Math 221: LINEAR ALGEBRA

Chapter 8. Orthogonality
§8-4. QR Factorization

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

QR Factorization

Algorithm for the QR Factorization

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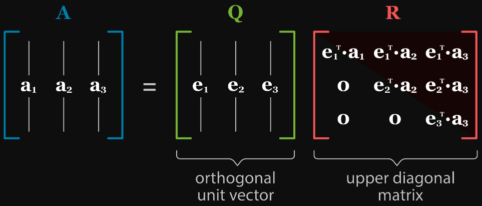
The QR Factorization

Definition

Let A be a real $m \times n$ matrix. Then a QR factorization of A can be written as

$$A = QR$$

where Q is an orthogonal matrix and R is an upper (or right) triangular matrix.



Theorem

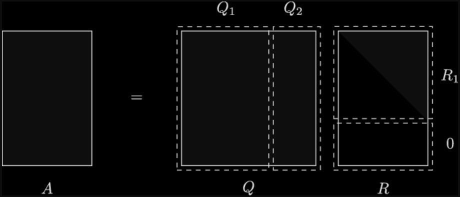
Let A be a real $m \times n$ matrix with linearly independent columns. Then A can be written

$$A = QR$$

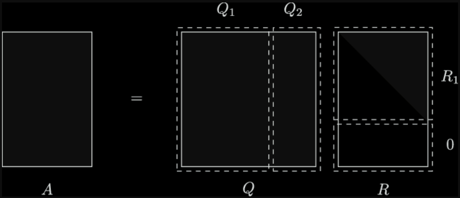
with Q orthogonal and R upper triangular with positive entries on the main diagonal.

Proof.

Using columns of A to carry out the Gram-Schmidt algorithm to find an orthonormal basis for im(A) or col(A) $\subseteq \mathbb{R}^m$ – columns of Q_1. One may further extend this basis to an orthonormal basis for the whole space $\mathbb{R}^m$ – columns of $Q = [Q_1, Q_2]$.



The Gram-Schmidt algorithm guarantees that the ith column of A is linear combinations of all jth columns of Q with $j = 1, \cdots, i$, which gives the upper triangular structure of R. ∎



Remark

$$A = QR = [Q_1, Q_2] \begin{bmatrix} R_1 \\ O \end{bmatrix} = Q_1 R_1 + Q_2 O = Q_1 R_1.$$

Both QR and $Q_1 R_1$ are called QR decompositions of A. The textbook refers $Q_1 R_1$.

Remark

Q is orthogonal matrix, namely, $QQ^T = Q^T Q = I_m$.
However, Q_1 is not orthogonal matrix (not a square matrix). But We have $Q_1^T Q_1 = I_n$ and $Q_1 Q_1^T \neq I_m$ (in general).

QR Factorization

Algorithm for the QR Factorization

Algorithm for QR Factorization

Algorithm 1: QR Factorization Algorithm

Input : Independent columns of A: $\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\} \in \text{col}(A) \subseteq \mathbb{R}^m$

for $j \leftarrow 1$ **to** n **do**

$$\vec{f}_j \leftarrow \vec{c}_j - \frac{\vec{c}_j \cdot \vec{f}_1}{||\vec{f}_1||^2}\vec{f}_1 - \frac{\vec{c}_j \cdot \vec{f}_2}{||\vec{f}_2||^2}\vec{f}_2 - \cdots - \frac{\vec{c}_j \cdot \vec{f}_{j-1}}{||\vec{f}_{j-1}||^2}\vec{f}_{j-1}.$$

$$\vec{q}_j \leftarrow \frac{\vec{f}_j}{||\vec{f}_j||}$$

for $i \leftarrow 1$ **to** j **do**

$$r_{ij} \leftarrow \vec{q}_i \cdot \vec{c}_j$$

end

end

Output: $Q = [\vec{q}_1, \cdots, \vec{q}_n]$ and $R = [r_{ij}]$

Problem

Let

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \\ 0 & 1 \end{bmatrix}$$

Find the QR factorization of A.

Solution

Set $A = [\vec{c}_1, \vec{c}_2]$. When $j = 1$,

$$\vec{f}_1 = \vec{c}_1 = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{q}_1 = \frac{\vec{f}_1}{||\vec{f}_1||} = \begin{bmatrix} \frac{4}{\sqrt{20}} \\ \frac{2}{\sqrt{20}} \\ 0 \end{bmatrix}.$$

For $i = 1$,

$$r_{11} = \vec{q}_1 \cdot \vec{c}_1 = \frac{\vec{f}_1}{||\vec{f}_1||} \cdot \vec{f}_1 = ||\vec{f}_1|| = \sqrt{20}.$$

Solution (continued)

When $j = 2$,

$$\vec{f}_2 = \vec{c}_2 - \frac{\vec{c}_2 \cdot \vec{f}_1}{||\vec{f}_1||^2}\vec{f}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - \frac{10}{20}\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{q}_2 = \frac{\vec{f}_2}{||\vec{f}_2||} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}.$$

For $i = 1$,

$$r_{12} = \vec{q}_1 \cdot \vec{c}_2 = \begin{bmatrix} \frac{4}{\sqrt{20}} \\ \frac{2}{\sqrt{20}} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \sqrt{5}.$$

and for $i = 2$,

$$r_{22} = \vec{q}_2 \cdot \vec{c}_2 = \frac{\vec{f}_2}{||\vec{f}_2||} \cdot \left(\vec{f}_2 + \frac{\vec{c}_2 \cdot \vec{f}_1}{||\vec{f}_1||^2}\vec{f}_1\right) = \frac{\vec{f}_2}{||\vec{f}_2||} \cdot \vec{f}_2 = ||\vec{f}_2|| = \sqrt{6}.$$

Solution (continued)

Therefore,

$$A = QR = [\vec{q}_1, \vec{q}_2] \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} 4 & 1 \\ 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{20} & \sqrt{5} \\ 0 & \sqrt{6} \end{bmatrix}$$

$\blacksquare$