Math 221: LINEAR ALGEBRA

Chapter 4. Vector Geometry

§4-3. More on the Cross Product

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

More on the Cross Product

NOTE: Much of this chapter is what you would learn in Multivariable Calculus.

You might find it interesting/useful to read.

But I will only cover the material important to this course.

Theorem

Given three vectors $\vec{u} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, it holds that

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \det \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} = \det \begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{bmatrix}.$$

Proof.

Let $\vec{u} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$. Then

$$\begin{aligned}
\vec{u} \cdot (\vec{v} \times \vec{w}) &= \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \cdot \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{bmatrix} \\
&= x_0(y_1 z_2 - z_1 y_2) - y_0(x_1 z_2 - z_1 x_2) + z_0(x_1 y_2 - y_1 x_2) \\
&= x_0 \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} - y_0 \begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix} + z_0 \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} \\
&= \begin{vmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{vmatrix}.
\end{aligned}$$

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Theorem (Properties of the Cross Product)

Let $\vec{u}, \vec{v}$ and $\vec{w}$ be in $\mathbb{R}^3$.

1. $\vec{u} \times \vec{v}$ is a vector.
2. $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.
3. $\vec{u} \times \vec{0} = \vec{0}$ and $\vec{0} \times \vec{u} = \vec{0}$.
4. $\vec{u} \times \vec{u} = \vec{0}$.
5. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$.
6. $(k\vec{u}) \times \vec{v} = k(\vec{u} \times \vec{v}) = \vec{u} \times (k\vec{v})$ for any scalar k.
7. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.
8. $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$.

Theorem (The Lagrange Identity)

If $\vec{u}, \vec{v} \in \mathbb{R}^3$, then

$$||\vec{u} \times \vec{v}||^2 = ||\vec{u}||^2 ||\vec{v}||^2 - (\vec{u} \cdot \vec{v})^2.$$

Proof.

Write $\vec{u} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, then both sides are equal to

$$(a_1 b_2 - a_2 b_1)^2 + (a_1 b_3 - a_3 b_1)^2 + (a_2 b_3 - a_3 b_2)^2 .$$

Work out these by yourself! ∎

As a consequence of the Lagrange Identity and the fact that

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| \, ||\vec{v}|| \cos \theta,$$

we have

$$\begin{aligned} ||\vec{u} \times \vec{v}||^2 &= ||\vec{u}||^2 ||\vec{v}||^2 - (\vec{u} \cdot \vec{v})^2 \\ &= ||\vec{u}||^2 ||\vec{v}||^2 - ||\vec{u}||^2 ||\vec{v}||^2 \cos^2 \theta \\ &= ||\vec{u}||^2 ||\vec{v}||^2 (1 - \cos^2 \theta) \\ &= ||\vec{u}||^2 ||\vec{v}||^2 \sin^2 \theta. \end{aligned}$$

Taking square roots on both sides yields,

$$||\vec{u} \times \vec{v}|| = ||\vec{u}|| \, ||\vec{v}|| \sin \theta.$$

Note that since $0 \leq \theta \leq \pi$, $\sin \theta \geq 0$.

If $\theta = 0$ or $\theta = \pi$, then $\sin \theta = 0$, and $||\vec{u} \times \vec{v}|| = 0$. This is consistent with our earlier observation that if $\vec{u}$ and $\vec{v}$ are parallel, then $\vec{u} \times \vec{v} = \vec{0}$.

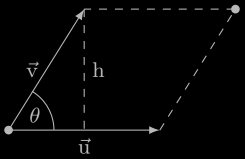
Theorem

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in $\mathbb{R}^3$, and let θ denote the angle between $\vec{u}$ and $\vec{v}$.

1. $||\vec{u} \times \vec{v}|| = ||\vec{u}|| \, ||\vec{v}|| \sin\theta$, and is the area of the parallelogram defined by $\vec{u}$ and $\vec{v}$.
2. $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} \times \vec{v} = \vec{0}$.

Proof. (area of parallelogram)

The area of the parallelogram defined by $\vec{u}$ and $\vec{v}$ is $||\vec{u}||h$, where h is the height of the parallelogram.



Since $\sin\theta = \frac{h}{||\vec{v}||}$, we see that $h = ||\vec{v}|| \sin\theta$. Therefore, the area is

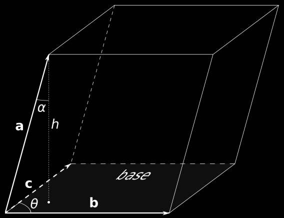
$$||\vec{u}|| \, ||\vec{v}|| \sin\theta.$$

■

Theorem

The volume of the parallelepiped determined by the three vectors $\vec{b}$, $\vec{c}$, and $\vec{a}$ in $\mathbb{R}^3$ is

$$|\vec{a} \cdot (\vec{b} \times \vec{c})|.$$



Proof.

Volume $=$ base area $\times \mathrm{h}$, where base area $= |\vec{b} \times \vec{c}|$ and the height $\mathrm{h} = |\vec{a}||\cos(\alpha)|$. Hence,

$$\text{Vol} = |\vec{b} \times \vec{c}|\, |\vec{a}||\cos(\alpha)| = |(\vec{b} \times \vec{c}) \cdot \vec{a}|.$$

∎

Problem

Find the area of the triangle having vertices $A(3, -1, 2)$, $B(1, 1, 0)$ and $C(1, 2, -1)$.

Solution

The area of the triangle is half the area of the parallelogram defined by $\overrightarrow{AB}$ and $\overrightarrow{AC}$. $\overrightarrow{AB} = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix}$. Therefore

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix},$$

so the area of the triangle is $\frac{1}{2}||\overrightarrow{AB} \times \overrightarrow{AC}|| = \sqrt{2}$. ■

Problem

Find the volume of the parallelepiped determined by the vectors

$$\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \text{ and } \vec{w} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

Solution

The volume of the parallelepiped is

$$|\vec{w} \cdot (\vec{u} \times \vec{v})| = \left| \det \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix} \right| = 2.$$