

Math 362: Mathematical Statistics II

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Chapter 12. The Analysis of Variance

§ 12.1 Introduction

§ 12.2 The F Test

§ 12.3 Multiple Comparisons: Turkey's Method

§ 12.4 Testing Subhypotheses with Contrasts

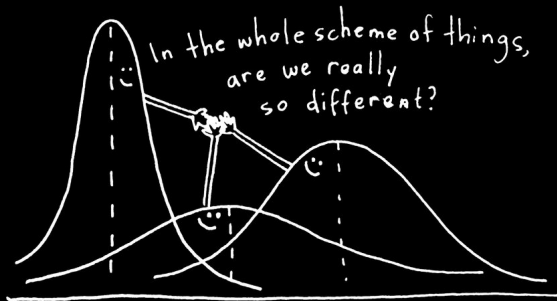
Chapter 12. The Analysis of Variance

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§ 12.4 Testing Subhypotheses with Contrasts



E.g. 1 Study the relation between smoking and heart rates.

Generations of athletes have been cautioned that cigarette smoking impedes performance. One measure of the truth of that warning is the effect of smoking on heart rate. In one study, six nonsmokers, six light smokers, six moderate smokers, and six heavy smokers each engaged in sustained physical exercise. Table 8.1.1 lists their heart rates after they had rested for three minutes.

	Nonsmokers	Light Smokers	Moderate Smokers	Heavy Smokers
	69	55	66	91
	52	60	81	72
	71	78	70	81
	58	58	77	67
	59	62	57	95
	65	66	79	84
<i>Averages:</i>	62.3	63.2	71.7	81.7

Show whether smoking affects heart rates at $\alpha = 0.05$.

E.g. 2 A certain fraction of antibiotics injected into the bloodstream are “bound” to serum proteins. This phenomenon bears directly on the effectiveness of the medication, because the binding decreases the systemic uptake of the drug. Table below lists the binding percentages in bovine serum measured for five widely prescribed antibiotics. Which antibiotics have similar binding properties, and which are different?

Table 12.3.1					
	Penicillin G	Tetra- cycline	Strepto- mycin	Erythro- mycin	Chloram- phenicol
	29.6	27.3	5.8	21.6	29.2
	24.3	32.6	6.2	17.4	32.8
	28.5	30.8	11.0	18.3	25.0
	32.0	34.8	8.3	19.0	24.2
T_j	114.4	125.5	31.3	76.3	111.2
\bar{Y}_j	28.6	31.4	7.8	19.1	27.8

Table 12.1.1				
	Treatment Level			
	1	2	...	k
	Y_{11}	Y_{12}		Y_{1k}
	Y_{21}	Y_{22}		
	\vdots	\vdots	...	\vdots
	$Y_{n_1 1}$	$Y_{n_2 2}$		$Y_{n_k k}$
Sample sizes:	n_1	n_2	...	n_k
Sample totals:	$T_{\cdot 1}$	$T_{\cdot 2}$		$T_{\cdot k}$
Sample means:	$\bar{Y}_{\cdot 1}$	$\bar{Y}_{\cdot 2}$		$\bar{Y}_{\cdot k}$
True means:	μ_1	μ_2		μ_k

- ▶ k treatment levels; k independent random sample of size n_1, \dots, n_k
- ▶ Total sample size: $n = \sum_{i=1}^k n_i$
- ▶ Y_{ij} : i -th observation for the j -th level.
- ▶ Sample total: $T_{\cdot j} = \sum_{i=1}^{n_j} Y_{ij}$
- ▶ Sample mean: $\bar{Y}_{\cdot j} = \frac{1}{n_j} \sum_{i=1}^{n_j} Y_{ij} = \frac{T_{\cdot j}}{n_j}$
- ▶ Overall total: $T_{\cdot \cdot} = \sum_{j=1}^k \sum_{i=1}^{n_j} Y_{ij} = \sum_{j=1}^k T_{\cdot j}$
- ▶ Overall mean: $\bar{Y}_{\cdot \cdot} = \frac{1}{n} \sum_{j=1}^k \sum_{i=1}^{n_j} Y_{ij} = \frac{1}{n} \sum_{j=1}^k n_j \bar{Y}_{\cdot j} = \frac{1}{n} \sum_{j=1}^k T_{\cdot j}$

Assume For $j = 1, \dots, k$, $Y_{ij} \sim N(\mu_j, \sigma_j^2)$ and $\sigma_1^2 = \dots = \sigma_k^2 = \sigma^2$ (unknown).

Problem Testing

$$H_0 : \mu_1 = \mu_2 = \dots = \mu_k$$

versus

$$H_1 : \text{not all the } \mu_j\text{'s are equal}$$

Or testing *subhypotheses* such as

$$H_0 : \mu_i = \mu_j \quad \text{or} \quad H_0 : \mu_3 = (\mu_1 + \mu_2)/2$$

ANOVA was developed by statistician and evolutionary biologist —



Ronald Fisher



Statistician

Sir Ronald Aylmer Fisher FRS was a British statistician and geneticist. For his work in statistics, he has been described as "a genius who almost single-handedly created the foundations for modern statistical science" and "the single most important figure in 20th century statistics". [Wikipedia](#)

Born: February 17, 1890, East Finchley, London, United Kingdom

Died: July 29, 1962, Adelaide, Australia

Known for: Fisher's principle, Fisher information

Residence: United Kingdom, Australia

Education: Gonville & Caius College, University of Cambridge, Harrow School

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Model assumptions

1. Independence of observations
2. Normality
3. Homogeneity of variances



Assume:

$\forall j = 1, \dots, k, \forall i = 1, \dots, n_j,$

1. Y_{ij} are independent.
2. $Y_{ij} \sim N(\mu_j, \sigma^2)$



Assume:

$\forall j = 1, \dots, k, \forall i = 1, \dots, n_j,$

$$Y_{ij} = \mu_j + \epsilon_{ij}$$

1. ϵ_{ij} are independent.
2. $\epsilon_{ij} \sim N(0, \sigma^2)$

Table 12.1.1

	<i>Treatment Level</i>			
	1	2	...	k
	Y_{11}	Y_{12}		Y_{1k}
	Y_{21}	Y_{22}		
	\vdots	\vdots	...	\vdots
	$Y_{n_1 1}$	$Y_{n_2 2}$		$Y_{n_k k}$
Sample sizes:	n_1	n_2	...	n_k
Sample totals:	$T_{\cdot 1}$	$T_{\cdot 2}$...	$T_{\cdot k}$
Sample means:	$\bar{Y}_{\cdot 1}$	$\bar{Y}_{\cdot 2}$		$\bar{Y}_{\cdot k}$
True means:	μ_1	μ_2		μ_k

Likelihood ratio test

1. The parameter spaces are

$$\Omega = \{(\mu_1, \dots, \mu_k, \sigma^2) : -\infty < \mu_1, \dots, \mu_k < \infty, \sigma^2 > 0\}$$

$$\omega = \{(\mu_1, \dots, \mu_k, \sigma^2) : -\infty < \mu_1 = \dots = \mu_k < \infty, \sigma^2 > 0\}$$

2. The likelihood functions are

$$L(\omega) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{j=1}^k \sum_{i=1}^{n_j} (y_{ij} - \mu)^2 \right\}$$

$$L(\Omega) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{j=1}^k \sum_{i=1}^{n_j} (y_{ij} - \mu_j)^2 \right\}$$

3. Now

$$\frac{\partial \ln L(\omega)}{\partial \mu} = \frac{1}{\sigma^2} \sum_{j=1}^k \sum_{i=1}^{\eta_j} (y_{ij} - \mu)$$

$$\frac{\partial \ln L(\omega)}{\partial (\sigma^2)} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{j=1}^k \sum_{i=1}^{\eta_j} (y_{ij} - \mu)^2$$

Setting the above derivatives to zero, the solutions for μ and σ^2 are,

$$\frac{1}{n} \sum_{j=1}^k \sum_{i=1}^{\eta_j} y_{ij} = \bar{y}_{..}$$

$$\frac{1}{n} \sum_{j=1}^k \sum_{i=1}^{\eta_j} (y_{ij} - \bar{y}_{..})^2 = v$$

3' Similarly,

$$\frac{\partial \ln L(\Omega)}{\partial \mu_j} = \frac{1}{\sigma^2} \sum_{j=1}^k \sum_{i=1}^{n_j} (y_{ij} - \mu_j), \quad j = 1, \dots, k$$

$$\frac{\partial \ln L(\Omega)}{\partial (\sigma^2)} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{j=1}^k \sum_{i=1}^{n_j} (y_{ij} - \mu_j)^2$$

Setting the above derivatives to zero, the solutions for μ_j and σ^2 are,

$$\frac{1}{n_j} \sum_{i=1}^{n_j} y_{ij} = \bar{y}_{\cdot j}$$

$$\frac{1}{n} \sum_{j=1}^k \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{\cdot j})^2 = w$$

4. Hence,

$$L(\hat{\omega}) = \left(\frac{n}{2\pi \sum_{j=1}^k \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{..})^2} \right)^{n/2} \exp \left\{ -\frac{n \sum_{j=1}^k \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{..})^2}{2 \sum_{j=1}^k \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{..})^2} \right\}$$

$$\|$$

$$\left(\frac{n}{2\pi \sum_{j=1}^k \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{..})^2} \right)^{n/2} e^{-n/2}$$

Similarly,

$$L(\hat{\Omega}) = \left(\frac{n}{2\pi \sum_{j=1}^k \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{.j})^2} \right)^{n/2} \exp \left\{ -\frac{n \sum_{j=1}^k \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{.j})^2}{2 \sum_{j=1}^k \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{.j})^2} \right\}$$

$$\|$$

$$\left(\frac{n}{2\pi \sum_{j=1}^k \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{.j})^2} \right)^{n/2} e^{-n/2}$$

5. Finally,

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \left(\frac{\sum_{j=1}^k \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{\cdot j})^2}{\sum_{j=1}^k \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{\cdot \cdot})^2} \right)^{n/2}$$

⇒ Test statistic:

$$\Lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \left(\frac{\sum_{j=1}^k \sum_{i=1}^{n_j} (Y_{ij} - \bar{Y}_{\cdot j})^2}{\sum_{j=1}^k \sum_{i=1}^{n_j} (Y_{ij} - \bar{Y}_{\cdot \cdot})^2} \right)^{n/2}$$

$$\begin{aligned}
SSTOT &:= \sum_{j=1}^k \sum_{i=1}^{n_j} (Y_{ij} - \bar{Y}_{..})^2 \\
&= \sum_{j=1}^k \sum_{i=1}^{n_j} \left[(Y_{ij} - \bar{Y}_{.j}) + (\bar{Y}_{.j} - \bar{Y}_{..}) \right]^2 \\
&= \sum_{j=1}^k \sum_{i=1}^{n_j} (Y_{ij} - \bar{Y}_{.j})^2 + \text{zero cross term} + \sum_{j=1}^k \sum_{i=1}^{n_j} (\bar{Y}_{.j} - \bar{Y}_{..})^2 \\
&= \underbrace{\sum_{j=1}^k \sum_{i=1}^{n_j} (Y_{ij} - \bar{Y}_{.j})^2}_{SSE} + \underbrace{\sum_{j=1}^k n_j (\bar{Y}_{.j} - \bar{Y}_{..})^2}_{SSTR}
\end{aligned}$$

⇓

$$\Lambda = \left(\frac{SSE}{SSTOT} \right)^{n/2} = \left(\frac{SSE}{SSE + SSTR} \right)^{n/2} = \left(\frac{1}{1 + SSTR/SSE} \right)^{n/2}$$

6. Critical regions: for some $\lambda_* \in (0, 1)$ close to 0,

$$\begin{aligned}\alpha &= \mathbb{P}(\Lambda \leq \lambda_*) \\ &= \mathbb{P}\left(\frac{1}{1 + SSTR/SSE} \leq \lambda_*^{2/n}\right) \\ &= \mathbb{P}\left(\frac{SSTR}{SSE} \leq \lambda_*^{-2/n} - 1\right) \\ &= \mathbb{P}\left(\frac{SSTR/(k-1)}{SSE/(n-k)} \leq \left(\lambda_*^{-2/n} - 1\right) \frac{n-k}{k-1}\right)\end{aligned}$$

7. We will prove that under H_0 , $\frac{SSTR/(k-1)}{SSE/(n-k)} \sim F$ -distr.
 $df_1 = k - 1, df_2 = n - k$

$$\Rightarrow \left(\lambda_*^{-2/n} - 1\right) \frac{n-k}{k-1} = F_{1-\alpha, k-1, n-k}.$$

□

Treatment sum of squares: SSTR

Sample size: (Weights)	n_1	n_2	\dots	n_k	$n = \sum_{j=1}^k n_j$ <i>Weighted average</i>
Sample means:	$\bar{Y}_{\cdot 1}$	$\bar{Y}_{\cdot 2}$	\dots	$\bar{Y}_{\cdot k}$	$\bar{Y}_{\cdot\cdot} = \frac{1}{n} \sum_{j=1}^k n_j \bar{Y}_{\cdot j}$
True means:	μ_1	μ_2	\dots	μ_k	$\mu = \frac{1}{n} \sum_{j=1}^k n_j \mu_j$
Squares:	$(\bar{Y}_{\cdot 1} - \bar{Y}_{\cdot\cdot})^2$	$(\bar{Y}_{\cdot 2} - \bar{Y}_{\cdot\cdot})^2$	\dots	$(\bar{Y}_{\cdot k} - \bar{Y}_{\cdot\cdot})^2$	SSTR

$$SSTR := \sum_{j=1}^k n_j (\bar{Y}_{\cdot j} - \bar{Y}_{\cdot\cdot})^2$$

1. When $k = 1$, $SSTR \equiv 0$.
2. When $k = 2$, say X_1, \dots, X_n and Y_1, \dots, Y_m :

$$\bar{Y}_{..} = \frac{1}{m+n} (n\bar{X} + m\bar{Y})$$

$$\begin{aligned} SSTR &= n \left[\bar{X} - \frac{1}{n+m} (n\bar{X} + m\bar{Y}) \right]^2 + m \left[\bar{Y} - \frac{1}{n+m} (n\bar{X} + m\bar{Y}) \right]^2 \\ &= n \left[\frac{m(\bar{X} - \bar{Y})}{n+m} \right]^2 + m \left[\frac{n(\bar{X} - \bar{Y})}{n+m} \right]^2 \\ &= \left[\frac{nm^2}{(n+m)^2} + \frac{n^2m}{(n+m)^2} \right] (\bar{X} - \bar{Y})^2 \\ &= \frac{nm}{n+m} (\bar{X} - \bar{Y})^2 \end{aligned}$$

$$SSTR = \frac{(\bar{X} - \bar{Y})^2}{\frac{1}{m} + \frac{1}{n}}$$

$$\begin{aligned}
SSTR &= \sum_{j=1}^k n_j (\bar{Y}_{.j} - \bar{Y}_{..})^2 = \sum_{j=1}^k n_j [(\bar{Y}_{.j} - \mu) - (\bar{Y}_{..} - \mu)]^2 \\
&= \sum_{j=1}^k n_j [(\bar{Y}_{.j} - \mu)^2 + (\bar{Y}_{..} - \mu)^2 - 2(\bar{Y}_{.j} - \mu)(\bar{Y}_{..} - \mu)] \\
&= \sum_{j=1}^k n_j (\bar{Y}_{.j} - \mu)^2 + \sum_{j=1}^k n_j (\bar{Y}_{..} - \mu)^2 - 2(\bar{Y}_{..} - \mu) \sum_{j=1}^k n_j (\bar{Y}_{.j} - \mu) \\
&= \sum_{j=1}^k n_j (\bar{Y}_{.j} - \mu)^2 + n(\bar{Y}_{..} - \mu)^2 - 2(\bar{Y}_{..} - \mu)n(\bar{Y}_{..} - \mu) \\
&= \sum_{j=1}^k n_j (\bar{Y}_{.j} - \mu)^2 - n(\bar{Y}_{..} - \mu)^2 \tag{12.2.1}
\end{aligned}$$

⇓

$$SSTR = \sum_{j=1}^k n_j \left[(\bar{Y}_{.j} - \mu_j)^2 - 2(\bar{Y}_{.j} - \mu_j)(\mu - \mu_j) + (\mu - \mu_j)^2 \right] - n(\bar{Y}_{..} - \mu)^2$$

Notice that

$$\bar{Y}_{.j} \sim N(\mu_j, \sigma^2/n_j) \quad \text{and} \quad \bar{Y}_{..} \sim N(\mu, \sigma^2/n)$$

\Rightarrow

$$\begin{aligned} \mathbb{E}[SSTR] &= \sum_{j=1}^k n_j \left[\frac{\sigma^2}{n_j} - 2 \times 0 + (\mu - \mu_j)^2 \right] - n \frac{\sigma^2}{n} \\ &= (k-1)\sigma^2 + \sum_{j=1}^k n_j (\mu - \mu_j)^2 \end{aligned}$$

Remark

When $\mu_1 = \dots = \mu_j$ then

0.1 $\mathbb{E}[SSTR] = (k-1)\sigma^2$

0.2 $MSTR := \frac{SSTR}{k-1}$ is an unbiased estimator for σ^2 .

0.3 $SSTR/\sigma^2 \sim \text{Chi square } (df = k-1)$.

(Homework)

Test $H_0 : \mu_1 = \cdots = \mu_k$ v.s. μ_j are not the same.

Case I. when σ^2 is known.

Reject H_0 if $SSTR/\sigma^2 \geq \chi_{1-\alpha, k-1}^2$.

Case II. when σ^2 is unknown.

.....

Sum of Squared Errors: SSE

1. Sum of squared error:

$$\begin{aligned}SSE &:= \sum_{j=1}^k \sum_{i=1}^{n_j} (Y_{ij} - \bar{Y}_{\cdot j})^2 \\&= \sum_{j=1}^k (n_j - 1) \left[\frac{1}{n_j - 1} \sum_{i=1}^{n_j} (Y_{ij} - \bar{Y}_{\cdot j})^2 \right] \\&= \sum_{j=1}^k (n_j - 1) S_j^2\end{aligned}$$

2. Pooled variance S_p^2 :

$$S_p^2 = \frac{SSE}{\sum_{j=1}^k (n_j - 1)} = \frac{SSE}{n - k}$$

Mean square of error $MSE = S_p^2$

Notice that

1. $(n_j - 1)S_j^2/\sigma^2 \sim \text{Chi square } (df = n_j - 1)$
2. S_j^2 's are independent
3. $SSE/\sigma^2 = (n - k)S_p^2/\sigma^2 = \sum_{j=1}^k (n_j - 1)S_j^2/\sigma^2$,
Sum of independent of Chi squares

↓

Thm. No matter $H_0 : \mu_1 = \dots = \mu_k$ is true or not

- a. $SSE/\sigma^2 = (n - k)S_p^2/\sigma^2 \sim \text{Chi square } (df = \sum_{j=1}^k (n_j - 1) = n - k)$
- b. $SSTR \perp SSE$.

Proof. We have shown part (a). Part (b) is trickier. Indeed, both parts are a consequence of **Cochran's theorem**¹ ... □

¹https://en.wikipedia.org/wiki/Cochran%27s_theorem

Let's see two special cases of

Thm. No matter $H_0 : \mu_1 = \dots = \mu_k$ is true or not

a. $SSE/\sigma^2 = (n - k)S_p^2/\sigma^2 \sim \text{Chi square } (df = \sum_{j=1}^k (n_j - 1) = n - k)$

b. $SSTR \perp SSE$.

Cases

1. $k = 1$, one sample case, S_p^2 is sample variance

Chapter 7

a. $(n - 1)S^2/\sigma^2 \sim \chi^2(n - 1)$

b. $SSTR \equiv 0$

2. $k = 2$, two sample case

Chapter 9

a. $(n - 2)S_p^2/\sigma^2 \sim \chi^2(n - 2)$

b. $\bar{X} - \bar{Y} \perp S_p^2 \iff SSTR \perp SSE$

Under $H_0 : \mu_1 = \dots = \mu_k$

1. $SSTR/\sigma^2 \sim \chi^2(k-1)$

2. $SSE/\sigma^2 \sim \chi^2(n-k)$

3. $SSTR \perp SSE$

$$\implies F = \frac{SSTR/(k-1)}{SSE/(n-k)} \sim F(df_1 = k-1, df_2 = n-k)$$

Reject H_0 if $F \geq F_{1-\alpha, k-1, n-k}$

Total Sum of Squares: SSTOT

$$SSTOT = SSE + SSTR$$

$$SSTOT := \sum_{j=1}^k \sum_{i=1}^{n_j} (Y_{ij} - \bar{Y}_{..})^2$$

||

$$\sum_{j=1}^k \sum_{i=1}^{n_j} \left[(Y_{ij} - \bar{Y}_{j.}) + (\bar{Y}_{j.} - \bar{Y}_{..}) \right]^2$$

||

$$\sum_{j=1}^k \sum_{i=1}^{n_j} (Y_{ij} - \bar{Y}_{j.})^2 + 2 \sum_{j=1}^k \sum_{i=1}^{n_j} (Y_{ij} - \bar{Y}_{j.}) (\bar{Y}_{j.} - \bar{Y}_{..}) + \sum_{j=1}^k \sum_{i=1}^{n_j} (\bar{Y}_{j.} - \bar{Y}_{..})^2$$

||

$$\sum_{j=1}^k \sum_{i=1}^{n_j} (Y_{ij} - \bar{Y}_{j.})^2 + 2 \sum_{j=1}^k (\bar{Y}_{j.} - \bar{Y}_{..}) \sum_{i=1}^{n_j} (Y_{ij} - \bar{Y}_{j.}) + \sum_{j=1}^k n_j (\bar{Y}_{j.} - \bar{Y}_{..})^2$$

||

$$SSE + 0 + SSTR$$

$$SSTOT = SSE + SSTR$$

↓

$$\frac{SSTOT}{\sigma^2} = \frac{SSE}{\sigma^2} + \frac{SSTR}{\sigma^2}$$

∝

∝

∝

$$\chi^2(n-1) \quad \chi^2(n-k) \quad \perp \quad \chi^2(k-1)$$

Under H_0

✓

Under H_0

One-way ANOVA Table

Source of Variance	Degree of Freedom (df)	Sum Square (SS)	Mean Square (MS)	F-ratio
Between Groups (Treatment)	k-1	$SSB = \sum_{j=1}^k \left(\frac{T_j^2}{n_j} \right) - \frac{T^2}{n}$ $SSB = \sum_{j=1}^k n_j (\bar{X}_j - \bar{X}_t)^2$	$MSB = \frac{SSB}{k-1}$	$F = \frac{MSB}{MSW}$
Within Groups (Error)	n-k	$SSW = \sum_{j=1}^k \sum_{i=1}^n X_{ij}^2 - \sum_{j=1}^k \left(\frac{T_j^2}{n_j} \right)$ $SSW = \sum_{j=1}^k \sum_{i=1}^n (X_{ij} - \bar{X}_j)^2$	$MSW = \frac{SSW}{n-k}$	
Total	n-1	$SST = \sum_{j=1}^k \sum_{i=1}^n X_{ij}^2 - \frac{T^2}{n}$ $SST = \sum_{j=1}^k \sum_{i=1}^n (X_{ij} - \bar{X}_t)^2$		

- $SST = SSB + SSW$

k: number of groups n: number of samples
df: degree of freedom

Source	df	SS	MS	F	P
Treatment	k - 1	SSTR	MSTR	$\frac{MSTR}{MSE}$	$P(F_{k-1, n-k} \geq \text{observed } F)$
Error	n - k	SSE	MSE		
Total	n - 1	SSTOT			

Common notation

d.f.

k-1 Error sum of squares
Mean square of error
(Pooled sample variance)

$$SSE = SSW = SS_{within}$$
$$MSE = MSW = MS_{within} = S_p^2$$

n-k Treatment sum of squares
Mean square of treatment

$$SSTR = SSB = SS_{between}$$
$$MSTR = MSB = MS_{between}$$

n-1 Total sum of squares:

$$SST = SSTOT$$

One way ANOVA v.s. Two sample t -test

Let X_1, \dots, X_n and Y_1, \dots, Y_m be samples from $N(\mu_X, \sigma^2)$ and $N(\mu_Y, \sigma^2)$, respectively.

Recall

$$1. SSTR/\sigma^2 = \frac{(\bar{X} - \bar{Y})^2}{\sigma^2 \left(\frac{1}{n} + \frac{1}{m}\right)} \sim \chi^2(1)$$

$$2. SSE/\sigma^2 = (n + m - 2)S_p^2/\sigma^2 \sim \chi^2(n + m - 2)$$

$$\implies F = \frac{SSTR/1}{SSE/(n + m - 2)} = \frac{(\bar{X} - \bar{Y})^2}{S_p^2 \left(\frac{1}{n} + \frac{1}{m}\right)} \sim F(df_1 = 1, df_2 = n + m - 2)$$

||
 T^2

$$\implies \alpha = \mathbb{P}(|T| \geq t_{\alpha/2, n+m-2}) = \mathbb{P}(T^2 \geq t_{\alpha/2, n+m-2}^2) = \mathbb{P}(F \geq F_{1-\alpha, 1, n+m-2})$$

Equivalent!

E.g. 1 Study the relation between smoking and heart rates.

Generations of athletes have been cautioned that cigarette smoking impedes performance. One measure of the truth of that warning is the effect of smoking on heart rate. In one study, six nonsmokers, six light smokers, six moderate smokers, and six heavy smokers each engaged in sustained physical exercise. Table 8.1.1 lists their heart rates after they had rested for three minutes.

	Nonsmokers	Light Smokers	Moderate Smokers	Heavy Smokers
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	65	66	79	84
<i>Averages:</i>	62.3	63.2	71.7	81.7

Show whether smoking affects heart rates at $\alpha = 0.05$.

Sol. Let μ_1, \dots, μ_4 be the true heart rates.

Test $H_0 : \mu_0 = \dots = \mu_4$ or not.

Critical region:

Let $\alpha = 0.05$. For these data, $k = 4$ and $n = 24$, so $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$ should be rejected if

$$F = \frac{SSTR/(4-1)}{SSE/(24-4)} \geq F_{1-0.05, 4-1, 24-4} = F_{.95, 3, 20} = 3.10$$

(see Figure 12.2.2).

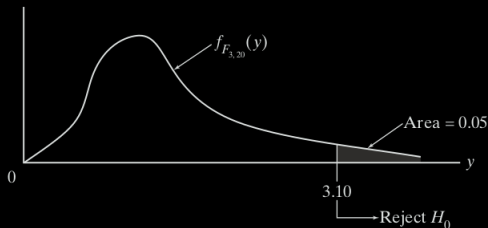


Figure 12.2.2

Table 12.2.1

	Non smokers	Light Smokers	Moderate Smokers	Heavy Smokers
	69	55	66	91
	52	60	81	72
	71	78	70	81
	58	58	77	67
	59	62	57	95
	65	66	79	84
T_j	374	379	430	490
\bar{Y}_j	62.3	63.2	71.7	81.7

The overall sample mean, $\bar{Y}_{..}$, is given by

$$\begin{aligned}\bar{Y}_{..} &= \frac{1}{n} \sum_{j=1}^k T_j = \frac{374 + 379 + 430 + 490}{24} \\ &= 69.7\end{aligned}$$

Therefore,

$$\begin{aligned}SSR &= \sum_{j=1}^4 n_j (\bar{Y}_{.j} - \bar{Y}_{..})^2 = 6[(62.3 - 69.7)^2 + \dots + (81.7 - 69.7)^2] \\ &= 1464.125\end{aligned}$$

Similarly,

$$\begin{aligned}SSE &= \sum_{j=1}^4 \sum_{i=1}^6 (Y_{ij} - \bar{Y}_{.j})^2 = [(69 - 62.3)^2 + \dots + (65 - 62.3)^2] \\ &\quad + \dots + [(91 - 81.7)^2 + \dots + (84 - 81.7)^2] \\ &= 1594.833\end{aligned}$$

The observed test statistic, then, equals 6.12:

$$F = \frac{1464.125/(4 - 1)}{1594.833/(24 - 4)} = 6.12$$

Since $6.12 > F_{.95,3,20} = 3.10$, $H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4$ should be rejected. These data support the contention that smoking influences a person's heart rate.

Figure 12.2.3 shows the analysis of these data summarized in the ANOVA table format. Notice that the small P -value ($= 0.004$) is consistent with the conclusion that H_0 should be rejected.

Source	df	SS	MS	F	P
Treatment	3	1464.125	488.04	6.12	0.004
Error	20	1594.833	79.74		
Total	23	3058.958			

Figure 12.2.3



```
1 > Input <-c("
2 + rates group
3 + 69 non
4 + 52 non
5 + 71 non
6 + 58 non
7 + 59 non
8 + 65 non
9 + 55 light
10 + 60 light
11 + 78 light
12 + 58 light
13 + 62 light
14 + 66 light
15 + 66 moderate
16 + 81 moderate
17 + 70 moderate
18 + 77 moderate
19 + 57 moderate
20 + 79 moderate
21 + 91 heavy
22 + 72 heavy
23 + 81 heavy
24 + 67 heavy
25 + 95 heavy
26 + 84 heavy
27 + ")
28 > Data = read.table(textConnection(
      Input),
29 + header=TRUE)
```

```
1 > Data
2   rates  group
3 1    69   non
4 2    52   non
5 3    71   non
6 4    58   non
7 5    59   non
8 6    65   non
9 7    55  light
10 8    60  light
11 9    78  light
12 10   58  light
13 11   62  light
14 12   66  light
15 13   66 moderate
16 14   81 moderate
17 15   70 moderate
18 16   77 moderate
19 17   57 moderate
20 18   79 moderate
21 19   91  heavy
22 20   72  heavy
23 21   81  heavy
24 22   67  heavy
25 23   95  heavy
26 24   84  heavy
```

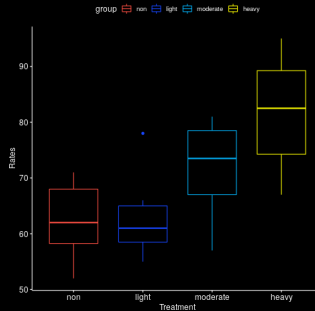
```
1 > # Check the levels
2 > levels(Data$group)
3 [1] "heavy" "light" "moderate" "non"
4 > # Order the groups
5 > Data$group <- ordered(Data$group, levels = c("non", "light", "moderate", "heavy")
6 )
7 > levels(Data$group)
8 [1] "non" "light" "moderate" "heavy"
```

```
1 > # Compute summary statistics by groups
2 > # including count, mean, sd:
3 > library(dplyr) # a grammar of data manipulation
4 > group_by(Data, group) %>%
5 + summarise(
6 +   count = n(),
7 +   mean = mean(rates, na.rm = TRUE),
8 +   sd = sd(rates, na.rm = TRUE)
9 + )
10 # A tibble: 4 x 4
11   group   count mean  sd
12   <ord> <int> <dbl> <dbl>
13 1 non         6  62.3  7.26
14 2 light        6  63.2  8.16
15 3 moderate    6  71.7  9.16
16 4 heavy        6  81.7 10.8
```

```

1 # Box plots
2 # ++++++
3 # Plot rates by group and color by group
4 library(ggpubr)
5 png("Case_12-2-1-ggboxplot.png")
6 ggboxplot(Data, x = "group", y = "rates",
7           color = "group", palette = c("#00AFBB", "#E7B800", "#FC4E07", "blue"),
8           order = c("non", "light", "moderate", "heavy"),
9           ylab = "Rates", xlab = "Treatment")
10 dev.off()

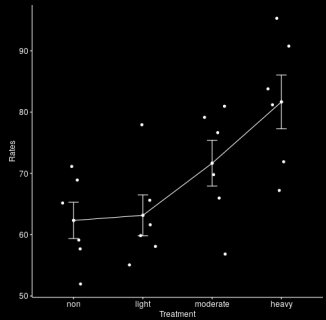
```




```

1 # Mean plots
2 # ++++++
3 # Plot rates by group
4 # Add error bars: mean_se
5 # (other values include: mean_sd, mean_ci, median_iqr, ....)
6 png("Case_12-2-1-ggline.png")
7 library(ggpubr)
8 ggline(Data, x = "group", y = "rates",
9         add = c("mean_se", "jitter"),
10        order = c("non", "light", "moderate", "heavy"),
11        ylab = "Rates", xlab = "Treatment")
12 dev.off()

```



```

1 > # Compute the analysis of variance
2 > res.aov <- aov(rates ~ group, data = Data)
3 > # Summary of the analysis
4 > summary(res.aov)
5           Df Sum Sq Mean Sq F value Pr(>F)
6 group      3  1464  488.0    6.12 0.00398 **
7 Residuals 20  1595   79.7
8 ---
9 Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

```

```

1 > # Tukey multiple multiple-comparisons
2 > TukeyHSD(res.aov)
3   Tukey multiple comparisons of means
4     95% family-wise confidence level
5
6 Fit: aov(formula = rates ~ group, data = Data)
7
8 $group
9           diff          lwr          upr          p adj
10 light-non    0.8333333 -13.596955  15.26362  0.9984448
11 moderate-non 9.3333333  -5.096955  23.76362  0.2978123
12 heavy-non   19.3333333  4.903045  33.76362  0.0063659
13 moderate-light 8.5000000  -5.930289  22.93029  0.3755571
14 heavy-light  18.5000000  4.069711  32.93029  0.0091463
15 heavy-moderate 10.0000000  -4.430289  24.43029  0.2438158

```

1. diff: difference between means of the two groups
2. lwr, upr: the lower and the upper end point of the C.I. at 95% (default)
3. p adj: p-value after adjustment for the multiple comparisons

Inferences

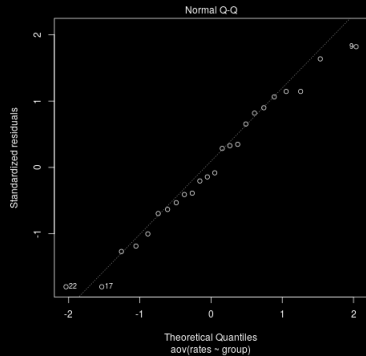
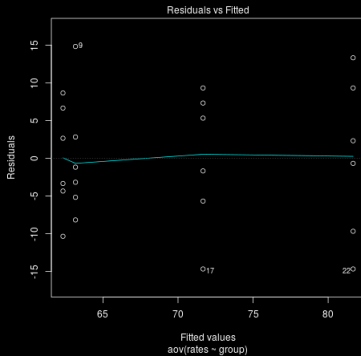
if p-value \leq 0.05 \iff if zero is in the C.I.

```

1 > # Or one may use multcomp package or multiple comparisons
2 > library(multcomp)
3 > summary(glht(res.aov, linfct = mcp(group = "Tukey")))
4
5 Simultaneous Tests for General Linear Hypotheses
6
7 Multiple Comparisons of Means: Tukey Contrasts
8
9
10 Fit: aov(formula = rates ~ group, data = Data)
11
12 Linear Hypotheses:
13             Estimate Std. Error t value Pr(>|t|)
14 light - non == 0    0.8333    5.1556  0.162 0.99844
15 moderate - non == 0  9.3333    5.1556  1.810 0.29776
16 heavy - non == 0   19.3333    5.1556  3.750 0.00629 **
17 moderate - light == 0 8.5000    5.1556  1.649 0.37544
18 heavy - light == 0  18.5000    5.1556  3.588 0.00901 **
19 heavy - moderate == 0 10.0000    5.1556  1.940 0.24382
20 ---
21 Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
22 (Adjusted p values reported -- single-step method)

```

```
1 # Check ANOVA assumptions: test validity?
2 # diagnostic plots
3 layout(matrix(c(1,2),1,2)) # optional 1x2 graphs/page
4 plot(res.aov,c(1,2))
```



1. Residuals vs Fitted: test homogeneity of variances

One can also use Levene's test for this purpose:

```
1 > # Use Levene's test to test homogeneity of variances
2 > library(car)
3 > leveneTest(rates ~ group, data = Data)
4 Levene's Test for Homogeneity of Variance (center = median
5   )
6   Df F value Pr(>F)
7   group 3  0.3885 0.7625
8   20
```

2. Normal Q-Q plot: Test normality. (It should be close to diagonal line.)

One can also use Shapiro-Wilk test:

```
1 # Extract the residuals
2 > aov_residuals <- residuals(object = res.aov )
3 > # Run Shapiro-Wilk test
4 > shapiro.test(x = aov_residuals )
5
6 Shapiro-Wilk normality test
7
8 data:  aov_residuals
9 W = 0.9741, p-value = 0.7677
```

Non-parametric alternative to one-way ANOVA test

```
1 > # Non-parametric alternative to one-way ANOVA test
2 > # a non-parametric alternative to one-way ANOVA
3 > # is Kruskal-Wallis rank sum test, which can be
4 > # used when ANNOVA assumptions are not met.
5 > kruskal.test(rates ~ group, data = Data)
6
7   Kruskal-Wallis rank sum test
8
9   data: rates by group
10  Kruskal-Wallis chi-squared = 10.729, df = 3, p-value =
    0.01329
```

See Section 4 of Chapter 14 for more details.

Chapter 12. The Analysis of Variance

§ 12.1 Introduction

§ 12.2 The F Test

§ 12.3 Multiple Comparisons: Turkey's Method

§ 12.4 Testing Subhypotheses with Contrasts



1. John Wilder Tukey (June 16, 1915 – July 26, 2000) was an American mathematician best known for development of the Fast Fourier Transform (FFT) algorithm and box plot.
2. The Tukey range test, the Tukey lambda distribution, the Tukey test of additivity, and the Teichmüller-Tukey lemma all bear his name.
3. He is also credited with coining the term 'bit'.

https://en.wikipedia.org/wiki/John_Tukey

$N(\mu_1, \sigma^2)$	$N(\mu_2, \sigma^2)$	\cdots	$N(\mu_k, \sigma^2)$
Y_{11}	Y_{12}	\cdots	Y_{1k}
Y_{21}	Y_{22}	\cdots	Y_{2k}
\vdots	\vdots	\vdots	\vdots
Y_{r1}	Y_{r2}	\cdots	Y_{rk}

Goal For any $i \neq j$, test

$$H_0 : \mu_i = \mu_j \quad \text{v.s.} \quad H_1 : \mu_i \neq \mu_j$$

at the α level of significance defined as

$$\mathbb{P} \left(\bigcup_{j=1}^{\binom{k}{2}} E_j \right) = \alpha$$

where there are $\binom{k}{2}$ pairs, and E_j is the event of making a type I error for the j -th pair.

Goal' Simultaneous C.I.'s for $\binom{k}{2}$ pairs of means:

Given α , find I_{ij} , the C.I. for $\mu_i - \mu_j$ (with $i, j = 1, \dots, k$ and $i \neq j$), s.t.

$$\mathbb{P}(\mu_i - \mu_j \in I_{ij}, \forall i, j = 1, \dots, k, i \neq j) = 1 - \alpha.$$

??? Why not the standard pair-wise two-sample t-test?

Suppose $\mathbb{P}(E_j) = \alpha_*$. Then

$$\alpha = \mathbb{P}\left(\bigcup_{j=1}^k E_j\right) = 1 - \mathbb{P}\left(\bigcap_{j=1}^k E_j^c\right) \approx 1 - \prod_{j=1}^k \mathbb{P}(E_j^c) = 1 - (1 - \alpha_*)^{\binom{k}{2}}$$

Hence,

$$\alpha_* \approx 1 - (1 - \alpha)^{1/\binom{k}{2}}$$

E.g., $\alpha = 0.05$

k	5	8	100
α_*	0.0051162	0.001830	0.00001036

Bonferroni's method

— A straightforward method

$$\mathbb{P}(\mu_i - \mu_j \in I_{ij}, \forall i \neq j)$$

||

$$\mathbb{P}\left(\bigcap_{i \neq j} \mu_i - \mu_j \in I_{ij}\right)$$

||

$$1 - \mathbb{P}\left(\bigcup_{i \neq j} \mu_i - \mu_j \notin I_{ij}\right)$$

∨

$$1 - \sum_{i \neq j} \mathbb{P}(\mu_i - \mu_j \notin I_{ij})$$

||

$$1 - \binom{k}{2} \alpha_*$$

1. If we choose $\alpha_* = \alpha / \binom{k}{2}$,
2. let I_{ij} be the $(1 - \alpha_*)100\%$ C.I. $i \neq j$

↓

$$\mathbb{P}(\mu_i - \mu_j \in I_{ij}, \forall i \neq j)$$

∨

$$1 - \binom{k}{2} \alpha_*$$

||

$$1 - \alpha.$$

Remark This is an approximation. The resulting C.I. are in general too wide.

The exact, and much more precise, solution is given by J.W. Turkey.

One can also construct simultaneous C.I. for all possible linear combinations of the parameters $\sum_{j=1}^k c_j \mu_j$, this can be achieved by **Scheffé's method**. A simple version is given in §12.4.

Tukey's HSD (honestly significant difference) test

Let's construct $(1 - \alpha)100\%$ C.I.'s simultaneously for all pairs.

$$\mathbb{P} \left(\left| (\bar{Y}_{.i} - \mu_i) - (\bar{Y}_{.j} - \mu_j) \right| \leq \mathcal{E}, \quad \forall i \neq j \right) = 1 - \alpha$$

||

$$\mathbb{P} \left(\max_i (\bar{Y}_{.i} - \mu_i) - \min_j (\bar{Y}_{.j} - \mu_j) \leq \mathcal{E} \right)$$

||

$$\mathbb{P} \left(\max_i \bar{Y}_{.i} - \min_j \bar{Y}_{.j} \leq \mathcal{E} \right)$$

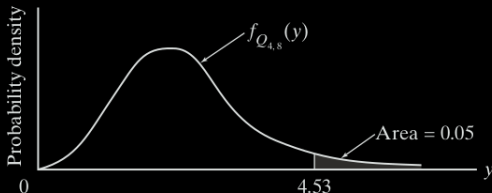
\implies Needs to study ...

Def. Let W_1, \dots, W_k be k i.i.d. r.v.'s from $N(\mu, \sigma^2)$. Let R denote their range:

$$R = \max_i W_i - \min_i W_i.$$

Let S^2 be an unbiased estimator for σ^2 independent of the W_i 's and based on ν df. Define the **Studentized range**, $Q_{k,\nu}$, to be the ratio:

$$Q_{k,\nu} := \frac{R}{S}.$$

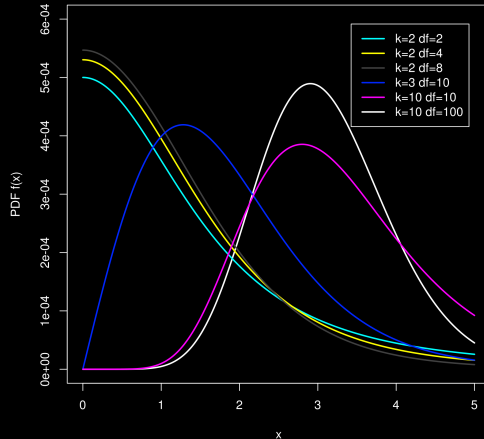


Remark 0.1 We need $R \perp S$ to mimic Student's t-distribution.
 0.2 In the following $\nu = n - k = rk - k = r(k - 1)$.

$Q_{k,\nu} \sim$ **Studentized range distribution** with parameters k and ν .

k : number of groups.

ν : degrees of freedom.



Let's find one example that all requirements of the $Q_{k,\nu}$ are satisfied.

1. Take $W_j = \bar{Y}_{.j} - \mu_j, j = 1, \dots, k \implies W_j \sim N(0, \sigma^2/r)$.

2. MSE or the pooled variance S_p^2 is an unbiased estimator for σ^2 is $\perp \{\bar{Y}_{.j}\}_{j=1, \dots, k}$, hence $\perp \{W_j\}_{j=1, \dots, k}$ MSE/r
 σ^2/r

3. df of MSE is equal to $n - k = kr - k = k(r - 1)$.

$$\implies \frac{\max_i W_i - \min_j W_j}{\sqrt{MSE/r}} \sim \text{Studentized range distribution}(k, rk - k)$$

$$\mathbb{P} \left(\frac{\max_i W_i - \min_j W_j}{\sqrt{MSE/r}} \leq Q_{\alpha, k, rk-k} \right) = 1 - \alpha$$

||

$$\mathbb{P} \left(\max_i W_i - \min_j W_j \leq \frac{Q_{\alpha, k, rk-k}}{\sqrt{r}} \sqrt{MSE} \right)$$

||

$$\mathbb{P} \left(|W_i - W_j| \leq \frac{Q_{\alpha, k, rk-k}}{\sqrt{r}} \sqrt{MSE}, \forall i \neq j \right)$$

||

$$\mathbb{P} \left(\left| (\bar{Y}_{.i} - \bar{Y}_{.j}) - (\mu_i - \mu_j) \right| \leq \frac{Q_{\alpha, k, rk-k}}{\sqrt{r}} \sqrt{MSE}, \forall i \neq j \right)$$

||

$$\mathbb{P} \left(\bar{Y}_{.i} - \bar{Y}_{.j} - \frac{Q_{\alpha, k, rk-k}}{\sqrt{r}} \sqrt{MSE} \leq \mu_i - \mu_j \leq \bar{Y}_{.i} - \bar{Y}_{.j} + \frac{Q_{\alpha, k, rk-k}}{\sqrt{r}} \sqrt{MSE}, \forall i \neq j \right)$$

Therefore, for all $i \neq j$, the $100(1 - \alpha)\%$ C.I. for $\mu_i - \mu_j$ is

$$\bar{Y}_{.i} - \bar{Y}_{.j} \pm \frac{Q_{\alpha, k, rk-k}}{\sqrt{2}} \sqrt{MSE} \sqrt{\frac{2}{r}}$$

To test $H_0 : \mu_i = \mu_j$ for specific $i \neq j$, reject H_0 in favor of $H_1 : \mu_i \neq \mu_j$ if the C.I. does NOT contain 0, at the α level of significance. \square

Note: When sample sizes are not equal, use the **Tukey-Kramer method**:

$$\bar{Y}_{.i} - \bar{Y}_{.j} \pm \frac{Q_{\alpha, k, rk-k}}{\sqrt{2}} \sqrt{MSE} \sqrt{\frac{1}{r_i} + \frac{1}{r_j}}$$

E.g. 2 A certain fraction of antibiotics injected into the bloodstream are “bound” to serum proteins. This phenomenon bears directly on the effectiveness of the medication, because the binding decreases the systemic uptake of the drug. Table below lists the binding percentages in bovine serum measured for five widely prescribed antibiotics. Which antibiotics have similar binding properties, and which are different?

	Penicillin G	Tetra- cycline	Strepto- mycin	Erythro- mycin	Chloram- phenicol
	29.6	27.3	5.8	21.6	29.2
	24.3	32.6	6.2	17.4	32.8
	28.5	30.8	11.0	18.3	25.0
	32.0	34.8	8.3	19.0	24.2
T_j	114.4	125.5	31.3	76.3	111.2
\bar{Y}_j	28.6	31.4	7.8	19.1	27.8

To answer that question requires that we make all $\binom{5}{2} = 10$ pairwise comparisons of μ_i versus μ_j . First, MSE must be computed. From the entries in Table 12.3.1,

$$SSE = \sum_{j=1}^5 \sum_{i=1}^4 (Y_{ij} - \bar{Y}_{.j})^2 = 135.83$$

so $MSE = 135.83/(20 - 5) = 9.06$. Let $\alpha = 0.05$. Since $n - k = 20 - 5 = 15$, the appropriate cutoff from the studentized range distribution is $Q_{.05,5,15} = 4.37$. Therefore, $D = 4.37/\sqrt{4} = 2.185$ and $D\sqrt{MSE} = 6.58$.

Table 12.3.2

Pairwise Difference	$\bar{Y}_{.i} - \bar{Y}_{.j}$	Tukey Interval	Conclusion
$\mu_1 - \mu_2$	-2.8	(-9.38, 3.78)	NS
$\mu_1 - \mu_3$	20.8	(14.22, 27.38)	Reject
$\mu_1 - \mu_4$	9.5	(2.92, 16.08)	Reject
$\mu_1 - \mu_5$	0.8	(-5.78, 7.38)	NS
$\mu_2 - \mu_3$	23.6	(17.02, 30.18)	Reject
$\mu_2 - \mu_4$	12.3	(5.72, 18.88)	Reject
$\mu_2 - \mu_5$	3.6	(-2.98, 10.18)	NS
$\mu_3 - \mu_4$	-11.3	(-17.88, -4.72)	Reject
$\mu_3 - \mu_5$	-20.0	(-26.58, -13.42)	Reject
$\mu_4 - \mu_5$	-8.7	(-15.28, -2.12)	Reject

```

1 > # Case Study 12.3.1
2 > # Input data first
3 > Input <- c("
4 + rates group
5 + 29.6 M1
6 + 24.3 M1
7 + 28.5 M1
8 + 32.0 M1
9 + 27.3 M2
10 + 32.6 M2
11 + 30.8 M2
12 + 34.8 M2
13 + 5.8 M3
14 + 6.2 M3
15 + 11.0 M3
16 + 8.3 M3
17 + 21.6 M4
18 + 17.4 M4
19 + 18.3 M4
20 + 19.0 M4
21 + 29.2 M5
22 + 32.8 M5
23 + 25.0 M5
24 + 24.2 M5
25 + ")
26 > Data = read.table(
27   textConnection(Input),
28   +   header=TRUE)

```

```

1 > # Compute one-way ANOVA test
2 > res.aov <- aov(rates ~ group, data = Data)
3 > # Summary of the analysis
4 > summary(res.aov)
5           Df Sum Sq Mean Sq F value Pr(>F)
6 group      4 1480.8   370.2   40.88 6.74e-08 ***
7 Residuals 15  135.8     9.1
8 ---
9 Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

```

```

1 > # Tukey multiple pairwise-comparisons
2 > TukeyHSD(res.aov)
3 Tukey multiple comparisons of means
4 95% family-wise confidence level
5
6 Fit: aov(formula = rates ~ group, data = Data)
7
8 $group
9      diff      lwr      upr    p adj
10 M2-M1  2.775 -3.795401  9.345401 0.6928357
11 M3-M1 -20.775 -27.345401 -14.204599 0.0000006
12 M4-M1  -9.525 -16.095401  -2.954599 0.0034588
13 M5-M1  -0.800  -7.370401  5.770401 0.9952758
14 M3-M2 -23.550 -30.120401 -16.979599 0.0000001
15 M4-M2 -12.300 -18.870401  -5.729599 0.0003007
16 M5-M2  -3.575 -10.145401  2.995401 0.4737713
17 M4-M3  11.250  4.679599  17.820401 0.0007429
18 M5-M3  19.975 13.404599  26.545401 0.0000010
19 M5-M4  8.725  2.154599  15.295401 0.0071611

```



```

1 > round(TukeyHSD(res.aov)$group,2)
2           diff      lwr      upr  p adj
3 M2-M1  2.78    -3.80    9.35  0.69
4 M3-M1 -20.77  -27.35  -14.20  0.00
5 M4-M1 -9.52   -16.10   -2.95  0.00
6 M5-M1 -0.80   -7.37    5.77  1.00
7 M3-M2 -23.55  -30.12  -16.98  0.00
8 M4-M2 -12.30  -18.87   -5.73  0.00
9 M5-M2 -3.58  -10.15    3.00  0.47
10 M4-M3 11.25    4.68   17.82  0.00
11 M5-M3 19.97   13.40   26.55  0.00
12 M5-M4  8.73    2.15   15.30  0.01
13 ---
14 Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 '
      ' 1
15 (Adjusted p values reported -- single-step
      method)

```

Table 12.3.2

Pairwise Difference	$\bar{Y}_i - \bar{Y}_j$	Tukey Interval	Conclusion
$\mu_1 - \mu_2$	-2.8	(-9.38, 3.78)	NS
$\mu_1 - \mu_3$	20.8	(14.22, 27.38)	Reject
$\mu_1 - \mu_4$	9.5	(2.92, 16.08)	Reject
$\mu_1 - \mu_5$	0.8	(-5.78, 7.38)	NS
$\mu_2 - \mu_3$	23.6	(17.02, 30.18)	Reject
$\mu_2 - \mu_4$	12.3	(5.72, 18.88)	Reject
$\mu_2 - \mu_5$	3.6	(-2.98, 10.18)	NS
$\mu_3 - \mu_4$	-11.3	(-17.88, -4.72)	Reject
$\mu_3 - \mu_5$	-20.0	(-26.58, -13.42)	Reject
$\mu_4 - \mu_5$	8.7	(2.15, 15.30)	Reject

```

1 > # Or one may use multcomp package or multiple
    comparisons
2 > library(multcomp)
3 > summary(glht(res.aov, linfct = mcp(group = "Tukey")))
4
5 Simultaneous Tests for General Linear Hypotheses
6
7 Multiple Comparisons of Means: Tukey Contrasts
8
9
10 Fit: aov(formula = rates ~ group, data = Data)
11
12 Linear Hypotheses:
13 Estimate Std. Error t value Pr(>|t|)
14 M2 - M1 == 0 2.775 2.128 1.304 0.69283
15 M3 - M1 == 0 -20.775 2.128 -9.764 < 0.001 ***
16 M4 - M1 == 0 -9.525 2.128 -4.477 0.00348 **
17 M5 - M1 == 0 -0.800 2.128 -0.376 0.99528
18 M3 - M2 == 0 -23.550 2.128 -11.068 < 0.001 ***
19 M4 - M2 == 0 -12.300 2.128 -5.781 < 0.001 ***
20 M5 - M2 == 0 -3.575 2.128 -1.680 0.47374
21 M4 - M3 == 0 11.250 2.128 5.287 < 0.001 ***
22 M5 - M3 == 0 19.975 2.128 9.388 < 0.001 ***
23 M5 - M4 == 0 8.725 2.128 4.101 0.00717 **
24 ---
25 Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
26 (Adjusted p values reported -- single-step method)

```

		Estimate	Std. Error	t value	Pr(> t)
1					
2	M2 - M1 == 0	2.775	2.128	1.304	0.69283
3	M3 - M1 == 0	-20.775	2.128	-9.764	< 0.001 ***
4	M4 - M1 == 0	-9.525	2.128	-4.477	0.00348 **
5	M5 - M1 == 0	-0.800	2.128	-0.376	0.99527
6	M3 - M2 == 0	-23.550	2.128	-11.068	< 0.001 ***
7	M4 - M2 == 0	-12.300	2.128	-5.781	< 0.001 ***
8	M5 - M2 == 0	-3.575	2.128	-1.680	0.47371
9	M4 - M3 == 0	11.250	2.128	5.287	< 0.001 ***
10	M5 - M3 == 0	19.975	2.128	9.388	< 0.001 ***
11	M5 - M4 == 0	8.725	2.128	4.101	0.00719 **

Table 12.3.2

Pairwise Difference	$\bar{Y}_i - \bar{Y}_j$	Tukey Interval	Conclusion
$\mu_1 - \mu_2$	-2.8	(-9.38, 3.78)	NS
$\mu_1 - \mu_3$	20.8	(14.22, 27.38)	Reject
$\mu_1 - \mu_4$	9.5	(2.92, 16.08)	Reject
$\mu_1 - \mu_5$	0.8	(-5.78, 7.38)	NS
$\mu_2 - \mu_3$	23.6	(17.02, 30.18)	Reject
$\mu_2 - \mu_4$	12.3	(5.72, 18.88)	Reject
$\mu_2 - \mu_5$	3.6	(-2.98, 10.18)	NS
$\mu_3 - \mu_4$	-11.3	(-17.88, -4.72)	Reject
$\mu_3 - \mu_5$	-20.0	(-26.58, -13.42)	Reject
$\mu_4 - \mu_5$	-8.7	(-15.28, -2.12)	Reject

Two more examples of ANOVA using R

E.g. 1 <http://www.sthda.com/english/wiki/one-way-anova-test-in-r>

E.g. 2 <https://datascienceplus.com/one-way-anova-in-r/>

Chapter 12. The Analysis of Variance

§ 12.1 Introduction

§ 12.2 The F Test

§ 12.3 Multiple Comparisons: Turkey's Method

§ 12.4 Testing Subhypotheses with Contrasts

