Math 221: LINEAR ALGEBRA

Chapter 8. Orthogonality

§8-2. Orthogonal Diagonalization

Le Chen[1]

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Orthogonal Matrices

Orthogonal Diagonalization and Symmetric Matrices

Quadratic Forms

Orthogonal Matrices

Definition

An $n \times n$ matrix A is a orthogonal if its inverse is equal to its transpose, i.e., $A^{-1} = A^T$.

Example

$$\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad \frac{1}{7}\begin{bmatrix} 2 & 6 & -3 \\ 3 & 2 & 6 \\ -6 & 3 & 2 \end{bmatrix}$$

are orthogonal matrices (verify).

Theorem

The following are equivalent for an $n \times n$ matrix A.

1. A is orthogonal.
2. The rows of A are orthonormal.
3. The columns of A are orthonormal.

Proof.

"(1) $\iff$ (3)": Write $A = [\vec{a}_1, \cdots \vec{a}_n]$.

$$\text{A is orthogonal} \iff A^T A = I_n \iff \begin{pmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_n \end{pmatrix} [\vec{a}_1, \cdots \vec{a}_n] = I_n$$

$$\iff \begin{bmatrix} \vec{a}_1 \cdot \vec{a}_1 & \vec{a}_1 \cdot \vec{a}_2 & \cdots & \vec{a}_1 \cdot \vec{a}_n \\ \vec{a}_2 \cdot \vec{a}_1 & \vec{a}_2 \cdot \vec{a}_2 & \cdots & \vec{a}_2 \cdot \vec{a}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{a}_n \cdot \vec{a}_1 & \vec{a}_n \cdot \vec{a}_2 & \cdots & \vec{a}_n \cdot \vec{a}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

"(1) $\iff$ (2)": Similarly (Try it yourself). ■

Example

$$A = \begin{bmatrix} 2 & 1 & -2 \\ -2 & 1 & 2 \\ 1 & 0 & 8 \end{bmatrix}$$

has orthogonal columns, but its rows are not orthogonal (verify).

Normalizing the columns of A gives us the matrix

$$A' = \begin{bmatrix} 2/3 & 1/\sqrt{2} & -1/3\sqrt{2} \\ -2/3 & 1/\sqrt{2} & 1/3\sqrt{2} \\ 1/3 & 0 & 4/3\sqrt{2} \end{bmatrix},$$

which has orthonormal columns. Therefore, A' is an orthogonal matrix.

If an $n \times n$ matrix has orthogonal rows (columns), then normalizing the rows (columns) results in an orthogonal matrix.

Example (Orthogonal Matrices: Products and Inverses)

Suppose A and B are orthogonal matrices.

1. Since
$$(AB)(B^TA^T) = A(BB^T)A^T = AA^T = I.$$
and AB is square, $B^TA^T = (AB)^T$ is the inverse of AB, so AB is invertible, and $(AB)^{-1} = (AB)^T$. Therefore, AB is orthogonal.
2. $A^{-1} = A^T$ is also orthogonal, since
$$(A^{-1})^{-1} = A = (A^T)^T = (A^{-1})^T.$$

Remark (Summary)

If A and B are orthogonal matrices, then AB is orthogonal and A^{-1} is orthogonal.

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Orthogonal Diagonalization and Symmetric Matrices

Definition

An $n \times n$ matrix A is orthogonally diagonalizable if there exists an orthogonal matrix, P, so that $P^{-1}AP = P^{T}AP$ is diagonal.

Theorem (Principal Axis Theorem)

Let A be an $n \times n$ matrix. The following conditions are equivalent.

1. A has an orthonormal set of n eigenvectors.
2. A is orthogonally diagonalizable.
3. A is symmetric.

Proof. ((1) $\Rightarrow$ (2))

Suppose $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ is an orthonormal set of n eigenvectors of A. Then $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ is a basis of $\mathbb{R}^n$, and hence $P = \left[\begin{array}{cccc} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{array}\right]$ is an orthogonal matrix such that $P^{-1}AP = P^TAP$ is a diagonal matrix. Therefore A is orthogonally diagonalizable.

Proof. ((2) $\Rightarrow$ (1))

Suppose that A is orthogonally diagonalizable. Then there exists an orthogonal matrix P such that P^TAP is a diagonal matrix. If P has columns $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$, then $B = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ is a set of n orthonormal vectors in $\mathbb{R}^n$. Since B is orthogonal, B is independent; furthermore, since $|B| = n = \dim(\mathbb{R}^n)$, B spans $\mathbb{R}^n$ and is therefore a basis of $\mathbb{R}^n$.
Let $P^TAP = \text{diag}(\ell_1, \ell_2, \ldots, \ell_n) = D$. Then $AP = PD$, so

$$A\begin{bmatrix} \vec{x_1} & \vec{x_2} & \cdots & \vec{x_n} \end{bmatrix} = \begin{bmatrix} \vec{x_1} & \vec{x_2} & \cdots & \vec{x_n} \end{bmatrix}\begin{bmatrix} \ell_1 & 0 & \cdots & 0 \\ 0 & \ell_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \ell_n \end{bmatrix}$$

$$\begin{bmatrix} A\vec{x_1} & A\vec{x_2} & \cdots & A\vec{x_n} \end{bmatrix} = \begin{bmatrix} \ell_1\vec{x_1} & \ell_2\vec{x_2} & \cdots & \ell_n\vec{x_n} \end{bmatrix}$$

Thus $A\vec{x}_i = \ell_i\vec{x}_i$ for each i, $1 \leq i \leq n$, implying that B consists of eigenvectors of A. Therefore, A has an orthonormal set of n eigenvectors.

Proof. ((2) $\Rightarrow$ (3))

Suppose A is orthogonally diagonalizable, that D is a diagonal matrix, and that P is an orthogonal matrix so that $P^{-1}AP = D$. Then $P^{-1}AP = P^{T}AP$, so

$$A = PDP^{T}.$$

Taking transposes of both sides of the equation:

$$\begin{aligned} A^{T} = (PDP^{T})^{T} &= (P^{T})^{T}D^{T}P^{T} \\ &= PD^{T}P^{T} \quad \text{(since } (P^{T})^{T} = P\text{)} \\ &= PDP^{T} \quad \text{(since } D^{T} = D\text{)} \\ &= A. \end{aligned}$$

Since $A^{T} = A$, A is symmetric.

Proof. ((3) $\Rightarrow$ (2))

If A is $n \times n$ symmetric matrix, we will prove by induction on n that A is orthogonal diagonalizable. If $n = 1$, A is already diagonalizable. If $n \geq 2$, assume that (3)$\Rightarrow$(2) for all $(n-1) \times (n-1)$ symmetric matrix.

First we know that all eigenvalues are real (because A is symmetric). Let λ_1 be one real eigenvalue and $\vec{x}_1$ be the normalized eigenvector. We can extend $\{\vec{x}_1\}$ to an orthonormal basis of $\mathbb{R}^n$, say $\{\vec{x}_1, \cdots, \vec{x}_n\}$ by adding vectors. Let $P_1 = [\vec{x}_1, \cdots, \vec{x}_n]$. So P is orthogonal.
Now we can apply the technical lemma proved in Section 5.5 to see that

$$P_1^T A P_1 = \begin{bmatrix} \lambda_1 & B \\ \vec{0} & A_1 \end{bmatrix}.$$

Since LHS is symmetric, so does the RHS. This implies that $B = O$ and A_1 is symmetric.

Proof. ((3) ⇒ (2) – continued)

By induction assumption, A_1 is orthogonal diagonalizable, i.e., for some orthogonal matrix Q and diagonal matrix D, $A_1 = QDQ^T$. Hence,

$$P_1^T A P_1 = \begin{bmatrix} \lambda_1 & \vec{0}^T \\ \vec{0} & QDQ^T \end{bmatrix} = \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & Q \end{bmatrix} \begin{bmatrix} \lambda_1 & \vec{0}^T \\ \vec{0} & D \end{bmatrix} \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & Q^T \end{bmatrix}$$

which is nothing but

$$\begin{aligned} A &= P_1 \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & Q \end{bmatrix} \begin{bmatrix} \lambda_1 & \vec{0}^T \\ \vec{0} & D \end{bmatrix} \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & Q^T \end{bmatrix} P_1^T \\ &= \left(P_1 \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & Q \end{bmatrix} \right) \begin{bmatrix} \lambda_1 & \vec{0}^T \\ \vec{0} & D \end{bmatrix} \left(P_1 \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & Q \end{bmatrix} \right)^T. \end{aligned}$$

Finally, it is ready to verify that the matrix

$$P_1 \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & Q \end{bmatrix}$$

is a diagonal matrix. This complete the proof of the theorem. ■

Definition

Let A be an $n \times n$ matrix. A set of n orthonormal eigenvectors of A is called a set of **principal axes** of A.

Problem

Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}.$$

Solution

- $c_A(x) = (x+3)(x-3)^2$, so A has eigenvalues $\lambda_1 = 3$ of multiplicity two, and $\lambda_2 = -3$.
- $\{\vec{x}_1, \vec{x}_2\}$ is a basis of $E_3(A)$, where $\vec{x}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.
- $\{\vec{x}_3\}$ is a basis of $E_{-3}(A)$, where $\vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.
- $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ a linearly independent set of eigenvectors of A, and a basis of $\mathbb{R}^3$.

Solution (continued)

- Orthogonalize $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ using the Gram-Schmidt orthogonalization algorithm.
- Let $\vec{f_1} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{f_2} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$ and $\vec{f_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Then $\{\vec{f_1}, \vec{f_2}, \vec{f_3}\}$ is an orthogonal basis of $\mathbb{R}^3$ consisting of eigenvectors of A.
- Since $||\vec{f_1}|| = \sqrt{2}$, $||\vec{f_2}|| = \sqrt{6}$, and $||\vec{f_3}|| = \sqrt{3}$,

$$P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

is an orthogonal diagonalizing matrix of A, and

$$P^T A P = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

■

Theorem

If A is a symmetric matrix, then the eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

Proof.

Suppose λ and μ are eigenvalues of A, $\lambda \neq \mu$, and let $\vec{x}$ and $\vec{y}$, respectively, be corresponding eigenvectors, i.e., $A\vec{x} = \lambda\vec{x}$ and $A\vec{y} = \mu\vec{y}$. Consider $(\lambda - \mu)\vec{x} \cdot \vec{y}$.

$$
\begin{aligned}
(\lambda - \mu)\vec{x} \cdot \vec{y} &= \lambda(\vec{x} \cdot \vec{y}) - \mu(\vec{x} \cdot \vec{y}) \\
&= (\lambda\vec{x}) \cdot \vec{y} - \vec{x} \cdot (\mu\vec{y}) \\
&= (A\vec{x}) \cdot \vec{y} - \vec{x} \cdot (A\vec{y}) \\
&= (A\vec{x})^T\vec{y} - \vec{x}^T(A\vec{y}) \\
&= \vec{x}^T A^T \vec{y} - \vec{x}^T A\vec{y} \\
&= \vec{x}^T A\vec{y} - \vec{x}^T A\vec{y} \qquad \text{since A is symmetric} \\
&= 0.
\end{aligned}
$$

Since $\lambda \neq \mu$, $\lambda - \mu \neq 0$, and therefore $\vec{x} \cdot \vec{y} = 0$, i.e., $\vec{x}$ and $\vec{y}$ are orthogonal.

■

Remark (Diagonalizing a Symmetric Matrix)

Let A be a symmetric $n \times n$ matrix.

1. Find the characteristic polynomial and distinct eigenvalues of A.
2. For each distinct eigenvalue λ of A, find an orthonormal basis of $E_A(\lambda)$, the eigenspace of A corresponding to λ. This requires using the Gram-Schmidt orthogonalization algorithm when $\dim(E_A(\lambda)) \geq 2$.
3. By the previous theorem, the eigenvectors of distinct eigenvalues produce orthogonal eigenvectors, so the result is an orthonormal basis of $\mathbb{R}^n$.

Problem

Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}.$$

Solution

1. Since row sum is 5, $\lambda_1 = 5$ is one eigenvalue, corresponding eigenvector should be $(1, 1, 1)^T$. After normalization it should be

$$\vec{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

Solution (continued)

2. Since last two rows are identical, $\det(A) = 0$, so $\lambda_2 = 0$ is another eigenvalue, corresponding eigenvector should be $(0, 1, -1)^T$. After normalization it should be

$$\vec{v}_2 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Solution (continued)

3. Since $\mathrm{tr}(A) = 7 = \lambda_1 + \lambda_2 + \lambda_3$, we see that $\lambda_3 = 7 - 5 - 0 = 2$. Its eigenvector should be orthogonal to both $\vec{v}_1$ and $\vec{v}_2$, hence, $\vec{v}_3 = (2, -1, -1)$. After normalization,

$$\begin{pmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Hence, we have

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Orthogonal Matrices

Orthogonal Diagonalization and Symmetric Matrices

Quadratic Forms

Quadratic Forms

Definitions

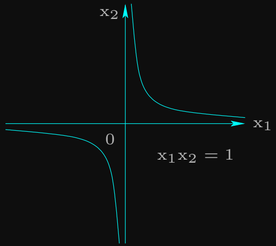
Let q be a real polynomial in variables x_1 and x_2 such that

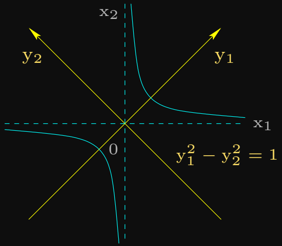
$$q(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2.$$

Then q is called a quadratic form in variables x_1 and x_2. The term bx_1x_2 is called the cross term. The graph of the equation $q(x_1, x_2) = 1$, is call a conic in variables x_1 and x_2.

Example

Below is the graph of the equation $x_1x_2 = 1$.





Let y_1 and y_2 be new variables such that

$$x_1 = y_1 + y_2 \quad \text{and} \quad x_2 = y_1 - y_2,$$

i.e., $y_1 = \frac{x_1+x_2}{2}$ and $y_2 = \frac{x_1-x_2}{2}$. Then $x_1x_2 = y_1^2 - y_2^2$, and $y_1^2 - y_2^2$ is a quadratic form with no cross terms, called a diagonal quadratic form; y_1 and y_2 are called principal axes of the quadratic form x_1x_2.

Principal axes of a quadratic form can be found by using orthogonal diagonalization.

Problem

Find principal axes of the quadratic form $q(x_1, x_2) = x_1^2 + 6x_1x_2 + x_2^2$, and transform $q(x_1, x_2)$ into a diagonal quadratic form.

Solution

Express $q(x_1, x_2)$ as a matrix product:

$$q(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{1}$$

We want a 2×2 symmetric matrix. Since $6x_1x_2 = 3x_1x_2 + 3x_2x_1$, we can rewrite (1) as

$$q(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2}$$

Setting $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, $q(x_1, x_2) = \vec{x}^T A \vec{x}$.

We now orthogonally diagonalize A.

Solution (continued)

$$c_A(z) = \begin{vmatrix} z-1 & -3 \\ -3 & z-1 \end{vmatrix} = (z-4)(z+2),$$

so A has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = -2$.

$$\vec{z}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{z}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

are eigenvectors corresponding to $\lambda_1 = 4$ and $\lambda_2 = -2$, respectively. Normalizing these eigenvectors gives us the orthogonal matrix

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ such that } P^T A P = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} = D.$$

Thus $A = PDP^T$, and

$$q(x_1, x_2) = \vec{x}^T A \vec{x} = \vec{x}^T (PDP^T)\vec{x} = (\vec{x}^T P) D (P^T \vec{x}) = (P^T \vec{x})^T D (P^T \vec{x}).$$

Solution (continued)

Let

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P^T\vec{x} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} x_1 + x_2 \\ x_2 - x_1 \end{bmatrix}.$$

Then

$$q(y_1, y_2) = \vec{y}^T D \vec{y} = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 4y_1^2 - 2y_2^2.$$

Therefore, the principal axes of $q(x_1, x_2) = x_1^2 + 6x_1x_2 + x_2^2$ are

$$y_1 = \frac{1}{\sqrt{2}}(x_1 + x_2)$$

and

$$y_2 = \frac{1}{\sqrt{2}}(x_2 - x_1),$$

yielding the diagonal quadratic form

$$q(y_1, y_2) = 4y_1^2 - 2y_2^2.$$

■

Problem

Find principal axes of the quadratic form

$$q(x_1, x_2) = 7x_1^2 - 4x_1x_2 + 4x_2^2,$$

and transform $q(x_1, x_2)$ into a diagonal quadratic form.

Solution (Final Answer)

$q(x_1, x_2)$ has principal axes

$$\begin{aligned} y_1 &= \frac{1}{\sqrt{5}}(-2x_1 + x_2), \\ y_2 &= \frac{1}{\sqrt{5}}(x_1 + 2x_2). \end{aligned}$$

yielding the diagonal quadratic form

$$q(y_1, y_2) = 8y_1^2 + 3y_2^2.$$

Theorem (Triangulation Theorem – Schur Decomposition)

Let A be an $n \times n$ matrix with n real eigenvalues. Then there exists an orthogonal matrix P such that $P^T AP$ is upper triangular.

Corollary

Let A be an $n \times n$ matrix with real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, not necessarily distinct. Then $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$ and $\operatorname{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

Proof.

By the theorem, there exists an orthogonal matrix P such that $P^T A P = U$, where U is an upper triangular matrix. Since P is orthogonal, $P^T = P^{-1}$, so U is similar to A; thus the eigenvalues of U are $\lambda_1, \lambda_2, \ldots, \lambda_n$. Furthermore, since U is (upper) triangular, the entries on the main diagonal of U are its eigenvalues, so $\det(U) = \lambda_1 \lambda_2 \cdots \lambda_n$ and $\operatorname{tr}(U) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. Since U and A are similar, $\det(A) = \det(U)$ and $\operatorname{tr}(A) = \operatorname{tr}(U)$, and the result follow. ■