Math 221: LINEAR ALGEBRA

Chapter 4. Vector Geometry
§4-1. Vectors and Lines

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary and those by James Nagy from Emory University.

Vector Norms (i.e., lengths)

Parallel Vectors

Length and Direction

Geometric Vectors

The Parallelogram Law

Lines in Space

NOTE: Much of this chapter is what you would learn in Multivariable Calculus.
You might find it interesting/useful to read.
But I will only cover the material important to this course.

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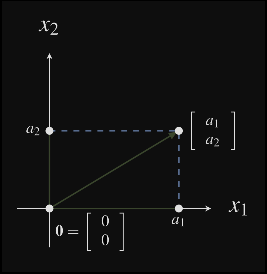
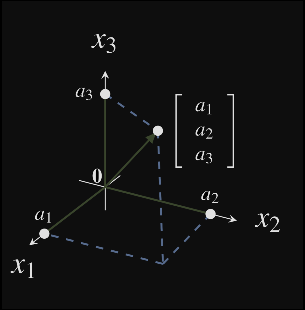
Vector Norms

- The word “norm” in linear algebra is used to mean “length”.
- There are actually many ways to define length, the most usual Euclidean:
  - In 2D and 3D:





  - In general, if $\vec{\mathrm{v}} \in \mathbb{R}^{\mathrm{n}}$, the Euclidean norm of $\vec{\mathrm{v}}$ is:

$$\|\vec{\mathrm{v}}\| = \sqrt{\vec{\mathrm{v}} \cdot \vec{\mathrm{v}}} = \sqrt{\vec{\mathrm{v}}^{\mathrm{T}} \vec{\mathrm{v}}} = \sqrt{\mathrm{v}_1^2 + \mathrm{v}_2^2 + \cdots + \mathrm{v}_{\mathrm{n}}^2}$$

Example:

If $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}$, find $\|\vec{v}\|$.

Example: Show that $\|c\vec{v}\| = |c|\|\vec{v}\|$ for any scalar c and any vector $\vec{v} \in \mathbb{R}^n$.

Definition

$\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is a vector norm if it satisfies the following properties:

1. $\|v\| \geq 0$ for all $v \in \mathbb{R}^n$, and $\|v\| = 0$ if and only if $v = 0$,
2. $\|v + w\| \leq \|v\| + \|w\|$ for all $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$,
3. $\|cv\| = |c|\|v\|$ for all vectors $v \in \mathbb{R}^n$ and all scalars c.

Remark

There many vector norms, so sometimes we include a subscript, such as $\|\cdot\|_p$, to indicate precisely which norm we are using. Here are some examples:

- The 2-norm is the standard Euclidean length:

$$\|\vec{v}\|_2 = \sqrt{\vec{v}^T\vec{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}\,.$$

- The 1-norm is defined as $\quad \|\vec{v}\|_1 = |v_1| + |v_2| + \cdots + |v_n|\,.$
- The ∞-norm is defined as $\quad \|\vec{v}\|_\infty = \max_{1\le i\le n}\{|v_i|\}\,.$
- In general, if $1 \le p < \infty$, then the p-norm is defined as

$$\|\vec{v}\|_p = \left(\sum_{i=1}^{n} |v_i|^p\right)^{1/p}\,.$$

Although other norms are used in certain applications, we usually use the 2-norm, and omit the subscript:

$$\|\vec{v}\| \equiv \|\vec{v}\|_2$$

Definition

A unit vector is a vector having norm equal to 1.

Example

Check if $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ are unit vectors.

Remark

We can scale any nonzero vector to have norm equal to 1.

If $\vec{v} \in \mathbb{R}^n$, $\vec{v} \neq \vec{0}$, then

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} \quad \text{is a unit vector}$$

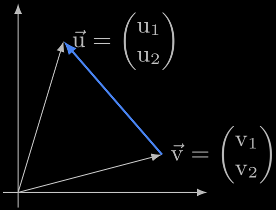
Problem

Scale $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ to a unit vector.

Definition

The distance between two vectors is defined as:

$$\mathrm{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$$



Vector Norms (i.e., lengths)

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Parallel Vectors

Definition

Two vectors are called parallel if they lie on the same line. Equivalently, two vectors are parallel if they are scalar multiples of each other.

Example

Determine if $\vec{v}$, $\vec{w}$, $\vec{z}$ are parallel to $\vec{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

$$\vec{v} = \begin{bmatrix} 6 \\ -4 \\ 2 \end{bmatrix}$$

$$\vec{w} = \begin{bmatrix} -6 \\ 4 \\ -2 \end{bmatrix}$$

$$\vec{z} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The following slides are for you to study by yourself

as reviewing matereial...

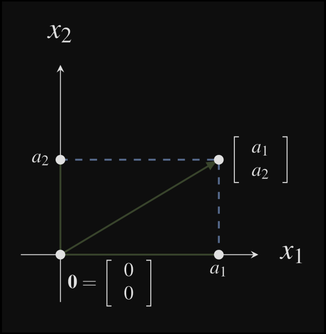
Scalar quantities versus vector quantities

- A scalar quantity has only magnitude; e.g. time, temperature.
- A vector quantity has both magnitude and direction; e.g. displacement, force, wind velocity.

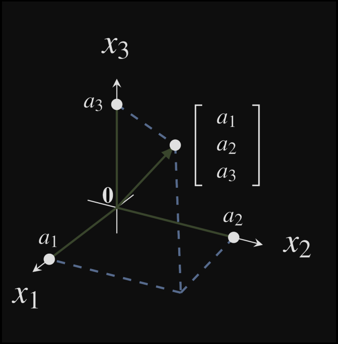
Whereas two scalar quantities are equal if they are represented by the same value, two vector quantities are equal if and only if they have the same magnitude and direction.

$\mathbb{R}^2$ and $\mathbb{R}^3$

Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ have convenient geometric representations as $\mathrm{position\ vectors}$ of points in the 2-dimensional (Cartesian) plane and in 3-dimensional space, respectively.



x_2
a_2
$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$
x_1
a_1
$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



x_3
a_3
$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$
$\mathbf{0}$
a_2
a_1
x_2
x_1

Notation

- If P is a point in $\mathbb{R}^3$ with coordinates $(\mathrm{x}, \mathrm{y}, \mathrm{x})$ we denote this by $\mathrm{P} = (\mathrm{x}, \mathrm{y}, \mathrm{z})$.
- If $\mathrm{P} = (\mathrm{x}, \mathrm{y}, \mathrm{z})$ is a point in $\mathbb{R}^3$, then

$$\overrightarrow{0\mathrm{P}} = \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{bmatrix}$$

  is often used to denote the position vector of the point.
- Instead of using a capital letter to denote the vector (as we generally do with matrices), we emphasize the importance of the geometry and the direction with an arrow over the name of the vector.

Notation and Terminology

- The notation $\overrightarrow{0P}$ emphasizes that this vector goes from the origin 0 to the point P. We can also use lower case letters for names of vectors. In this case, we write $\overrightarrow{0P} = \vec{p}$.
- Any vector
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ in } \mathbb{R}^3$$
is associated with the point (x_1, x_2, x_3).
- Often, there is no distinction made between the vector $\vec{x}$ and the point (x_1, x_2, x_3), and we say that both $(x_1, x_2, x_3) \in \mathbb{R}^3$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$.

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Length and Direction

Theorem

Let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ be vectors in $\mathbb{R}^3$. Then

1. $\vec{v} = \vec{w}$ if and only if $x = x_1$, $y = y_1$, and $z = z_1$.
2. $||\vec{v}|| = \sqrt{x^2 + y^2 + z^2}$.
3. $\vec{v} = \vec{0}$ if and only if $||\vec{v}|| = 0$.
4. For any scalar a, $||a\vec{v}|| = |a| \cdot ||\vec{v}||$.

Remark

Analogous results hold for $\vec{v}, \vec{w} \in \mathbb{R}^2$, i.e.,

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}, \vec{w} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

In this case, $||\vec{v}|| = \sqrt{x^2 + y^2}$.

Example

Let $\vec{\mathrm{p}} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, $\vec{\mathrm{q}} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$, and $-2\vec{\mathrm{q}} = \begin{bmatrix} -6 \\ 2 \\ 4 \end{bmatrix}$,

Then

$$||\vec{\mathrm{p}}|| = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = 5,$$

$$||\vec{\mathrm{q}}|| = \sqrt{(3)^2 + (-1)^2 + (-2)^2} = \sqrt{9 + 1 + 3} = \sqrt{14},$$

and

$$\begin{aligned} ||-2\vec{\mathrm{q}}|| &= \sqrt{(-6)^2 + 2^2 + 4^2} \\ &= \sqrt{36 + 4 + 16} \\ &= \sqrt{56} = \sqrt{4 \times 14} \\ &= 2\sqrt{14} = 2||\vec{\mathrm{q}}||. \end{aligned}$$

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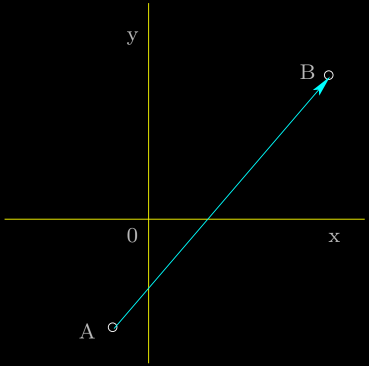
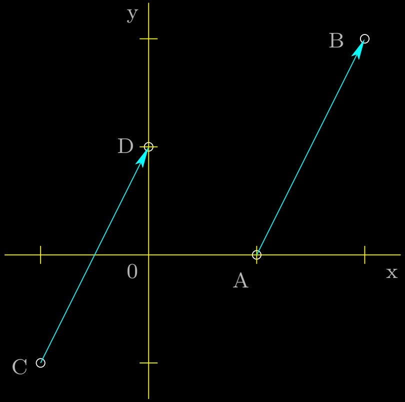
Geometric Vectors

Let A and B be two points in $\mathbb{R}^2$ or $\mathbb{R}^3$.



- $\overrightarrow{AB}$ is the geometric vector from A to B.
- A is the tail of $\overrightarrow{AB}$.
- B is the tip of $\overrightarrow{AB}$.
- the magnitude of $\overrightarrow{AB}$ is its length, and is denoted $||\overrightarrow{AB}||$.



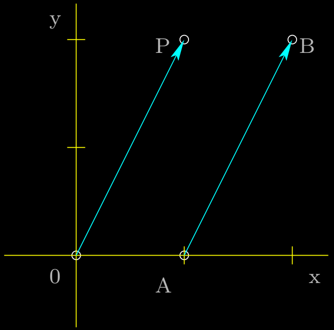
- $\overrightarrow{AB}$ is the vector from $A(1,0)$ to $B(2,2)$.
- $\overrightarrow{CD}$ is the vector from $C(-1,-1)$ to $D(0,1)$.

- $\overrightarrow{AB} = \overrightarrow{CD}$ because the vectors have the same length and direction.

Definition

A vector is in standard position if its tail is at the origin.

We co-ordinatize vectors by putting them in standard position, and then identifying them with their tips.



Thus $\overrightarrow{AB} = \overrightarrow{0P}$ where $P = P(1, 2)$, and we write $\overrightarrow{0P} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \overrightarrow{AB}$.

$\overrightarrow{0P}$ is the position vector for $P(1, 2)$.

More generally, if $P(x, y, z)$ is a point in $\mathbb{R}^3$, then $\overrightarrow{0P} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is the position vector for P.

If we aren't concerned with the locations of the tail and tip, we simply write $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

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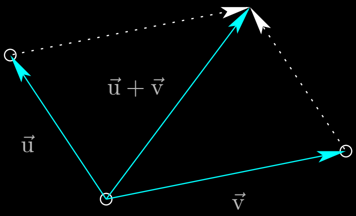
Intrinsic Description of Vectors

- vector equality: same length and direction.
- $\vec{0}$: the vector with length zero and no direction.
- scalar multiplication: if $\vec{v} \neq \vec{0}$ and $a \in \mathbb{R}$, $a \neq 0$, then $a\vec{v}$ has length $|a| \cdot ||\vec{v}||$ and
  - the same direction as $\vec{v}$ if $a > 0$;
  - direction opposite to $\vec{v}$ if $a < 0$.
- addition: $\vec{u} + \vec{v}$ is the diagonal of the parallelogram defined by $\vec{u}$ and $\vec{v}$, and having the same tail as $\vec{u}$ and $\vec{v}$.



parallelogram law

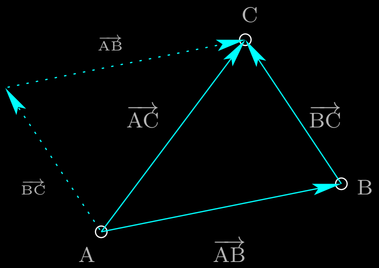
If we have a coordinate system, then

- vector equality: $\vec{u} = \vec{v}$ if and only if $\vec{u}$ and $\vec{v}$ are equal as matrices.
- $\vec{0}$: has all coordinates equal to zero.
- scalar multiplication: $a\vec{v}$ is obtained from $\vec{v}$ by multiplying each entry of $\vec{v}$ by a (matrix scalar multiplication).
- addition: $\vec{u} + \vec{v}$ is represented by the matrix sum of the columns $\vec{u}$ and $\vec{v}$.

Tip-to-Tail Method for Vector Addition

For points A, B and C,

$$\overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{BC}} = \overrightarrow{\mathrm{AC}}.$$

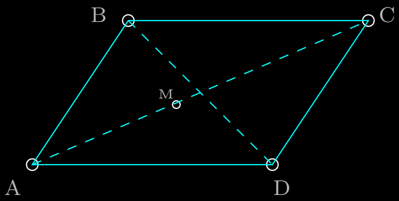


Problem

Show that the diagonals of any parallelogram bisect each other.

Proof.

Denote the parallelogram by its vertices, ABCD.



- Let M denote the midpoint of $\overrightarrow{\mathrm{AC}}$. Then $\overrightarrow{\mathrm{AM}} = \overrightarrow{\mathrm{MC}}$.
- It now suffices to show that $\overrightarrow{\mathrm{BM}} = \overrightarrow{\mathrm{MD}}$.

$$\overrightarrow{\mathrm{BM}} = \overrightarrow{\mathrm{BA}} + \overrightarrow{\mathrm{AM}} = \overrightarrow{\mathrm{CD}} + \overrightarrow{\mathrm{MC}} = \overrightarrow{\mathrm{MC}} + \overrightarrow{\mathrm{CD}} = \overrightarrow{\mathrm{MD}}.$$

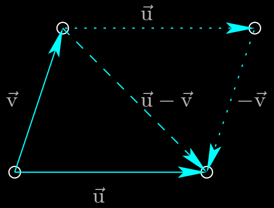
Since $\overrightarrow{\mathrm{BM}} = \overrightarrow{\mathrm{MD}}$, these vectors have the same magnitude and direction, implying that M is the midpoint of $\overrightarrow{\mathrm{BD}}$.

Therefore, the diagonals of ABCD bisect each other. ∎

Vector Subtraction

- If we have a coordinate system, then subtract the vectors as you would subtract matrices.

- For the intrinsic description:



$\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$ and is the diagonal from the tip of $\vec{v}$ to the tip of $\vec{u}$ in the parallelogram defined by $\vec{u}$ and $\vec{v}$.

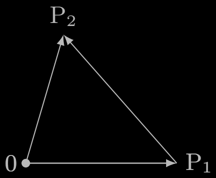
Theorem

Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two points. Then

1.
$$\overrightarrow{P_1P_2} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}.$$

2. The distance between P_1 and P_2 is
$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Proof.



$$\overrightarrow{0P_1} + \overrightarrow{P_1P_2} = \overrightarrow{0P_2}, \text{ so } \overrightarrow{P_1P_2} = \overrightarrow{0P_2} - \overrightarrow{0P_1}$$

and the distance between P_1 and P_2 is $||\overrightarrow{P_1P_2}||$.

∎

Example

For $\mathrm{P}(1,-1,3)$ and $\mathrm{Q}(3,1,0)$

$$\overrightarrow{\mathrm{PQ}} = \begin{bmatrix} 3-1 \\ 1-(-1) \\ 0-3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$$

and the distance between P and Q is $||\overrightarrow{\mathrm{PQ}}|| = \sqrt{2^2+2^2+(-3)^2} = \sqrt{17}$.

Definition

A unit vector is a vector of length one.

Example

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$, are examples of unit vectors.

Example

$\vec{v} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$ is not a unit vector, since $||\vec{v}|| = \sqrt{14}$. However,

$$\vec{u} = \frac{1}{\sqrt{14}}\vec{v} = \begin{bmatrix} \frac{-1}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \end{bmatrix}$$

is a unit vector in the same direction as $\vec{v}$, i.e.,

$$||\vec{u}|| = \frac{1}{\sqrt{14}}||\vec{v}|| = \frac{1}{\sqrt{14}}\sqrt{14} = 1.$$

Example

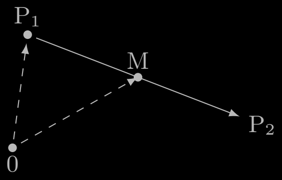
If $\vec{v} \neq \vec{0}$, then

$$\frac{1}{||\vec{v}||}\vec{v}$$

is a unit vector in the same direction as $\vec{v}$.

Problem

Find the point, M, that is midway between $P_1(-1,-4,3)$ and $P_2(5,0,-3)$.



Solution

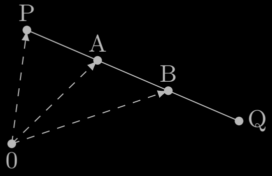
$$\overrightarrow{0M} = \overrightarrow{0P_1} + \overrightarrow{P_1M} = \overrightarrow{0P_1} + \frac{1}{2}\overrightarrow{P_1P_2} = \begin{bmatrix} -1 \\ -4 \\ 3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 6 \\ 4 \\ -6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -4 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}.$$

Therefore, $M = M(2,-2,0)$.

Problem

Find the two points trisecting the segment between $P(2, 3, 5)$ and $Q(8, -6, 2)$.



Solution

$\overrightarrow{0A} = \overrightarrow{0P} + \frac{1}{3}\overrightarrow{PQ}$ and $\overrightarrow{0B} = \overrightarrow{0P} + \frac{2}{3}\overrightarrow{PQ}$. Since $\overrightarrow{PQ} = \begin{bmatrix} 6 \\ -9 \\ -3 \end{bmatrix}$,

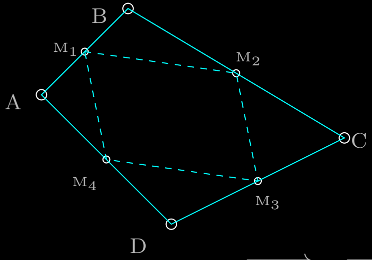
$\overrightarrow{0A} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}$, and $\overrightarrow{0B} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 4 \\ -6 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}$.

Therefore, the two points are $A(4, 0, 4)$ and $B(6, -3, 3)$.

Problem

Let ABCD be an arbitrary quadrilateral. Show that the midpoints of the four sides of ABCD are the vertices of a parallelogram.

Proof.



Let M_1 denote the midpoint of $\overrightarrow{AB}$,

M_2 the midpoint of $\overrightarrow{BC}$,

M_3 the midpoint of $\overrightarrow{CD}$, and

M_4 the midpoint of $\overrightarrow{DA}$.

We need to prove that $\overrightarrow{M_1M_2} = \overrightarrow{M_4M_3}$ and $\overrightarrow{M_1M_4} = \overrightarrow{M_2M_3}$.

Proof. (continued)

We will show $\overrightarrow{M_1M_4} = \overrightarrow{M_2M_3}$, the other relation can be shown in the same way. Notice that

$$\overrightarrow{0M_1} = \overrightarrow{0A} + \frac{1}{2}\overrightarrow{AB} \qquad \overrightarrow{0M_2} = \overrightarrow{0C} + \frac{1}{2}\overrightarrow{CB}$$
$$\overrightarrow{0M_4} = \overrightarrow{0A} + \frac{1}{2}\overrightarrow{AD} \qquad \overrightarrow{0M_3} = \overrightarrow{0C} + \frac{1}{2}\overrightarrow{CD}$$

Hence,

$$\overrightarrow{M_1M_4} = \overrightarrow{0M_4} - \overrightarrow{0M_1} = \frac{1}{2}\left(\overrightarrow{AD} - \overrightarrow{AB}\right) = \frac{1}{2}\overrightarrow{BD}$$

and

$$\overrightarrow{M_2M_3} = \overrightarrow{0M_3} - \overrightarrow{0M_2} = \frac{1}{2}\left(\overrightarrow{CD} - \overrightarrow{CB}\right) = \frac{1}{2}\overrightarrow{BD}$$

Therefore, $\overrightarrow{M_1M_4} = \overrightarrow{M_2M_3}$. ■

Definition

Two nonzero vectors are called parallel if and only if they have the same direction or opposite directions.

Theorem

Two nonzero vectors $\vec{v}$ and $\vec{w}$ are parallel if and only if one is a scalar multiple of the other.

In particular, if $\vec{v}$ and $\vec{w}$ are nonzero and have the same direction, then $\vec{v} = \frac{||\vec{v}||}{||\vec{w}||}\vec{w}$; if $\vec{v}$ and $\vec{w}$ have opposite directions, then $\vec{v} = -\frac{||\vec{v}||}{||\vec{w}||}\vec{w}$.

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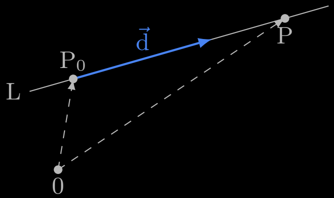
Lines in Space

Equations of lines

Let L be a line, $P_0(x_0, y_0, z_0)$ a fixed point on L, $P(x, y, z)$ an arbitrary point on L, and $\vec{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ a direction vector for L, i.e., a vector parallel to L.

Then $\overrightarrow{0P} = \overrightarrow{0P_0} + \overrightarrow{P_0P}$, and $\overrightarrow{P_0P}$ is parallel to $\vec{d}$, so $\overrightarrow{P_0P} = t\vec{d}$ for some $t \in \mathbb{R}$.



Definition

Vector Equation of a Line: $\boxed{\overrightarrow{0P} = \overrightarrow{0P_0} + t\,\vec{d}, \quad t \in \mathbb{R}.}$

Remark

Notation in the text: $\vec{p} = \overrightarrow{0P}$, $\vec{p}_0 = \overrightarrow{0P_0}$, so $\vec{p} = \vec{p}_0 + t\vec{d}$.

In component form, this is written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}, t \in \mathbb{R}.$$

Parametric Equations of a Line

$$\begin{array}{rcl} x & = & x_0 + ta \\ y & = & y_0 + tb \\ z & = & z_0 + tc \end{array}, \quad t \in \mathbb{R}.$$

Problem

Find an equation for the line through two points $P(2, -1, 7)$ and $Q(-3, 4, 5)$.

Solution

A direction vector for this line is

$$\vec{d} = \overrightarrow{PQ} = \begin{bmatrix} -5 \\ 5 \\ -2 \end{bmatrix}.$$

Therefore, a vector equation of this line is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix} + t \begin{bmatrix} -5 \\ 5 \\ -2 \end{bmatrix}.$$

Problem

Find an equation for the line through $Q(4, -7, 1)$ and parallel to the line

$$L : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}.$$

Solution

The line has equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}, t \in \mathbb{R}.$$

Problem

Given two lines L_1 and L_2, find the point of intersection, if it exists.

$$L_1: \begin{array}{rcl} x & = & 3 + t \\ y & = & 1 - 2t \\ z & = & 3 + 3t \end{array} \qquad\qquad L_2: \begin{array}{rcl} x & = & 4 + 2s \\ y & = & 6 + 3s \\ z & = & 1 + s \end{array}$$

Solution

Lines L_1 and L_2 intersect if and only if there are values $s, t \in \mathbb{R}$ such that

$$\begin{array}{rcl} 3 + t & = & 4 + 2s \\ 1 - 2t & = & 6 + 3s \\ 3 + 3t & = & 1 + s \end{array}$$

i.e., if and only if the system

$$\begin{array}{rcl} 2s - t & = & -1 \\ 3s + 2t & = & -5 \\ s - 3t & = & 2 \end{array}$$

is consistent.

Solution (continued)

$$\left[\begin{array}{rr|r} 2 & -1 & -1 \\ 3 & 2 & -5 \\ 1 & -3 & 2 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rr|r} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

L_1 and L_2 intersect when $s = -1$ and $t = -1$.

Using the equation for L_1 and setting $t = -1$, the point of intersection is

$$P(3 + (-1), 1 - 2(-1), 3 + 3(-1)) = P(2, 3, 0).$$

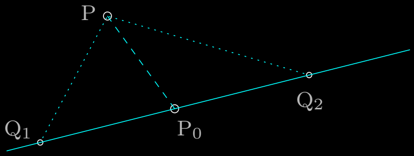
Note. You can check your work by setting $s = -1$ in the equation for L_2.

Problem

Find equations for the lines through $P(1, 0, 1)$ that meet the line

$$L : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

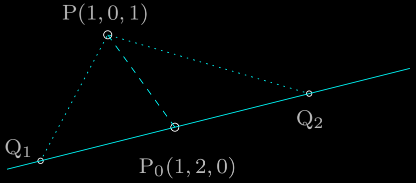
at points distance three from $P_0(1, 2, 0)$.



Solution

Find points Q_1 and Q_2 on L that are distance three from P_0, and then find equations for the lines through P and Q_1, and through P and Q_2.

Solution (continued)



$$\vec{d} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

First, $||\vec{d}|| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{9} = 3$, so

$$\overrightarrow{0Q_1} = \overrightarrow{0P_0} + 1\vec{d}, \quad \text{and} \quad \overrightarrow{0Q_2} = \overrightarrow{0P_0} - 1\vec{d}.$$

$$\overrightarrow{0Q_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \overrightarrow{0Q_2} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix},$$

so $Q_1 = Q_1(3,1,2)$ and $Q_2 = Q_2(-1,3,-2)$.

Solution (continued)

Equations for the lines:

- the line through $P(1,0,1)$ and $Q_1(3,1,2)$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \overrightarrow{0P} + \overrightarrow{PQ_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

- the line through $P(1,0,1)$ and $Q_2(-1,3,-2)$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \overrightarrow{0P} + \overrightarrow{PQ_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix}.$$

$\blacksquare$