\mathrm{x}} = \vec{0}$.

$$\left[\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution (continued)

Eigenvectors for $\lambda_1 = 1$: solve $(I - A)\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\vec{x} = \begin{bmatrix} s \\ t \\ 0 \end{bmatrix}$, $s, t \in \mathbb{R}$ so basic eigenvectors corresponding to $\lambda_1 = 1$ are

Solution (continued)

Eigenvectors for $\lambda_1 = 1$: solve $(I - A)\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccr|c} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\vec{x} = \begin{bmatrix} s \\ t \\ 0 \end{bmatrix}$, $s, t \in \mathbb{R}$ so basic eigenvectors corresponding to $\lambda_1 = 1$ are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Solution (continued)

Eigenvectors for $\lambda_2 = -3$: solve $(-3\mathrm{I} - \mathrm{A})\vec{\mathrm{x}} = \vec{0}$.

$$\left[\begin{array}{ccc|c} -4 & 0 & -1 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution (continued)

Eigenvectors for $\lambda_2 = -3$: solve $(-3I - A)\vec{x} = \vec{0}$.

$$\left[\begin{array}{rrr|r} -4 & 0 & -1 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\vec{x} = \left[\begin{array}{c} -\frac{1}{4}t \\ 0 \\ t \end{array}\right]$, $t \in \mathbb{R}$ so a basic eigenvector corresponding to $\lambda_2 = -3$ is

$$\left[\begin{array}{r} -1 \\ 0 \\ 4 \end{array}\right]$$

Solution (continued)

Let

$$P = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & 0 \end{bmatrix}.$$

Solution (continued)

Let

$$P = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & 0 \end{bmatrix}.$$

Then P is invertible,

Solution (continued)

Let

$$P = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & 0 \end{bmatrix}.$$

Then P is invertible, and

$$P^{-1}AP = \mathrm{diag}(-3, 1, 1) = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

■

Theorem (Matrix Diagonalization Test)

A square matrix A is diagonalizable if and only if every eigenvalue λ of multiplicity m yields exactly m basic eigenvectors, i.e., the solution to $(\lambda I - A)\vec{x} = \vec{0}$ has m parameters.

Theorem (Matrix Diagonalization Test)

A square matrix A is diagonalizable if and only if every eigenvalue λ of multiplicity m yields exactly m basic eigenvectors, i.e., the solution to $(\lambda I - A)\vec{x} = \vec{0}$ has m parameters.

A special case of this is:

Theorem (Distinct Eigenvalues and Diagonalization)

An $n \times n$ matrix with distinct eigenvalues is diagonalizable.

Problem

Show that $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is not diagonalizable.

Problem

Show that $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is not diagonalizable.

Solution

First,

$$c_A(x) = \begin{vmatrix} x-1 & -1 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x-2 \end{vmatrix} = (x-1)^2(x-2),$$

so A has eigenvalues $\lambda_1 = 1$ of multiplicity two; $\lambda_2 = 2$ (of multiplicity one).

Solution (continued)

Eigenvectors for $\lambda_1 = 1$: solve $(\mathrm{I} - \mathrm{A})\vec{\mathrm{x}} = \vec{0}$.

$$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution (continued)

Eigenvectors for $\lambda_1 = 1$: solve $(\mathrm{I} - \mathrm{A})\vec{\mathrm{x}} = \vec{0}$.

$$\left[\begin{array}{rrr|r} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, $\vec{\mathrm{x}} = \left[\begin{array}{c} \mathrm{s} \\ 0 \\ 0 \end{array}\right]$, $\mathrm{s} \in \mathbb{R}$.

Solution (continued)

Eigenvectors for $\lambda_1 = 1$: solve $(I - A)\vec{x} = \vec{0}$.

$$\left[\begin{array}{rrr|r} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, $\vec{x} = \left[\begin{array}{c} s \\ 0 \\ 0 \end{array}\right]$, $s \in \mathbb{R}$.

Since $\lambda_1 = 1$ has multiplicity two, but has only one basic eigenvector, we can conclude that A is NOT diagonalizable. ■

Why Diagonalization?

Eigenvalues and Eigenvectors

Geometric Interpretation of Eigenvalues and Eigenvectors

Diagonalization

Linear Dynamical Systems

Linear Dynamical Systems

Linear Dynamical Systems

Definition

A linear dynamical system consists of

- an $n \times n$ matrix A and an n-vector $\vec{v}_0$;

Linear Dynamical Systems

Definition

A linear dynamical system consists of

- an $n \times n$ matrix A and an n-vector $\vec{v}_0$;
- a matrix recursion defining $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots$ by $\vec{v}_{k+1} = A\vec{v}_k$; i.e.,

Linear Dynamical Systems

Definition

A linear dynamical system consists of

- an $n \times n$ matrix A and an n-vector $\vec{v}_0$;
- a matrix recursion defining $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots$ by $\vec{v}_{k+1} = A\vec{v}_k$; i.e.,

$$\begin{aligned}
\vec{v}_1 &= A\vec{v}_0 \\
\vec{v}_2 &= A\vec{v}_1 = A(A\vec{v}_0) = A^2\vec{v}_0 \\
\vec{v}_3 &= A\vec{v}_2 = A(A^2\vec{v}_0) = A^3\vec{v}_0 \\
\vdots \;\; & \;\; \vdots \;\; \vdots \\
\vec{v}_k &= A^k\vec{v}_0.
\end{aligned}$$

Linear Dynamical Systems

Definition

A linear dynamical system consists of

- an $n \times n$ matrix A and an n-vector $\vec{v}_0$;
- a matrix recursion defining $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots$ by $\vec{v}_{k+1} = A\vec{v}_k$; i.e.,

$$\begin{aligned}
\vec{v}_1 &= A\vec{v}_0 \\
\vec{v}_2 &= A\vec{v}_1 = A(A\vec{v}_0) = A^2\vec{v}_0 \\
\vec{v}_3 &= A\vec{v}_2 = A(A^2\vec{v}_0) = A^3\vec{v}_0 \\
\vdots &\quad \vdots \quad \vdots \\
\vec{v}_k &= A^k\vec{v}_0.
\end{aligned}$$

Remark

Linear dynamical systems are used, for example, to model the evolution of populations over time.

If A is diagonalizable, then

$$\mathrm{P}^{-1}\mathrm{AP} = \mathrm{D} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_\mathrm{n}),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_\mathrm{n}$ are the (not necessarily distinct) eigenvalues of A.

If A is diagonalizable, then

$$P^{-1}AP = D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the (not necessarily distinct) eigenvalues of A.

Thus $A = PDP^{-1}$, and $A^k = PD^kP^{-1}$. Therefore,

$$\vec{v}_k = A^k\vec{v}_0 = PD^kP^{-1}\vec{v}_0.$$

Problem

Consider the linear dynamical system $\vec{v}_{k+1} = A\vec{v}_k$ with

$$A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}, \quad \text{and} \quad \vec{v}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find a formula for $\vec{v}_k$.

Problem

Consider the linear dynamical system $\vec{v}_{k+1} = A\vec{v}_k$ with

$$A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}, \quad \text{and} \quad \vec{v}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find a formula for $\vec{v}_k$.

Solution

First, $c_A(x) = (x-2)(x+1)$, so A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$, and thus is diagonalizable.

Problem

Consider the linear dynamical system $\vec{v}_{k+1} = A\vec{v}_k$ with

$$A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}, \quad \text{and} \quad \vec{v}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find a formula for $\vec{v}_k$.

Solution

First, $c_A(x) = (x-2)(x+1)$, so A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$, and thus is diagonalizable.

Solve $(2I - A)\vec{x} = \vec{0}$:

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -3 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

has general solution $\vec{x} = \begin{bmatrix} s \\ s \end{bmatrix}$, $s \in \mathbb{R}$, and basic solution $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Solution (continued)

Solve $(-I - A)\vec{x} = \vec{0}$:

$$\left[\begin{array}{rr|r} -3 & 0 & 0 \\ -3 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

has general solution $\vec{x} = \left[\begin{array}{c} 0 \\ t \end{array}\right]$, $t \in \mathbb{R}$, and basic solution $\vec{x}_2 = \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$.

Solution (continued)

Solve $(-I - A)\vec{x} = \vec{0}$:

$$\left[\begin{array}{cc|c} -3 & 0 & 0 \\ -3 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

has general solution $\vec{x} = \left[\begin{array}{c} 0 \\ t \end{array}\right]$, $t \in \mathbb{R}$, and basic solution $\vec{x}_2 = \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$.

Thus, $P = \left[\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array}\right]$ is a diagonalizing matrix for A,

$$P^{-1} = \left[\begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array}\right], \quad \text{and} \quad P^{-1}AP = \left[\begin{array}{cc} 2 & 0 \\ 0 & -1 \end{array}\right].$$

Solution (continued)

Therefore,

$$\begin{aligned}
\vec{v}_k &= A^k \vec{v}_0 \\
&= PD^kP^{-1}\vec{v}_0 \\
&= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}^k \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & (-1)^k \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 2^k & 0 \\ 2^k & (-1)^k \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 2^k \\ 2^k - 2(-1)^k \end{bmatrix}.
\end{aligned}$$

■

Remark

Often, instead of finding an exact formula for $\vec{v}_k$, it suffices to estimate $\vec{v}_k$ as k gets large.

This can easily be done if A has a dominant eigenvalue with multiplicity one: an eigenvalue λ_1 with the property that

$$|\lambda_1| > |\lambda_j| \text{ for } j = 2, 3, \ldots, n.$$

Remark

Often, instead of finding an exact formula for $\vec{v}_k$, it suffices to estimate $\vec{v}_k$ as k gets large.

This can easily be done if A has a dominant eigenvalue with multiplicity one: an eigenvalue λ_1 with the property that

$$|\lambda_1| > |\lambda_j| \text{ for } j = 2, 3, \ldots, n.$$

Suppose that

$$\vec{v}_k = PD^kP^{-1}\vec{v}_0,$$

and assume that A has a dominant eigenvalue, λ_1, with corresponding basic eigenvector $\vec{x}_1$ as the first column of P.
For convenience, write $P^{-1}\vec{v}_0 = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^T$.

Then

$$
\begin{aligned}
\vec{v}_k &= PD^kP^{-1}\vec{v}_0 \\
&= \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \\
&= b_1\lambda_1^k\vec{x}_1 + b_2\lambda_2^k\vec{x}_2 + \cdots + b_n\lambda_n^k\vec{x}_n \\
&= \lambda_1^k\left(b_1\vec{x}_1 + b_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\vec{x}_2 + \cdots + b_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\vec{x}_n\right)
\end{aligned}
$$

Then

$$
\begin{aligned}
\vec{v}_k &= PD^kP^{-1}\vec{v}_0 \\
&= \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \\
&= b_1\lambda_1^k\vec{x}_1 + b_2\lambda_2^k\vec{x}_2 + \cdots + b_n\lambda_n^k\vec{x}_n \\
&= \lambda_1^k\left(b_1\vec{x}_1 + b_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\vec{x}_2 + \cdots + b_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\vec{x}_n\right)
\end{aligned}
$$

Now, $\left|\frac{\lambda_j}{\lambda_1}\right| < 1$ for $j = 2, 3, \ldots n$, and thus $\left(\frac{\lambda_j}{\lambda_1}\right)^k \to 0$ as $k \to \infty$.

Then

$$
\begin{aligned}
\vec{v}_k &= PD^kP^{-1}\vec{v}_0 \\
&= \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \\
&= b_1\lambda_1^k\vec{x}_1 + b_2\lambda_2^k\vec{x}_2 + \cdots + b_n\lambda_n^k\vec{x}_n \\
&= \lambda_1^k\left(b_1\vec{x}_1 + b_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\vec{x}_2 + \cdots + b_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\vec{x}_n\right)
\end{aligned}
$$

Now, $\left|\frac{\lambda_j}{\lambda_1}\right| < 1$ for $j = 2, 3, \ldots n$, and thus $\left(\frac{\lambda_j}{\lambda_1}\right)^k \to 0$ as $k \to \infty$.

Therefore, for large values of k, $\vec{v}_k \approx \lambda_1^k b_1\vec{x}_1$.

Problem

If

$$A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}, \quad \text{and} \quad \vec{v}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

estimate $\vec{v}_k$ for large values of k.

Problem

If

$$A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}, \quad \text{and} \quad \vec{v}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

estimate $\vec{v}_k$ for large values of k.

Solution

In our previous example, we found that A has eigenvalues 2 and -1. This means that $\lambda_1 = 2$ is a dominant eigenvalue; let $\lambda_2 = -1$.

Problem

If

$$A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}, \quad \text{and} \quad \vec{v}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

estimate $\vec{v}_k$ for large values of k.

Solution

In our previous example, we found that A has eigenvalues 2 and -1. This means that $\lambda_1 = 2$ is a dominant eigenvalue; let $\lambda_2 = -1$.

As before $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a basic eigenvector for $\lambda_1 = 2$, and $\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a basic eigenvector for $\lambda_2 = -1$, giving us

Problem

If

$$A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}, \quad \text{and} \quad \vec{v}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

estimate $\vec{v}_k$ for large values of k.

Solution

In our previous example, we found that A has eigenvalues 2 and -1. This means that $\lambda_1 = 2$ is a dominant eigenvalue; let $\lambda_2 = -1$.

As before $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a basic eigenvector for $\lambda_1 = 2$, and $\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a basic eigenvector for $\lambda_2 = -1$, giving us

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad P^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Solution (continued)

$$P^{-1}\vec{v}_0 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Solution (continued)

$$P^{-1}\vec{v}_0 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

For large values of k,

$$\vec{v}_k \approx \lambda_1^k b_1 \vec{x}_1 = 2^k(1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2^k \\ 2^k \end{bmatrix}.$$

■

Solution (continued)

$$P^{-1}\vec{v}_0 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

For large values of k,

$$\vec{v}_k \approx \lambda_1^k b_1 \vec{x}_1 = 2^k(1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2^k \\ 2^k \end{bmatrix}.$$

■

Remark

Let's compare this to the exact formula for $\vec{v}_k$ that we obtained earlier:

$$\vec{v}_k = \begin{bmatrix} 2^k \\ 2^k - 2(-1/2)^k \end{bmatrix} \approx \begin{bmatrix} 2^k \\ 2^k \end{bmatrix}.$$