Math 221: LINEAR ALGEBRA

Chapter 5. Vector Space $\mathbb{R}^n$

§5-2. Independence and Dimension

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

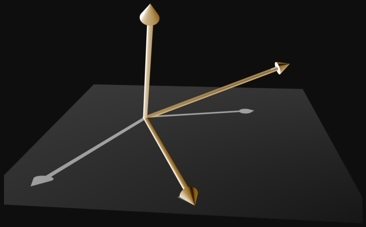
Linear Independence

Definition

Let $S = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ be a subset of $\mathbb{R}^n$. The set S is linearly independent (or simply independent) if the following condition is satisfied:

$$t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k = \vec{0}_n \quad \Rightarrow \quad t_1 = t_2 = \cdots = t_k = 0$$

i.e., the only linear combination of vectors of S that vanishes (is equal to the zero vector) is the trivial one (all coefficients equal to zero).
A set that is not linearly independent is called dependent.



$\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_k\}$		$t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k = \vec{0}_n$
Linearly Independent	$\Longleftrightarrow$	Trivial Solution
Linearly Dependent	$\Longleftrightarrow$	Nontrivial Solution

Example

Is $S = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \right\}$ linearly independent?

Suppose that a linear combination of these vectors vanishes, i.e., there exist $a, b, c \in \mathbb{R}$ so that

$$a \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Example (continued)

Solve the homogeneous system of three equation in three variables:

$$\left[\begin{array}{rrr|r} -1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 1 & 5 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The system has solutions $a = -2r, b = -3r, c = r$ for $r \in \mathbb{R}$, so it has nontrivial solutions. Therefore S is dependent. In particular, when $r = 1$ we see that

$$-2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

i.e., this is a nontrivial linear combination that vanishes.

Example

Consider the set $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\} \subseteq \mathbb{R}^n$, and suppose $t_1, t_2, \ldots, t_n \in \mathbb{R}$ are such that

$$t_1\vec{e}_1 + t_2\vec{e}_2 + \cdots t_n\vec{e}_n = \vec{0}_n.$$

Since

$$t_1\vec{e}_1 + t_2\vec{e}_2 + \cdots t_n\vec{e}_n = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix},$$

the only linear combination that vanishes is the trivial one, i.e., the one with $t_1 = t_2 = \cdots = t_n = 0$. Therefore, $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is linearly independent.

Problem

Let $\{\vec{u}, \vec{v}, \vec{w}\}$ be an independent subset of $\mathbb{R}^n$. Is $\{\vec{u+v}, 2\vec{u+w}, \vec{v} - 5\vec{w}\}$ linearly independent?

Solution

In order to show the $\{\vec{u+v}, 2\vec{u+w}, \vec{v} - 5\vec{w}\}$ is linearly independent, we need to show that

$$a(\vec{u} + \vec{v}) + b(2\vec{u} + \vec{w}) + c(\vec{v} - 5\vec{w}) = \vec{0}_n \quad \Rightarrow \quad a = b = c = 0.$$

$$\Updownarrow$$

$$(a + 2b)\vec{u} + (a + c)\vec{v} + (b - 5c)\vec{w} = \vec{0}_n.$$

because $\{\vec{u}, \vec{v}, \vec{w}\}$ is independent $\Downarrow$

$$\begin{aligned} a + 2b &= 0 \\ a + c &= 0 \\ b - 5c &= 0. \end{aligned}$$

$$\Downarrow$$

$$a = b = c = 0$$

$\blacksquare$

Problem

Let $X \subseteq \mathbb{R}^n$ and suppose that $\vec{0}_n \in X$. Show that X linearly dependent.

Solution

Let $X = \{\vec{x_1}, \vec{x_2}, \ldots, \vec{x_k}\}$ for some $k \geq 1$, and suppose $\vec{x}_1 = \vec{0_n}$. Then

$$1\vec{x}_1 + 0\vec{x}_2 + \cdots + 0\vec{x}_k = 1\vec{0} + 0\vec{x}_2 + \cdots + 0\vec{x}_k = \vec{0},$$

i.e., we have found a nontrivial linear combination of the vectors of X that vanishes. Therefore, X is dependent. ■

Example

Let $\vec{u} \in \mathbb{R}^n$ and let $S = \{\vec{u}\}$.

1. If $\vec{u} = \vec{0}_n$, then S is dependent (see the previous Problem).
2. If $\vec{u} \neq \vec{0}_n$, then S is independent: if $t\vec{u} = \vec{0}_n$ for some $t \in \mathbb{R}$, then $t = 0$.

As a consequence,

$$S = \{\vec{u}\} \text{ is independent} \qquad \Longleftrightarrow \qquad \vec{u} \neq \vec{0}_n$$

Example

$A = \begin{bmatrix} 0 & 1 & -1 & 2 & 5 & 1 \\ 0 & 0 & 1 & -3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is a row-echelon matrix. Treat the nonzero rows of A as transposes of vectors in $\mathbb{R}^6$:

$$\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \\ 5 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -2 \end{bmatrix},$$

and suppose that $a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3 = \vec{0}_6$ for some $a, b, c \in \mathbb{R}$.

Example (continued)

This results in a system of six equations in three variables, whose augmented matrix is

$$\left[\begin{array}{rrr|r} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ 5 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

The solution to the system is easily determined to be $a = b = c = 0$, so the set $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is independent. Hence, nonzero rows of A are independent.

Remark

In general, the nonzero rows of any row-echelon matrix form an independent set of (row) vectors.

Theorem

Let $U = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \subseteq \mathbb{R}^n$ be an independent set. Then any vector $\vec{x} \in \text{span}(U)$ has a unique representation as a linear combination of vectors of U.

Proof.

Suppose that there is a vector $\vec{x} \in \text{span}(U)$ such that

$$\begin{aligned} \vec{x} &= s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_k\vec{v}_k, \text{ for some } s_1, s_2, \ldots, s_k \in \mathbb{R}, \text{ and} \\ \vec{x} &= t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k, \text{ for some } t_1, t_2, \ldots, t_k \in \mathbb{R}. \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} \vec{0}_n = \vec{x} - \vec{x} &= (s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_k\vec{v}_k) - (t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k) \\ &= (s_1 - t_1)\vec{v}_1 + (s_2 - t_2)\vec{v}_2 + \cdots + (s_k - t_k)\vec{v}_k. \end{aligned}$$

U is independent $\Downarrow$

$$s_1 - t_1 = 0, \quad s_2 - t_2 = 0, \quad \cdots, s_k - t_k = 0$$

$$\Updownarrow$$

$$s_1 = t_1, \quad s_2 = t_2, \quad \cdots, s_k = t_k.$$

■

Two Geometric Examples

Problem

Suppose that $\vec{u}$ and $\vec{v}$ are nonzero vectors in $\mathbb{R}^3$. Prove that $\{\vec{u}, \vec{v}\}$ is dependent if and only if $\vec{u}$ and $\vec{v}$ are parallel.

Solution

($\Rightarrow$) If $\{\vec{u}, \vec{v}\}$ is dependent, then there exist $a, b \in \mathbb{R}$ so that $a\vec{u} + b\vec{v} = \vec{0}_3$ with a and b not both zero. By symmetry, we may assume that $a \neq 0$. Then $\vec{u} = -\frac{b}{a}\vec{v}$, so $\vec{u}$ and $\vec{v}$ are scalar multiples of each other, i.e., $\vec{u}$ and $\vec{v}$ are parallel.

($\Leftarrow$) Conversely, if $\vec{u}$ and $\vec{v}$ are parallel, then there exists a $t \in \mathbb{R}$, $t \neq 0$, so that $\vec{u} = t\vec{v}$. Thus $\vec{u} - t\vec{v} = \vec{0}_3$, so we have a nontrivial linear combination of $\vec{u}$ and $\vec{v}$ that vanishes. Therefore, $\{\vec{u}, \vec{v}\}$ is dependent. ∎

Problem

Suppose that $\vec{u}, \vec{v}$ and $\vec{w}$ are nonzero vectors in $\mathbb{R}^3$, and that $\{\vec{v}, \vec{w}\}$ is independent. Prove that $\{\vec{u}, \vec{v}, \vec{w}\}$ is independent if and only if $\vec{u} \notin \text{span}\{\vec{v}, \vec{w}\}$.

Solution

($\Rightarrow$) If $\vec{u} \in \text{span}\{\vec{v}, \vec{w}\}$, then there exist $a, b \in \mathbb{R}$ so that $\vec{u} = a\vec{v} + b\vec{w}$. This implies that $\vec{u} - a\vec{v} - b\vec{w} = \vec{0}_3$, so $\vec{u} - a\vec{v} - b\vec{w}$ is a nontrivial linear combination of $\{\vec{u}, \vec{v}, \vec{w}\}$ that vanishes, and thus $\{\vec{u}, \vec{v}, \vec{w}\}$ is dependent.

($\Leftarrow$) Now suppose that $\vec{u} \notin \text{span}\{\vec{v}, \vec{w}\}$, and suppose that there exist $a, b, c \in \mathbb{R}$ such that $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}_3$. If $a \neq 0$, then $\vec{u} = -\frac{b}{a}\vec{v} - \frac{c}{a}\vec{w}$, and $\vec{u} \in \text{span}\{\vec{v}, \vec{w}\}$, a contradiction. Therefore, $a = 0$, implying that $b\vec{v} + c\vec{w} = \vec{0}_3$. Since $\{\vec{v}, \vec{w}\}$ is independent, $b = c = 0$, and thus $a = b = c = 0$, i.e., the only linear combination of $\vec{u}, \vec{v}$ and $\vec{w}$ that vanishes is the trivial one. Therefore, $\{\vec{u}, \vec{v}, \vec{w}\}$ is independent. ■

Independence, spanning, and matrices

Theorem

Suppose A is an $m \times n$ matrix with columns $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n \in \mathbb{R}^m$. Then

1. $\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$ is independent if and only if $A\vec{x} = \vec{0}_m$ with $\vec{x} \in \mathbb{R}^n$ implies $\vec{x} = \vec{0}_n$.
2. $\mathbb{R}^m = \operatorname{span}\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$ if and only if $A\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^m$.

Problem

Let $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k \in \mathbb{R}^n$.

1. Are $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k$ linearly independent?
2. Do $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k$ span $\mathbb{R}^n$?

Solution

To answer both question, simply let A be a matrix whose columns are the vectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k \in \mathbb{R}^n$. Find R, a row-echelon form of A.

1. "yes" if and only if each column of R has a leading one.
2. "yes" if and only if each row of R has a leading one.

Problem (first seen earlier)

Let $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$.

Show that $\text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\} \neq \mathbb{R}^4$.

Solution

Let $A = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \end{bmatrix}$. Apply row operations to get R, a row-echelon form of A:

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the last row of R consists only of zeros, $R\vec{x} = \vec{e}_4$ has no solution $\vec{x} \in \mathbb{R}^4$, implying that there is a $\vec{b} \in \mathbb{R}^4$ so that $A\vec{x} = \vec{b}$ has no solution $\vec{x} \in \mathbb{R}^4$. By previous Theorem, $\mathbb{R}^4 \neq \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$. ∎

Theorem

Let A be an $n \times n$ matrix. The following are equivalent.

1. A is invertible.
2. The columns of A are independent.
3. The columns of A span $\mathbb{R}^n$.
4. The rows of A are independent, i.e., the columns of A^T are independent.
5. The rows of A span the set of all $1 \times n$ rows, i.e., the columns of A^T span $\mathbb{R}^n$.

Problem (revisited)

Let $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$.

Show that $\text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\} \neq \mathbb{R}^4$.

Solution

Let $A = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}$.

By the previous Theorem, the columns of A span $\mathbb{R}^4$ if and only if A is invertible. Since $\det(A) = 0$ (row 2 is (-1) times row 1), A is not invertible, and thus $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ does not span $\mathbb{R}^4$. ■

Problem

Let

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix}.$$

Is $\{\vec{u}, \vec{v}, \vec{w}\}$ independent?

Solution

Let $A = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$. From the previous Theorem, $\{\vec{u}, \vec{v}, \vec{w}\}$ is independent if and only if A is invertible.

Since

$$\det(A) = \det \begin{bmatrix} 1 & 3 & 3 \\ -1 & 2 & 5 \\ 0 & -1 & -2 \end{bmatrix} = -2,$$

and $-2 \neq 0$, A is invertible, and therefore $\{\vec{u}, \vec{v}, \vec{w}\}$ is an independent subset of $\mathbb{R}^3$. ∎

Remark

Notice that $\{\vec{u}, \vec{v}, \vec{w}\}$ also spans $\mathbb{R}^3$.

Bases and Dimension

Theorem (Fundamental Theorem)

Let U be a subspace of $\mathbb{R}^n$ that is spanned by m vectors. If U contains a subset of k linearly independent vectors, then $k \leq m$.

Definition

Let U be a subspace of $\mathbb{R}^n$. A set $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$ is a basis of U if

1. $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$ is linearly independent;
2. $U = \text{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$.

As a consequence of all this, if $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$ is a basis of a subspace U, then every $\vec{u} \in U$ has a unique representation as a linear combination of the vectors $\vec{x}_i$, $1 \leq i \leq m$.

Example

The subset $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is a basis of $\mathbb{R}^n$, called the standard basis of $\mathbb{R}^n$. (We've already seen that $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is linearly independent and that $\mathbb{R}^n = \text{span}\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$.)

Example

In a previous problem, we saw that $\mathbb{R}^4 = \text{span}(S)$ where

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

S is also linearly independent (prove this). Therefore, S is a basis of $\mathbb{R}^4$.

Theorem (Invariance Theorem)

If $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$ and $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_k\}$ are bases of a subspace U of $\mathbb{R}^n$, then $m = k$.

Proof.

Let $S = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$ and $T = \{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_k\}$. Since S spans U and T is independent, it follows from the Fundamental Theorem that $k \leq m$. Also, since T spans U and S is independent, it follows from the Fundamental Theorem that $m \leq k$. Since $k \leq m$ and $m \leq k$, $k = m$. ■

Definition

The dimension of a subspace U of $\mathbb{R}^n$ is the number of vectors in any basis of U, and is denoted $\dim(U)$.

Problem

In $\mathbb{R}^n$, what is the dimension of the subspace $\{\vec{0}_n\}$?

Solution

The only basis of the zero subspace is the empty set, $\emptyset$:
(i) the empty set is (trivially) independent, and
(ii) any linear combination of no vectors is the zero vector.
Therefore, the zero subspace has dimension zero.

Example

Since $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is a basis of $\mathbb{R}^n$, $\mathbb{R}^n$ has dimension n.
This is why the Cartesian plane, $\mathbb{R}^2$, is called 2-dimensional, and $\mathbb{R}^3$ is called 3-dimensional.

Problem

Let

$$U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; a - b = d - c \right\}.$$

Show that U is a subspace of $\mathbb{R}^4$, find a basis of U, and find $\dim(U)$.

Solution

The condition $a - b = d - c$ is equivalent to the condition $a = b - c + d$, so we may write

$$U = \left\{ \begin{bmatrix} b - c + d \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \right\} = \left\{ b \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \middle| \; b, c, d \in \mathbb{R} \right\}$$

This shows that U is a subspace of $\mathbb{R}^4$, since $U = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ where

$$\begin{aligned} \vec{x}_1 &= \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T \\ \vec{x}_2 &= \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}^T \\ \vec{x}_3 &= \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T. \end{aligned}$$

Solution (continued)

Furthermore,

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is linearly independent, as can be seen by taking the reduced row-echelon (RRE) form of the matrix whose columns are $\vec{x}_1, \vec{x}_2$ and $\vec{x}_3$.

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since every column of the RRE matrix has a leading one, the columns are linearly independent.

Therefore $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is linearly independent and spans U, so is a basis of U, and hence U has dimension three. ■

Example (Important!)

Suppose that $B = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ is a basis of $\mathbb{R}^n$ and that A is an $n \times n$ invertible matrix. Let $D = \{A\vec{x}_1, A\vec{x}_2, \ldots, A\vec{x}_n\}$, and let

$$X = \left[\begin{array}{cccc} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{array} \right].$$

Since B is a basis of $\mathbb{R}^n$, B is independent (also a spanning set of $\mathbb{R}^n$); thus X is invertible. Now, because A and X are invertible, so is

$$AX = \left[\begin{array}{cccc} A\vec{x}_1 & A\vec{x}_2 & \cdots & A\vec{x}_n \end{array} \right].$$

Therefore, the columns of AX are independent and span $\mathbb{R}^n$. Since the columns of AX are the vectors of D, D is a basis of $\mathbb{R}^n$.

Finding Bases and Dimension

Theorem

Let U be a subspace of $\mathbb{R}^n$. Then

1. U has a basis, and $\dim(U) \leq n$.
2. Any independent set of U can be extended (by adding vectors) to a basis of U.
3. Any spanning set of U can be cut down (by deleting vectors) to a basis of U.

Example

Previously, we showed that

$$U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; a - b = d - c \right\}$$

is a subspace of $\mathbb{R}^4$, and that $\dim(U) = 3$. Also, it is easy to verify that

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix} \right\},$$

is an independent subset of U.

By a previous Theorem, S can be extended to a basis of U. To do so, find a vector in U that is not in span(S).

Example (continued)

$$\begin{bmatrix} 1 & 2 & ? \\ 1 & 3 & ? \\ 1 & 3 & ? \\ 1 & 2 & ? \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 3 & -1 \\ 1 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, S can be extended to the basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\} \text{ of U.}$$

Problem

Let

$$\vec{u}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \\ -1 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 4 \\ 4 \\ 11 \\ -3 \end{bmatrix}, \quad \vec{u}_4 = \begin{bmatrix} 3 \\ -2 \\ 2 \\ -1 \end{bmatrix},$$

and let $U = \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$. Find a basis of U that is a subset of $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$, and find $\dim(U)$.

Solution

Suppose $a_1\vec{u}_1 + a_2\vec{u}_2 + a_3\vec{u}_3 + a_4\vec{u}_4 = \vec{0}$. Solve for a_1, a_2, a_3, a_4; if some $a_i \neq 0$, $1 \leq i \leq 4$, then $\vec{u}_i$ can be removed from the set $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$, and the resulting set still spans U. Repeat this on the resulting set until a linearly independent set is obtained.

One solution is $B = \{\vec{u}_1, \vec{u}_2\}$. Then $U = \text{span}(B)$ and B is linearly independent. Therefore B is a basis of U, and thus $\dim(U) = 2$. ■

Remark

In the next section, we will learn an efficient technique for solving this type of problem.

Theorem

Let U be a subspace of $\mathbb{R}^n$ with $\dim(U) = m$, and let $B = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$ be a subset of U. Then B is linearly independent if and only if B spans U.

Proof.

$(\Rightarrow)$ Suppose B is linearly independent. If $\operatorname{span}(B) \neq U$, then extend B to a basis B' of U by adding appropriate vectors from U. Then B' is a basis of size more than $m = \dim(U)$, which is impossible. Therefore, $\operatorname{span}(B) = U$, and hence B is a basis of U.

$(\Leftarrow)$ Conversely, suppose $\operatorname{span}(B) = U$. If B is not linearly independent, then cut B down to a basis B' of U by deleting appropriate vectors. But then B' is a basis of size less than $m = \dim(U)$, which is impossible. Therefore, B is linearly independent, and hence B is a basis of U. ∎

Remark

Let U be a subspace of $\mathbb{R}^n$ and suppose $B \subseteq U$.

- If B spans U and $|B| = \dim(U)$, then B is also independent, and hence B is a basis of U.
- If B is independent and $|B| = \dim(U)$, then B also spans U, and hence B is a basis of U.

Therefore, if $|B| = \dim(U)$, in order to prove that B is a basis, it is sufficient to prove either of the following two statements:

1. B is independent
2. B spans U

Theorem

Let U and W be subspace of $\mathbb{R}^n$, and suppose that $U \subseteq W$. Then

1. $\dim(U) \leq \dim(W)$.
2. If $\dim(U) = \dim(W)$, then $U = W$.

Proof.

Let $\dim(W) = k$, and let B be a basis of U.

1. If $\dim(U) > k$, then B is a subset of independent vectors of W with $|B| = \dim(U) > k$, which contradicts the Fundamental Theorem.
2. If $\dim(U) = \dim(W)$, then B is an independent subset of W containing $k = \dim(W)$ vectors. Therefore, B spans W, so B is a basis of W, and $U = \operatorname{span}(B) = W$.

$\blacksquare$

Example

Any subspace U of $\mathbb{R}^2$, other than $\{\vec{0}_2\}$ and $\mathbb{R}^2$ itself, must have dimension one, and thus has a basis consisting of one nonzero vector, say $\vec{u}$. Thus $U = \mathrm{span}\{\vec{u}\}$, and hence is a line through the origin.

Example

Any subspace U of $\mathbb{R}^3$, other than $\{\vec{0}_3\}$ and $\mathbb{R}^3$ itself, must have dimension one or two. If $\dim(U) = 1$, then, as in the previous example, U is a line through the origin. Otherwise $\dim(U) = 2$, and U has a basis consisting of two linearly independent vectors, say $\vec{u}$ and $\vec{v}$. Thus $U = \mathrm{span}\{\vec{u}, \vec{v}\}$, and hence is a plane through the origin.