

Math 221: LINEAR ALGEBRA

Chapter 4. Vector Geometry

§4-1. Vectors and Lines

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Emory University, 2021 Spring

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¹Slides are adapted from those by Karen Seyffarth from University of Calgary and those by James Nagy from Emory University.

Vector Norms (i.e., lengths)

Parallel Vectors

Length and Direction

Geometric Vectors

The Parallelogram Law

Lines in Space

NOTE: Much of this chapter is what you would learn in Multivariable Calculus.
You might find it interesting/useful to read.
But I will only cover the material important to this course.

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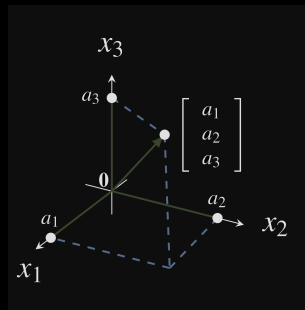
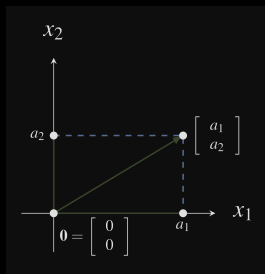
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Vector Norms

- ▶ The word “norm” in linear algebra is used to mean “length”.
- ▶ There are actually many ways to define length, the most usual Euclidean:
 - In 2D and 3D:



- In general, if $\vec{v} \in \mathbb{R}^n$, the Euclidean **norm** of \vec{v} is:

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{\vec{v}^T \vec{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Example:

$$\text{If } \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \text{ find } \|\vec{v}\|.$$

Example: Show that $\|c\vec{v}\| = |c|\|\vec{v}\|$ for any scalar c and any vector $\vec{v} \in \mathbb{R}^n$.

Definition

$\| \cdot \| : \mathbb{R}^n \rightarrow \mathbb{R}$ is a **vector norm** if it satisfies the following properties:

1. $\|v\| \geq 0$ for all $v \in \mathbb{R}^n$, and $\|v\| = 0$ if and only if $v = 0$,
2. $\|v + w\| \leq \|v\| + \|w\|$ for all $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$,
3. $\|cv\| = |c|\|v\|$ for all vectors $v \in \mathbb{R}^n$ and all scalars c .

Remark

There are many vector norms, so sometimes we include a subscript, such as $\|\cdot\|_p$, to indicate precisely which norm we are using. Here are some examples:

- ▶ The **2-norm** is the standard Euclidean length:

$$\|\vec{v}\|_2 = \sqrt{\vec{v}^T \vec{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

- ▶ The **1-norm** is defined as $\|\vec{v}\|_1 = |v_1| + |v_2| + \cdots + |v_n|$.
- ▶ The **∞ -norm** is defined as $\|\vec{v}\|_\infty = \max_{1 \leq i \leq n} \{|v_i|\}$.
- ▶ In general, if $1 \leq p < \infty$, then the **p-norm** is defined as

$$\|\vec{v}\|_p = \left(\sum_{i=1}^n |v_i|^p \right)^{1/p}.$$

Although other norms are used in certain applications, we usually use the 2-norm, and omit the subscript:

$$\|\vec{v}\| \equiv \|\vec{v}\|_2$$

Definition

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Example

Check if $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ are unit vectors.

Remark

We can scale any nonzero vector to have norm equal to 1.

If $\vec{v} \in \mathbb{R}^n$, $\vec{v} \neq \vec{0}$, then

$$\vec{u} = \frac{1}{\|\vec{v}\|} \vec{v} \quad \text{is a unit vector}$$

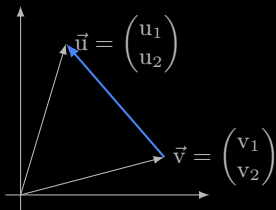
Problem

Scale $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ to a unit vector.

Definition

The **distance between two vectors** is defined as:

$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$$



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Parallel Vectors

Definition

Two vectors are called **parallel** if they lie on the same line. Equivalently, two vectors are parallel if they are scalar multiples of each other.

Example

Determine if \vec{v} , \vec{w} , \vec{z} are parallel to $\vec{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

$$\vec{v} = \begin{bmatrix} 6 \\ -4 \\ 2 \end{bmatrix}$$

$$\vec{w} = \begin{bmatrix} -6 \\ 4 \\ -2 \end{bmatrix}$$

$$\vec{z} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The following slides are for you to study by yourself
as reviewing material...

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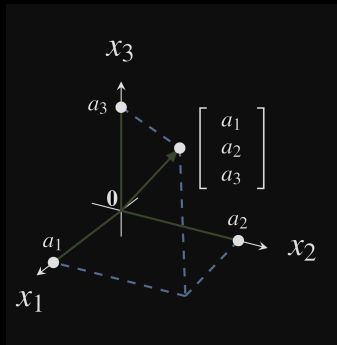
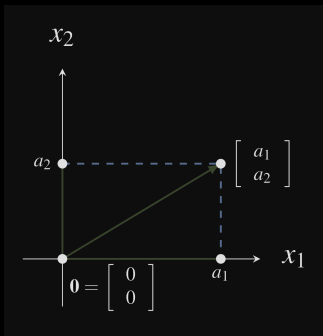
Scalar quantities versus vector quantities

- ▶ A **scalar** quantity has only magnitude; e.g. time, temperature.
- ▶ A **vector** quantity has both magnitude and direction; e.g. displacement, force, wind velocity.

Whereas two scalar quantities are equal if they are represented by the same value, two vector quantities are equal if and only if they have the same **magnitude** and **direction**.

\mathbb{R}^2 and \mathbb{R}^3

Vectors in \mathbb{R}^2 and \mathbb{R}^3 have convenient geometric representations as position vectors of points in the 2-dimensional (Cartesian) plane and in 3-dimensional space, respectively.



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- ▶ Instead of using a capital letter to denote the vector (as we generally do with matrices), we emphasize the importance of the geometry and the direction with an arrow over the name of the vector.

Notation and Terminology

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- ▶ Any vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ in } \mathbb{R}^3$$

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- ▶ Often, **there is no distinction made between the vector \vec{x} and the point (x_1, x_2, x_3)** , and we say that both $(x_1, x_2, x_3) \in \mathbb{R}^3$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$.

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Theorem

Let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ be vectors in \mathbb{R}^3 . Then

1. $\vec{v} = \vec{w}$ if and only if $x = x_1$, $y = y_1$, and $z = z_1$.

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2. $\|\vec{v}\| = \sqrt{x^2 + y^2 + z^2}$.

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3. $\vec{v} = \vec{0}$ if and only if $\|\vec{v}\| = 0$.
4. For any scalar a , $\|a\vec{v}\| = |a| \cdot \|\vec{v}\|$.

Remark

Analogous results hold for $\vec{v}, \vec{w} \in \mathbb{R}^2$, i.e.,

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}, \vec{w} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

In this case, $\|\vec{v}\| = \sqrt{x^2 + y^2}$.

Example

$$\text{Let } \vec{p} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}, \vec{q} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}, \text{ and } -2\vec{q} = \begin{bmatrix} -6 \\ 2 \\ 4 \end{bmatrix},$$

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and

$$\begin{aligned} \|-2\vec{q}\| &= \sqrt{(-6)^2 + 2^2 + 4^2} \\ &= \sqrt{36 + 4 + 16} \\ &= \sqrt{56} = \sqrt{4 \times 14} \\ &= 2\sqrt{14} = 2\|\vec{q}\|. \end{aligned}$$

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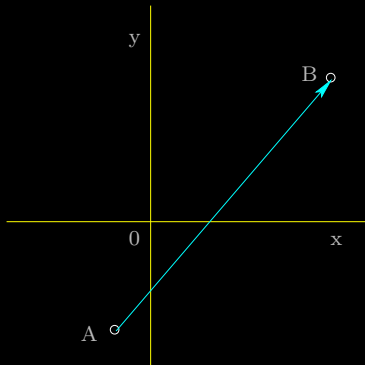
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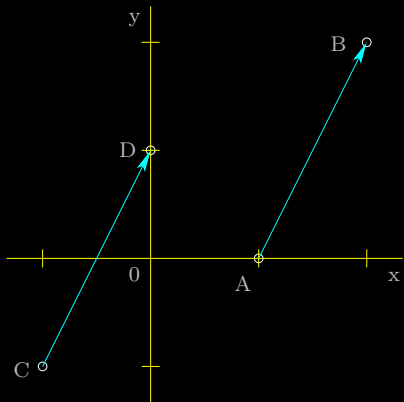
Lines in Space

Geometric Vectors

Let A and B be two points in \mathbb{R}^2 or \mathbb{R}^3 .



- \vec{AB} is the **geometric vector** from A to B .
- A is the **tail** of \vec{AB} .
- B is the **tip** of \vec{AB} .
- the **magnitude** of \vec{AB} is its length, and is denoted $\|\vec{AB}\|$.



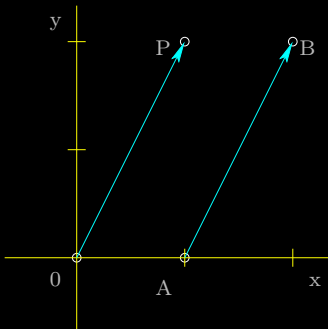
- \overrightarrow{AB} is the vector from $A(1, 0)$ to $B(2, 2)$.
- \overrightarrow{CD} is the vector from $C(-1, -1)$ to $D(0, 1)$.

- $\overrightarrow{AB} = \overrightarrow{CD}$ because the vectors have the same **length** and **direction**.

Definition

A vector is in **standard position** if its tail is at the origin.

We co-ordinatize vectors by putting them in standard position, and then identifying them with their tips.



Thus $\overrightarrow{AB} = \overrightarrow{OP}$ where $P = P(1, 2)$, and we write $\overrightarrow{OP} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \overrightarrow{AB}$.

\overrightarrow{OP} is the **position vector** for $P(1, 2)$.

More generally, if $P(x, y, z)$ is a point in \mathbb{R}^3 , then $\vec{0P} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is the position vector for P .

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If we aren't concerned with the locations of the tail and tip, we simply write $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

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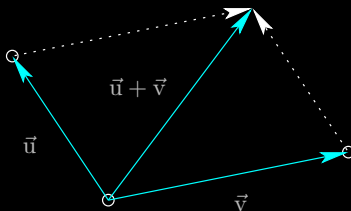
- ▶ **vector equality**: same length and direction.
- ▶ $\vec{0}$: the vector with length zero and **no direction**.
- ▶ **scalar multiplication**: if $\vec{v} \neq \vec{0}$ and $a \in \mathbb{R}$, $a \neq 0$, then $a\vec{v}$ has length $|a| \cdot \|\vec{v}\|$ and
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parallelogram law

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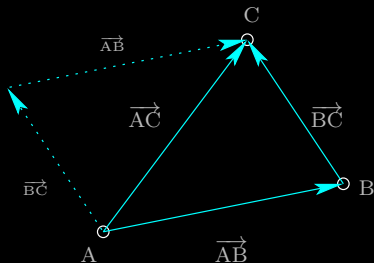
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- ▶ **addition:** $\vec{u} + \vec{v}$ is represented by the matrix sum of the columns \vec{u} and \vec{v} .

Tip-to-Tail Method for Vector Addition

For points A, B and C,

$$\vec{AB} + \vec{BC} = \vec{AC}.$$



Problem

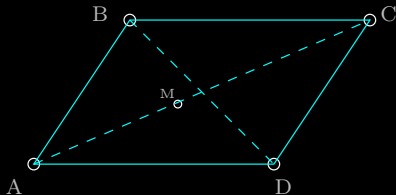
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Proof.

Denote the parallelogram by its vertices, ABCD.



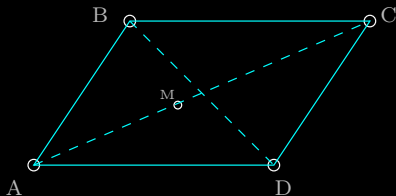
- Let M denote the midpoint of \overrightarrow{AC} .
Then $\overrightarrow{AM} = \overrightarrow{MC}$.
- It now suffices to show that $\overrightarrow{BM} = \overrightarrow{MD}$.

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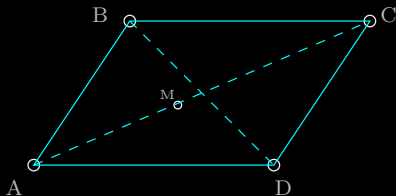
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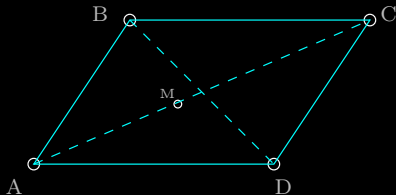
Since $\overrightarrow{BM} = \overrightarrow{MD}$, these vectors have the same **magnitude** and **direction**, implying that M is the midpoint of \overrightarrow{BD} .

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Show that the diagonals of any parallelogram bisect each other.

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$$\overrightarrow{BM} = \overrightarrow{BA} + \overrightarrow{AM} = \overrightarrow{CD} + \overrightarrow{MC} = \overrightarrow{MC} + \overrightarrow{CD} = \overrightarrow{MD}.$$

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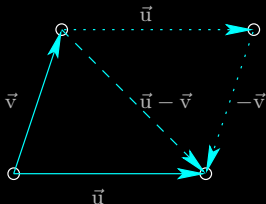
Therefore, the diagonals of ABCD bisect each other. ■

Vector Subtraction

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- ▶ For the intrinsic description:



$\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$ and is the diagonal from the tip of \vec{v} to the tip of \vec{u} in the parallelogram defined by \vec{u} and \vec{v} .

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Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two points. Then

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2. The distance between P_1 and P_2 is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Theorem

Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two points. Then

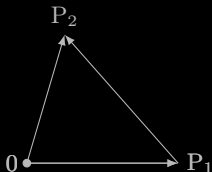
1.

$$\overrightarrow{P_1P_2} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}.$$

2. The distance between P_1 and P_2 is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Proof.



$$\overrightarrow{0P_1} + \overrightarrow{P_1P_2} = \overrightarrow{0P_2}, \text{ so } \overrightarrow{P_1P_2} = \overrightarrow{0P_2} - \overrightarrow{0P_1}$$

and the distance between P_1 and P_2 is $\|\overrightarrow{P_1P_2}\|$.



Example

For $P(1, -1, 3)$ and $Q(3, 1, 0)$

$$\overrightarrow{PQ} = \begin{bmatrix} 3 - 1 \\ 1 - (-1) \\ 0 - 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$$

and the distance between P and Q is $\|\overrightarrow{PQ}\| = \sqrt{2^2 + 2^2 + (-3)^2} = \sqrt{17}$.

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Definition

A **unit vector** is a vector of length one.

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A **unit vector** is a vector of length one.

Example

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$, are examples of unit vectors.

Example

$\vec{v} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$ is not a unit vector, since $\|\vec{v}\| = \sqrt{14}$.

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Example

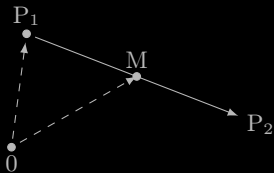
If $\vec{v} \neq \vec{0}$, then

$$\frac{1}{\|\vec{v}\|} \vec{v}$$

is a unit vector in the same direction as \vec{v} .

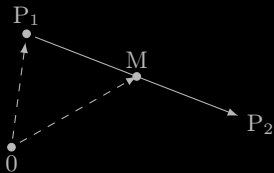
Problem

Find the point, M , that is midway between $P_1(-1, -4, 3)$ and $P_2(5, 0, -3)$.



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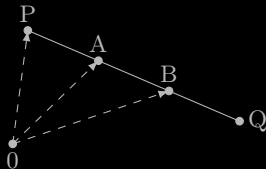
Solution

$$\begin{aligned}\overrightarrow{0M} &= \overrightarrow{0P_1} + \overrightarrow{P_1M} = \overrightarrow{0P_1} + \frac{1}{2}\overrightarrow{P_1P_2} = \begin{bmatrix} -1 \\ -4 \\ 3 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 6 \\ 4 \\ -6 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ -4 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}.\end{aligned}$$

Therefore, $M = M(2, -2, 0)$.

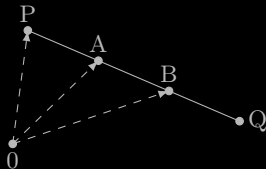
Problem

Find the two points trisecting the segment between $P(2, 3, 5)$ and $Q(8, -6, 2)$.



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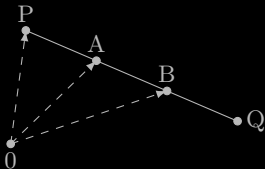


Solution

$$\vec{OA} = \vec{OP} + \frac{1}{3}\vec{PQ} \text{ and } \vec{OB} = \vec{OP} + \frac{2}{3}\vec{PQ}.$$

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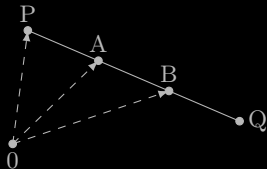
Solution

$$\vec{OA} = \vec{OP} + \frac{1}{3}\vec{PQ} \text{ and } \vec{OB} = \vec{OP} + \frac{2}{3}\vec{PQ}. \text{ Since } \vec{PQ} = \begin{bmatrix} 6 \\ -9 \\ -3 \end{bmatrix},$$

$$\vec{OA} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}, \text{ and } \vec{OB} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 4 \\ -6 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}.$$

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Find the two points trisecting the segment between $P(2, 3, 5)$ and $Q(8, -6, 2)$.



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Therefore, the two points are $A(4, 0, 4)$ and $B(6, -3, 3)$.

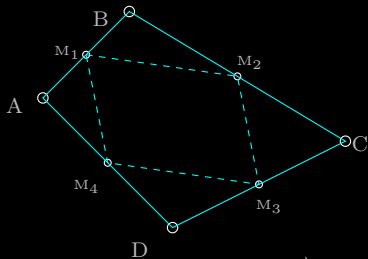
Problem

Let $ABCD$ be an arbitrary quadrilateral. Show that the midpoints of the four sides of $ABCD$ are the vertices of a parallelogram.

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Proof.



Let M_1 denote the midpoint of \overrightarrow{AB} ,
 M_2 the midpoint of \overrightarrow{BC} ,
 M_3 the midpoint of \overrightarrow{CD} , and
 M_4 the midpoint of \overrightarrow{DA} .

We need to prove that $\overrightarrow{M_1M_2} = \overrightarrow{M_4M_3}$ and $\overrightarrow{M_1M_4} = \overrightarrow{M_2M_3}$.

Proof. (continued)

We will show $\overrightarrow{M_1M_4} = \overrightarrow{M_2M_3}$, the other relation can be shown in the same way.

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$$\overrightarrow{OM_1} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$$

$$\overrightarrow{OM_4} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AD}$$

$$\overrightarrow{OM_2} = \overrightarrow{OC} + \frac{1}{2}\overrightarrow{CB}$$

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Hence,

$$\overrightarrow{M_1M_4} = \overrightarrow{OM_4} - \overrightarrow{OM_1} = \frac{1}{2}(\overrightarrow{AD} - \overrightarrow{AB}) = \frac{1}{2}\overrightarrow{BD}$$

and

$$\overrightarrow{M_2M_3} = \overrightarrow{OM_3} - \overrightarrow{OM_2} = \frac{1}{2}(\overrightarrow{CD} - \overrightarrow{CB}) = \frac{1}{2}\overrightarrow{BD}$$

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We will show $\overrightarrow{M_1M_4} = \overrightarrow{M_2M_3}$, the other relation can be shown in the same way. Notice that

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Therefore, $\overrightarrow{M_1M_4} = \overrightarrow{M_2M_3}$. ■

Definition

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In particular, if \vec{v} and \vec{w} are nonzero and have the same direction, then $\vec{v} = \frac{\|\vec{v}\|}{\|\vec{w}\|} \vec{w}$; if \vec{v} and \vec{w} have opposite directions, then $\vec{v} = -\frac{\|\vec{v}\|}{\|\vec{w}\|} \vec{w}$.

Vector Norms (i.e., lengths)

Parallel Vectors

Length and Direction

Geometric Vectors

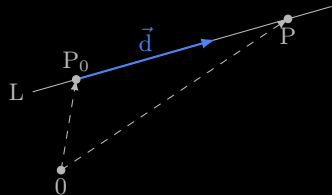
The Parallelogram Law

Lines in Space

Equations of lines

Let L be a line, $P_0(x_0, y_0, z_0)$ a fixed point on L , $P(x, y, z)$ an arbitrary point on L , and $\vec{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ a **direction vector** for L , i.e., a vector parallel to L .

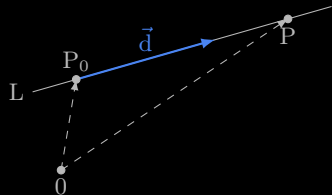
Then $\vec{OP} = \vec{OP}_0 + \vec{P_0P}$, and $\vec{P_0P}$ is parallel to \vec{d} , so $\vec{P_0P} = t\vec{d}$ for some $t \in \mathbb{R}$.



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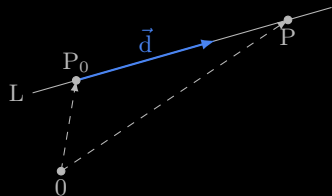
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Vector Equation of a Line: $\vec{OP} = \vec{OP}_0 + t\vec{d}, \quad t \in \mathbb{R}.$

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Remark

Notation in the text: $\vec{p} = \vec{OP}$, $\vec{p}_0 = \vec{OP}_0$, so $\vec{p} = \vec{p}_0 + t\vec{d}$.

In component form, this is written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}, t \in \mathbb{R}.$$

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Parametric Equations of a Line

$$\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb, \quad t \in \mathbb{R}. \\ z &= z_0 + tc \end{aligned}$$

Problem

Find an equation for the line through two points $P(2, -1, 7)$ and $Q(-3, 4, 5)$.

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A direction vector for this line is

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Therefore, a vector equation of this line is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix} + t \begin{bmatrix} -5 \\ 5 \\ -2 \end{bmatrix}.$$

Problem

Find an equation for the line through $Q(4, -7, 1)$ and parallel to the line

$$L : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}.$$

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Find an equation for the line through $Q(4, -7, 1)$ and parallel to the line

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Solution

The line has equation

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Problem

Given two lines L_1 and L_2 , find the point of intersection, if it exists.

$$\begin{aligned}L_1 : \quad x &= 3 + t \\ y &= 1 - 2t \\ z &= 3 + 3t\end{aligned}$$

$$\begin{aligned}L_2 : \quad x &= 4 + 2s \\ y &= 6 + 3s \\ z &= 1 + s\end{aligned}$$

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Solution

Lines L_1 and L_2 intersect if and only if there are values $s, t \in \mathbb{R}$ such that

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i.e., if and only if the system

$$\begin{aligned} 2s - t &= -1 \\ 3s + 2t &= -5 \\ s - 3t &= 2 \end{aligned}$$

is consistent.

Solution (continued)

$$\left[\begin{array}{cc|c} 2 & -1 & -1 \\ 3 & 2 & -5 \\ 1 & -3 & 2 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array} \right]$$

Solution (continued)

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L_1 and L_2 intersect when $s = -1$ and $t = -1$.

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L_1 and L_2 intersect when $s = -1$ and $t = -1$.

Using the equation for L_1 and setting $t = -1$, the point of intersection is

$$P(3 + (-1), 1 - 2(-1), 3 + 3(-1)) = P(2, 3, 0).$$

Solution (continued)

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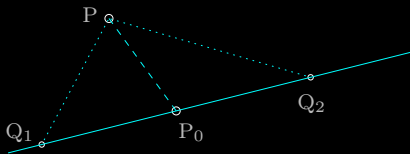
Note. You can check your work by setting $s = -1$ in the equation for L_2 .

Problem

Find equations for the lines through $P(1, 0, 1)$ that meet the line

$$L : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

at points distance three from $P_0(1, 2, 0)$.

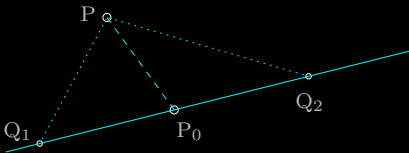


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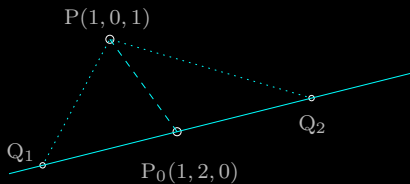
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Solution

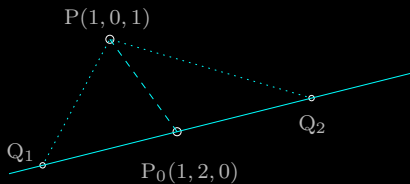
Find points Q_1 and Q_2 on L that are distance three from P_0 , and then find equations for the lines through P and Q_1 , and through P and Q_2 .

Solution (continued)



$$\vec{d} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

Solution (continued)



$$\vec{d} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

First, $\|\vec{d}\| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{9} = 3$, so

$$\overrightarrow{OQ_1} = \overrightarrow{OP_0} + 1\vec{d}, \quad \text{and} \quad \overrightarrow{OQ_2} = \overrightarrow{OP_0} - 1\vec{d}.$$

Solution (continued)

Equations for the lines:

- ▶ the line through $P(1, 0, 1)$ and $Q_1(3, 1, 2)$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0P} + \vec{PQ}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

- ▶ the line through $P(1, 0, 1)$ and $Q_2(-1, 3, -2)$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0P} + \vec{PQ}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix}.$$

