# Math 362: Mathematical Statistics II

Le Chen
le.chen@emory.edu

Emory University
Atlanta, GA

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# Chapter 7. Inference Based on The Normal Distribution

§ 7.1 Introduction

§ 7.2 Comparing $\frac{\overline{Y}-\mu}{\sigma/\sqrt{n}}$ and $\frac{\overline{Y}-\mu}{S/\sqrt{n}}$

§ 7.3 Deriving the Distribution of $\frac{\overline{Y}-\mu}{S/\sqrt{n}}$

§ 7.4 Drawing Inferences About $\mu$

§ 7.5 Drawing Inferences About $\sigma^2$

# Plan

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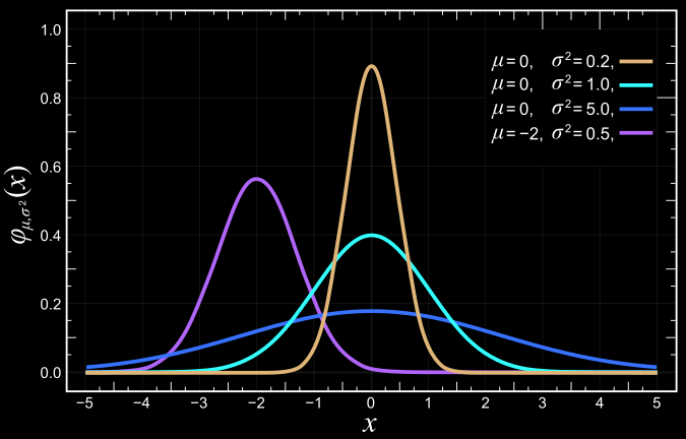
Carl Friedrich Gauss discovered the normal distribution in 1809 as a way to rationalize the method of least squares.

(1777-1855)

Marquis de Laplace proved the central limit theorem in 1810, consolidating the importance of the normal distribution in statistics.

(1749-1827)




| | |
|---|---|
| **Notation** | $\mathcal{N}(\mu, \sigma^2)$ |
| **Parameters** | $\mu \in \mathbb{R}$ = mean (location)<br>$\sigma^2 > 0$ = variance (squared scale) |
| **Support** | $x \in \mathbb{R}$ |
| **PDF** | $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ |
| **CDF** | $\frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$ |
| **Quantile** | $\mu + \sigma\sqrt{2}\,\operatorname{erf}^{-1}(2p-1)$ |
| **Mean** | $\mu$ |
| **Median** | $\mu$ |
| **Mode** | $\mu$ |
| **Variance** | $\sigma^2$ |
| **MAD** | $\sigma\sqrt{2/\pi}$ |
| **Skewness** | $0$ |
| **Ex. kurtosis** | $0$ |
| **Entropy** | $\frac{1}{2}\log(2\pi e\sigma^2)$ |
| **MGF** | $\exp(\mu t + \sigma^2 t^2/2)$ |
| **CF** | $\exp(i\mu t - \sigma^2 t^2/2)$ |
| **Fisher information** | $\mathcal{I}(\mu, \sigma) = \begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 2/\sigma^2 \end{pmatrix}$<br>$\mathcal{I}(\mu, \sigma^2) = \begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 1/(2\sigma^4) \end{pmatrix}$ |
| **Kullback-Leibler divergence** | $D_{\mathrm{KL}}(\mathcal{N}_0 \parallel \mathcal{N}_1) = \frac{1}{2}\left\{\left(\frac{\sigma_0}{\sigma_1}\right)^2 + \frac{(\mu_1-\mu_0)^2}{\sigma_1^2} - 1 + 2\ln\frac{\sigma_1}{\sigma_0}\right\}$ |

https://en.wikipedia.org/wiki/Normal_distribution

# Test for normal parameters (one sample test)

Let $Y_1, \cdots, Y_n$ be a random sample from $N(\mu, \sigma^2)$.

Prob. 1 Find a test statistic $\Lambda$ in order to test $H_0 : \mu = \mu_0$ v.s. $H_1 : \mu \neq \mu_0$.

When $\sigma^2$ is known: $\Lambda = \dfrac{\overline{Y} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$

When $\sigma^2$ is unknown: $\Lambda = ?$ $\Lambda = \dfrac{\overline{Y} - \mu_0}{S/\sqrt{n}} \sim t$

Prob. 2 Find a test statistic $\Lambda$ in order to test $H_0 : \sigma^2 = \sigma_0^2$ v.s. $H_1 : \sigma^2 \neq \sigma_0^2$.

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**Sol.** Composite-vs-composite test with:

$$\omega = \{(\mu, \sigma^2) : \mu = \mu_0,\ \sigma^2 > 0\}$$

$$\Omega = \{(\mu, \sigma^2) : \mu \in \mathbb{R},\ \sigma^2 > 0\}$$

The MLE under the two spaces are:

$$\omega_e = (\mu_e, \sigma_e^2): \qquad \mu_e = \mu_0 \quad \text{and} \quad \sigma_e^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \mu_0)^2 \quad \text{(Under } \omega\text{)}$$

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$$L(\omega_e) = \cdots = \left[ \frac{ne^{-1}}{2\pi \sum_{i=1}^{n} (y_i - \mu_0)^2} \right]^{n/2}$$

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Hence,

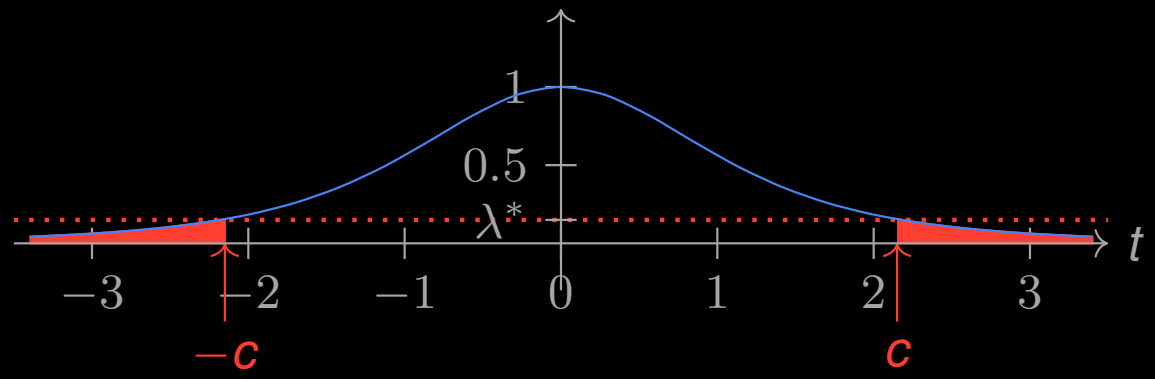
$$\lambda = \frac{L(\omega_e)}{L(\Omega_e)} = \left[\frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{\sum_{i=1}^{n}(y_i - \mu_0)^2}\right]^{n/2} = \cdots = \left[1 + \frac{n(\bar{y} - \mu_0)^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}\right]^{-n/2}$$

$$= \left[1 + \frac{1}{n-1}\left(\frac{\bar{y} - \mu_0}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2} \Big/ \sqrt{n}}\right)^2\right]^{-n/2}$$

$$= \left[1 + \frac{1}{n-1}\left(\frac{\bar{y} - \mu_0}{s \,/\, \sqrt{n}}\right)^2\right]^{-n/2}$$

$$= \left[1 + \frac{t^2}{n-1}\right]^{-n/2}, \quad t = \frac{\bar{y} - \mu_0}{s \,/\, \sqrt{n}}$$

$$\lambda(t) = (1 + \tfrac{t^2}{n-1})^{-\frac{n}{2}}$$




$$\lambda \in (0, \lambda^*] \qquad \Leftrightarrow \qquad |t| \geq c.$$

Finally, the test statistic is

$$\boxed{T = \frac{\overline{Y} - \mu_0}{S/\sqrt{n}}}$$

$$\text{with} \quad \overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i \quad \text{and} \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i - \overline{Y}\right)^2.$$

The critical region takes the form: $|t| \geq c$.

**Question:** Find the exact distribution of $T$.

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$$= \left[\frac{\frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2}{\frac{n}{n-1}\sigma_0^2}\right]^{n/2} \exp\left(-\frac{n-1}{2\sigma_0^2}\frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2 + \frac{n}{2}\right)$$

$$= \left[\frac{s^2}{\frac{n}{n-1}\sigma_0^2}\right]^{n/2} \exp\left(-\frac{n-1}{2\sigma_0^2}s^2 + \frac{n}{2}\right)$$

$$\Downarrow$$

$$\lambda(s^2) = \left[\frac{s^2}{\frac{n}{n-1}\sigma_0^2}\right]^{n/2} \exp\left(-\frac{n-1}{2\sigma_0^2}s^2 + \frac{n}{2}\right) \iff v(s^2) = (s^2)^{\frac{n}{2}} e^{-\lambda s^2}$$

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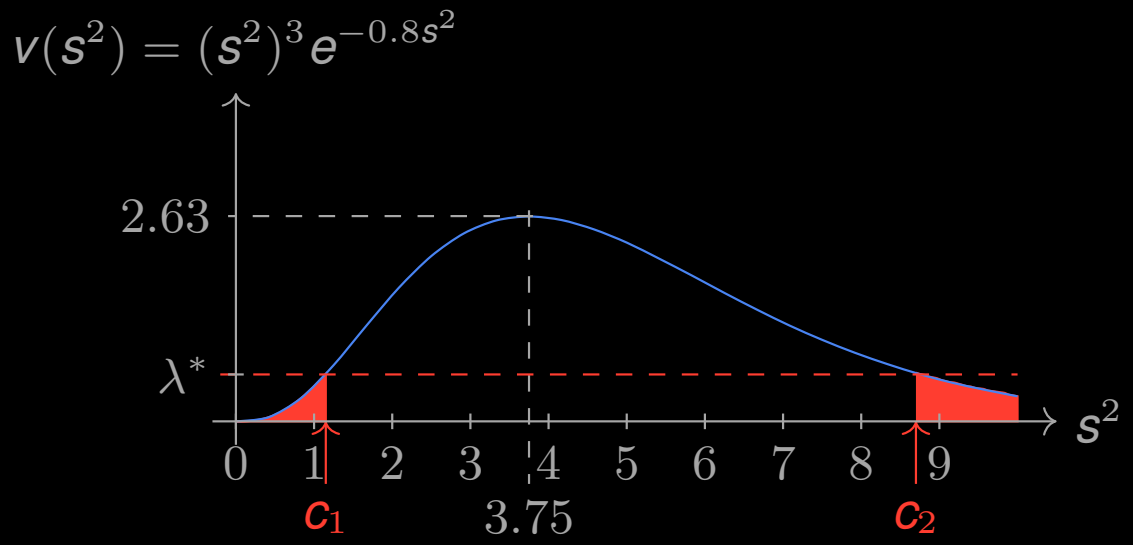
By setting $n = 6$ and $\lambda = 0.8$, we see ...




This suggests that the critical region should be of the form in terms of $s^2$:

$$(0, c_1) \cup (c_2, \infty)$$

For convenience, we put $\alpha/2$ mass on each tails of $S^2$:

Find $c_1$ and $c_2$ such that

$$\int_0^{c_1} f_{S^2}(z)dz = \int_{c_2}^{\infty} f_{S^2}(z)dz = \frac{\alpha}{2}.$$

Finally, the test statistic is

$$\boxed{S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i - \overline{Y}\right)^2} \quad \text{with} \quad \overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

**Question:** Find the exact distribution of $S^2$.

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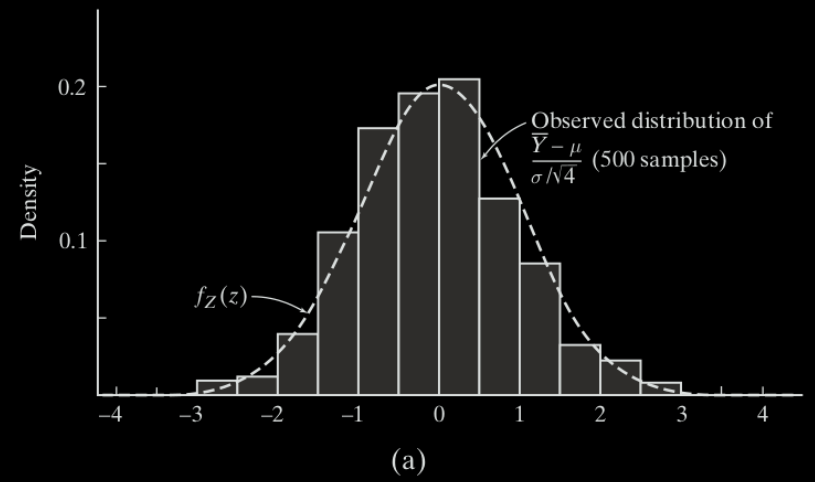
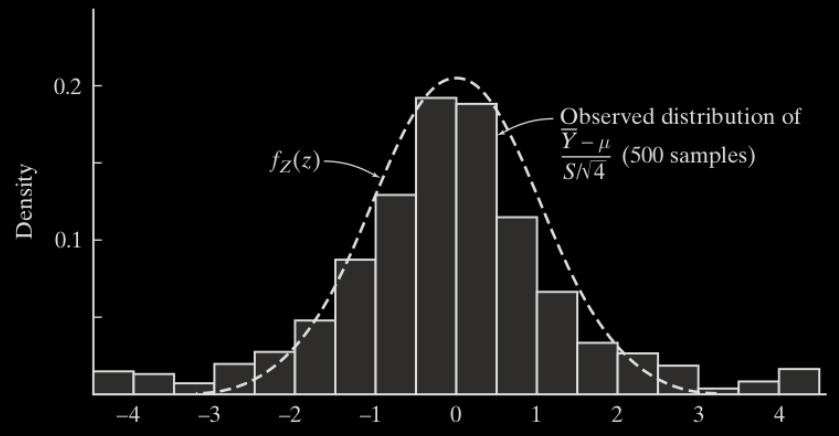

Density
0.2
0.1
–4
–3
–2
–1
0
1
2
3
4
$f_Z(z)$
Observed distribution of
$\frac{\overline{Y}-\mu}{\sigma/\sqrt{4}}$ (500 samples)

(a)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(b)

Statistician William Sealy Gosset, known as "Student"

Ref. Student's t distribution comes from William Sealy Gosset's 1908 paper in Biometrika under the pseudonym "Student".

Gosset worked at the Guinness Brewery in Dublin, Ireland, and was interested in the problems of small samples – for example, the chemical properties of barley where sample sizes might be as few as 3.

V1 One version of the origin of the pseudonym is that Gosset's employer preferred staff to use pen names when publishing scientific papers instead of their real name, so he used the name "Student" to hide his identity.

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# Plan

§ 7.1 Introduction

§ 7.2 Comparing $\frac{\overline{Y}-\mu}{\sigma/\sqrt{n}}$ and $\frac{\overline{Y}-\mu}{S/\sqrt{n}}$

§ 7.3 Deriving the Distribution of $\frac{\overline{Y}-\mu}{S/\sqrt{n}}$

§ 7.4 Drawing Inferences About $\mu$

§ 7.5 Drawing Inferences About $\sigma^2$

# Chapter 7. Inference Based on The Normal Distribution

§ 7.1 Introduction

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§ 7.5 Drawing Inferences About $\sigma^2$

Def. **Sampling distributions**

Distributions of *functions of random sample* of given size.

statistics / estimators

E.g. A random sample of size $n$ from $N(\mu, \sigma^2)$ with $\sigma^2$ known.

Sample mean $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i \sim N(\mu, \sigma^2/n)$

Aim: Determine distributions for

| | |
|---|---|
| Sample variance $S^2 := \frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i - \overline{Y}\right)^2$ | *Chi square distr.* |
| $T := \frac{\overline{Y} - \mu}{S/\sqrt{n}}$ | *Student t distr.* |
| $\frac{S_1^2}{\sigma_1^2} \Big/ \frac{S_2^2}{\sigma_2^2}$ | *F distr.* |

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**Thm 7.3.1.** Let $U = \sum_{i=1}^{m} Z_j^2$, where $Z_j$ are independent $N(0,1)$ normal r.v.s. Then

$$U \sim \text{Gamma}(\text{shape}=m/2,\ \text{rate}=1/2).$$

namely,

$$f_U(u) = \frac{1}{2^{m/2}\Gamma(m/2)} u^{\frac{m}{2}-1} e^{-u/2}, \qquad u \geq 0.$$

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**Def 7.3.1.** $U$ in Thm 7.3.1 is called **chi square distribution** with $m$ dgs of freedom.

**Proof.** We first consider the case when $m = 1$. In this case,

$$\begin{aligned} F_{Z^2}(u) &= \mathbb{P}\left(Z^2 \leq u\right) \\ &= \mathbb{P}\left(-\sqrt{u} \leq Z \leq \sqrt{u}\right) \\ &= 2\mathbb{P}(0 \leq Z \leq \sqrt{u}) \\ &= \frac{2}{\sqrt{2\pi}} \int_0^{2\pi} e^{-z^2/2} \mathrm{d}z \end{aligned}$$

Differentiating both sides of the above eq. in order to obtain the pdf:

$$\begin{aligned} f_{Z^2}(u) &= \frac{\mathrm{d}}{\mathrm{d}u} F_{Z^2}(u) \\ &= \frac{2}{\sqrt{2\pi}} \frac{1}{2\sqrt{u}} e^{-u/2} \\ &= \frac{1}{\sqrt{2}\Gamma(1/2)} u^{(1/2)-1} e^{-u/2}, \end{aligned}$$

which is the pdf of a gamma distribution with $r = \lambda = 1/2$.

Then adding $m$ independent copies of gamma distributions gives anther gamma distribution with $r = m/2$ and $\lambda = 1/2$ (See Theorem 4.6.4). $\square$

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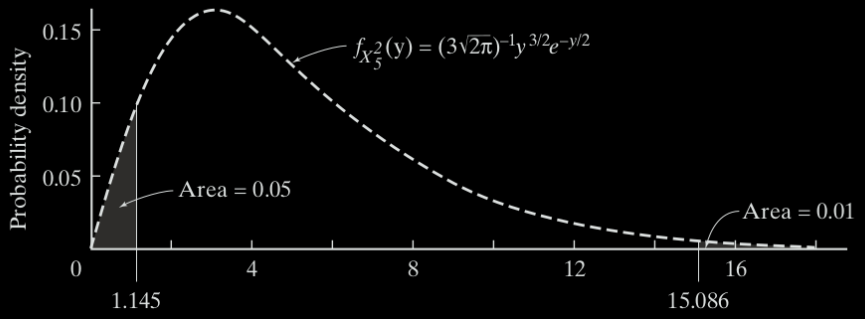
# Chi Square Table

| | $p$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| df | .01 | .025 | .05 | .10 | .90 | .95 | .975 | .99 |
| 1 | 0.000157 | 0.000982 | 0.00393 | 0.0158 | 2.706 | 3.841 | 5.024 | 6.635 |
| 2 | 0.0201 | 0.0506 | 0.103 | 0.211 | 4.605 | 5.991 | 7.378 | 9.210 |
| 3 | 0.115 | 0.216 | 0.352 | 0.584 | 6.251 | 7.815 | 9.348 | 11.345 |
| 4 | 0.297 | 0.484 | 0.711 | 1.064 | 7.779 | 9.488 | 11.143 | 13.277 |
| 5 | 0.554 | 0.831 | 1.145 | 1.610 | 9.236 | 11.070 | 12.832 | 15.086 |
| 6 | 0.872 | 1.237 | 1.635 | 2.204 | 10.645 | 12.592 | 14.449 | 16.812 |
| 7 | 1.239 | 1.690 | 2.167 | 2.833 | 12.017 | 14.067 | 16.013 | 18.475 |
| 8 | 1.646 | 2.180 | 2.733 | 3.490 | 13.362 | 15.507 | 17.535 | 20.090 |
| 9 | 2.088 | 2.700 | 3.325 | 4.168 | 14.684 | 16.919 | 19.023 | 21.666 |
| 10 | 2.558 | 3.247 | 3.940 | 4.865 | 15.987 | 18.307 | 20.483 | 23.209 |
| 11 | 3.053 | 3.816 | 4.575 | 5.578 | 17.275 | 19.675 | 21.920 | 24.725 |
| 12 | 3.571 | 4.404 | 5.226 | 6.304 | 18.549 | 21.026 | 23.336 | 26.217 |




$$\mathbb{P}(\chi^2_5 \leq 1.145) = 0.05 \iff \chi^2_{0.05,5} = 1.145$$

$$\mathbb{P}(\chi^2_5 \leq 15.086) = 0.99 \iff \chi^2_{0.99,5} = 15.086$$

```
> pchisq(1.145, df = 5)
[1] 0.04995622
> pchisq(15.086, df = 5)
[1] 0.9899989
```

```
> qchisq(0.05, df = 5)
[1] 1.145476
> qchisq(0.99, df = 5)
[1] 15.08627
```

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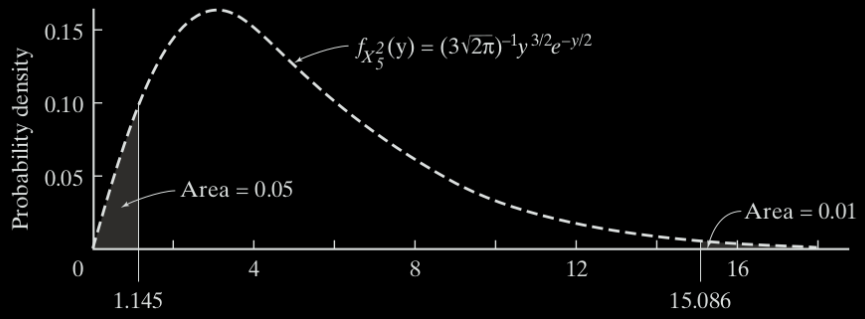


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```
> scipy.stats.chi2.cdf(1.145, 5)
[1]: 0.049956221552077 28
> scipy.stats.chi2.cdf(15.086, 5)
[1]: 0.9899988752378142
```

```
> scipy.stats.chi2.ppf(0.05, 5)
[1]: 1.1454762260617692
> scipy.stats.chi2.ppf(0.99, 5)
[1]: 15.08627246938899
```

**Thm 7.3.2.** Let $Y_1, \cdots, Y_n$ be a random sample from $N(\mu, \sigma^2)$. Then

(a) $S^2$ and $\overline{Y}$ are independent.

(b) $$\frac{(n-1)S^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(Y_i - \overline{Y}\right)^2 \quad \sim \quad \text{Chi Square}(n-1).$$

*Proof.* We will prove the case $n = 2$.

$$\overline{Y} = \frac{Y_1 + Y_2}{2}, \qquad Y_1 - \overline{Y} = \frac{Y_1 - Y_2}{2}, \qquad Y_2 - \overline{Y} = \frac{Y_2 - Y_1}{2}$$

$$S^2 = \ldots = \frac{1}{2}(Y_1 - Y_2)^2$$

(a) It is equivalanet to show $Y_1 + Y_2 \perp Y_1 - Y_2$. Since they are normal, it suffices to show that

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**Def 7.3.2.** If $U \sim$ Chi Square($n$) and $V \sim$ Chi Square($m$), and $U \perp V$, then

$$F := \frac{V/m}{U/n}$$

follows the **(Snedecor's) F distribution** with $m$ and $n$ degrees of freedom.

**Thm 7.3.3.** Let $F_{m,n} = \frac{V/m}{U/n}$ be an $F$ r.v. with $m$ and $n$ degrees of freedom. Then

$$f_{F_{m,n}}(w) = \frac{\Gamma\left(\frac{m+n}{2}\right) m^{m/2} n^{n/2}}{\Gamma(m/2)\Gamma(n/2)} \times \frac{w^{m/2-1}}{(n+mw)^{(m+n)/2}}, \quad w \geq 0$$

Equivalently,

$$f_{F_{m,n}}(w) = B(m/2, n/2)^{-1} \left(\frac{m}{n}\right)^{\frac{m}{2}} w^{\frac{m}{2}-1} \left(1 + \frac{m}{n} w\right)^{-\frac{m+n}{2}}$$

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**Thm 7.3.3.** Let $F_{m,n} = \frac{V/m}{U/n}$ be an $F$ r.v. with $m$ and $n$ degrees of freedom. Then

$$f_{F_{m,n}}(w) = \frac{\Gamma\left(\frac{m+n}{2}\right) m^{m/2} n^{n/2}}{\Gamma(m/2)\Gamma(n/2)} \times \frac{w^{m/2-1}}{(n+mw)^{(m+n)/2}}, \quad w \geq 0$$

Equivalently,

$$f_{F_{m,n}}(w) = B(m/2, n/2)^{-1} \left(\frac{m}{n}\right)^{\frac{m}{2}} w^{\frac{m}{2}-1} \left(1 + \frac{m}{n} w\right)^{-\frac{m+n}{2}}$$

where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$.

**Def 7.3.2.** If $U \sim$ Chi Square($n$) and $V \sim$ Chi Square($m$), and $U \perp V$, then

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## Recall

**Thm 3.8.4** Let $X$ and $Y$ be independent continuous random variables, with pdf $f_X(x)$ and $f_Y(y)$, respectively.

Assume that $X$ is zero for at most a set of isolated points.

Then $W = Y/X$ follows a distribution with pdf:

$$f_W(w) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(wx) \mathrm{d}x.$$

**Thm 3.8.2** Suppose $X$ is a continuous random variable and $a \neq 0$.

Then $Y = aX + b$ follows a distribution with pdf:

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right).$$

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$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right).$$

**Proof.** Let us first find the pdf for $W := V/U$. By Theorem 7.3.1,

$$f_V(v) = \frac{1}{2^{m/2}\Gamma(m/2)} v^{(m/2)-1} e^{-v/2},$$
$$f_U(u) = \frac{1}{2^{n/2}\Gamma(n/2)} u^{(n/2)-1} e^{-u/2}.$$

Then by Theorem 3.8.4, we see that the pdf of $W$ is

$$f_W(w) = \int_{-\infty}^{\infty} |u| f_U(u)\, f_V(uw) \mathrm{d}u$$
$$= \int_0^{\infty} u \frac{1}{2^{n/2}\Gamma(n/2)} u^{(n/2)-1} e^{-u/2} \frac{1}{2^{m/2}\Gamma(m/2)} (uw)^{(m/2)-1} e^{-uw/2} \mathrm{d}u$$
$$= \frac{1}{2^{(n+m)/2}\Gamma(n/2)\Gamma(m/2)} w^{(m/2)-1} \int_0^{\infty} u^{\frac{n+m}{2}-1} e^{-\frac{1+w}{2}u} \mathrm{d}u$$

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$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} |u| f_U(u)\, f_V(uw) \mathrm{d}u \\ &= \int_0^{\infty} u \frac{1}{2^{n/2}\Gamma(n/2)} u^{(n/2)-1} e^{-u/2} \frac{1}{2^{m/2}\Gamma(m/2)} (uw)^{(m/2)-1} e^{-uw/2} \mathrm{d}u \\ &= \frac{1}{2^{(n+m)/2}\Gamma(n/2)\Gamma(m/2)} w^{(m/2)-1} \int_0^{\infty} u^{\frac{n+m}{2}-1} e^{-\frac{1+w}{2}u} \mathrm{d}u \end{aligned}$$

Then by the change of variables, $y = \frac{1+w}{2}u$, we see that

$$f_W(w) = \frac{1}{2^{(n+m)/2}\Gamma(n/2)\Gamma(m/2)} w^{(m/2)-1} \left(\frac{2}{1+w}\right)^{\frac{n+m}{2}} \int_0^\infty y^{\frac{n+m}{2}-1} e^{-y} \mathrm{d}y$$

$$= \frac{1}{2^{(n+m)/2}\Gamma(n/2)\Gamma(m/2)} w^{(m/2)-1} \left(\frac{2}{1+w}\right)^{\frac{n+m}{2}} \Gamma\left(\frac{n+m}{2}\right)$$

where the last equality is due to the definition of the Gamma function.

Finally, by Theorem 3.8.2, we see that $F = \frac{V/m}{U/n} = \frac{n}{m}W$ follows a distribution with pdf

$$f_F(y) = \frac{m}{n} f_W\left(\frac{m}{n}y\right)$$

$$= \frac{m}{n} \frac{1}{2^{(n+m)/2}\Gamma(n/2)\Gamma(m/2)} \left(\frac{m}{n}y\right)^{(m/2)-1} \left(\frac{2}{1+\frac{m}{n}y}\right)^{\frac{n+m}{2}} \Gamma\left(\frac{n+m}{2}\right)$$

$$= \cdots \qquad y \geq 0.$$

□

Then by the change of variables, $y = \frac{1+w}{2}u$, we see that

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$$= \frac{1}{2^{(n+m)/2}\Gamma(n/2)\Gamma(m/2)} w^{(m/2)-1} \left(\frac{2}{1+w}\right)^{\frac{n+m}{2}} \Gamma\left(\frac{n+m}{2}\right)$$

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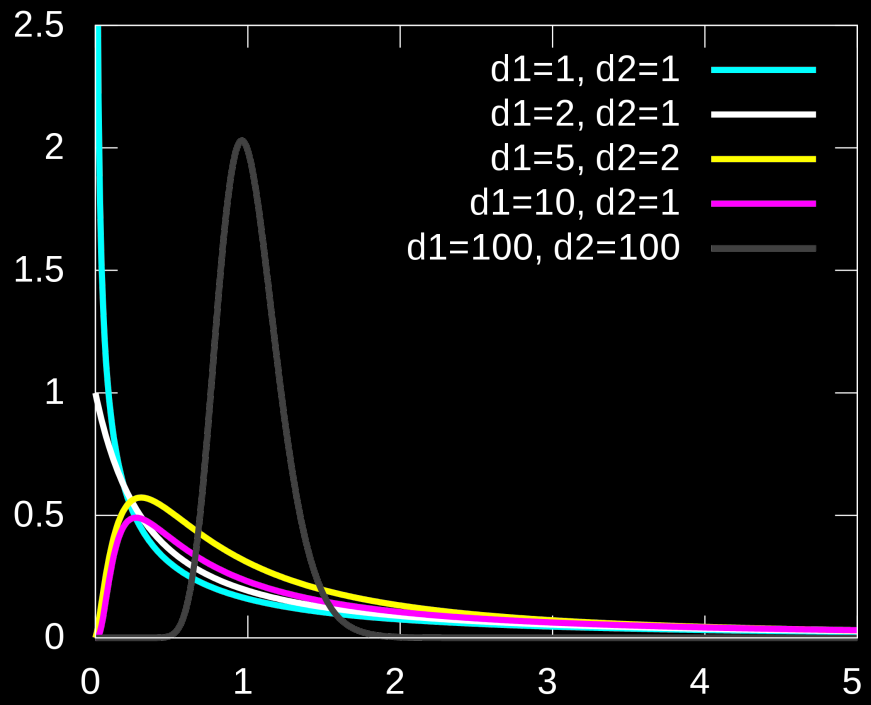
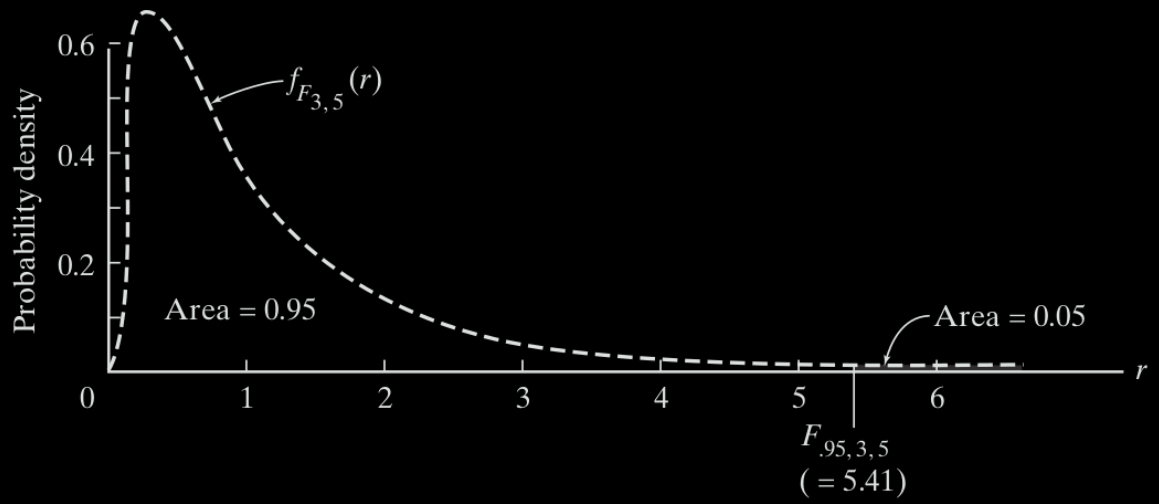
□




```
# Draw F density
x=seq(0,5,0.01)
pdf= cbind(df(x, df1 = 1, df2 = 1),
df(x, df1 = 2, df2 = 1),
df(x, df1 = 5, df2 = 2),
df(x, df1 = 10, df2 = 1),
df(x, df1 = 100, df2 = 100))
matplot(x,pdf, type = "l")
title("F with various dgrs of freedom")
```

# F- Table




$$\mathbb{P}(F_{3,5} \leq 5.41) = 0.95 \iff F_{0.95,3,5} = 5.41$$

```
> pf(5.41, df1 = 3, df2 = 5)
[1] 0.9500093
```

```
> qf(0.95, df1 = 3, df2 = 5)
[1] 5.409451
```

```
> scipy.stats.f.cdf(5.41, 3, 5)
[1] 0.9500092950699683
```

```
> scipy.stats.f.ppf(0.95, 3, 5)
[1] 5.40945131805649
```

**Def 7.3.3.** Suppose $Z \sim N(0,1)$, $U \sim$ Chi Square$(n)$, and $Z \perp U$. Then

$$T_n = \frac{Z}{\sqrt{U/n}}$$

follows the **Student's t-distribution** of $n$ degrees of freedom.

Remark $T_n^2 \sim F$-distribution with 1 and $n$ degrees of freedom.

**Thm 7.3.4.** The pdf of the Student t of degree $n$ is

$$f_{T_n}(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)} \times \left(1 + \frac{t^2}{n}\right)^{-\frac{n+2}{2}}, \quad t \in \mathbb{R}.$$

**Def 7.3.3.** Suppose $Z \sim N(0,1)$, $U \sim$ Chi Square$(n)$, and $Z \perp U$. Then

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**Proof.** Note that $T_n^2 = \frac{Z^2}{U/n}$ follows an $F(1, n)$ distribution. Hence,

$$f_{T_n^2}(t) = \frac{n^{\frac{n}{2}}\Gamma(\frac{n+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n}{2})} t^{-\frac{1}{2}} \frac{1}{(n+t)^{\frac{n+1}{2}}}, \quad t > 0.$$

Therefore,

$$F_{T_n}(t) = \mathbb{P}(T_n \leq t) = \mathbb{P}(-\infty < T_n \leq 0) + \mathbb{P}(0 \leq T_n \leq t).$$

The term $\mathbb{P}(-\infty < T_n \leq 0)$ is a constant which will disappear upon differentiation.

Notice that

$$\begin{aligned}\left\{T_n^2 \leq t^2\right\} &= \{-t \leq T_n \leq t\} = \{-t \leq T_n \leq 0\} \cup \{0 \leq T_n \leq t\} \\ &= \left\{-t\sqrt{U/n} \leq Z \leq 0\right\} \cup \left\{0 \leq Z \leq t\sqrt{U/n}\right\}\end{aligned}$$

**Proof.** Note that $T_n^2 = \frac{Z^2}{U/n}$ follows an $F(1, n)$ distribution. Hence,

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By symmetry of the distribution of $Z$,

$$\mathbb{P}\left(-t\sqrt{U/n} \le Z \le 0\right) = \mathbb{P}\left(0 \le Z \le t\sqrt{U/n}\right)$$

Therefore,

$$\begin{aligned}\mathbb{P}\left(T_n^2 \le t^2\right) &= \mathbb{P}\left(-t\sqrt{U/n} \le Z \le 0\right) + \mathbb{P}\left(0 \le Z \le t\sqrt{U/n}\right) \\ &= 2\mathbb{P}\left(0 \le Z \le t\sqrt{U/n}\right) \\ &= 2\mathbb{P}(0 \le T_n \le t).\end{aligned}$$

Hence,

$$F_{T_n}(t) = const. + \frac{1}{2}\mathbb{P}\left(T_n^2 \le t^2\right)$$

Finally, differentiation gives the density:

$$f_{T_n}(t) = \frac{d}{dt}F_{T_n}(t) = \frac{d}{dt}\frac{1}{2}F_{T_n^2}(t^2) = t \cdot f_{T_n^2}(t^2) = \cdots.$$

□

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Finally, differentiation gives the density:

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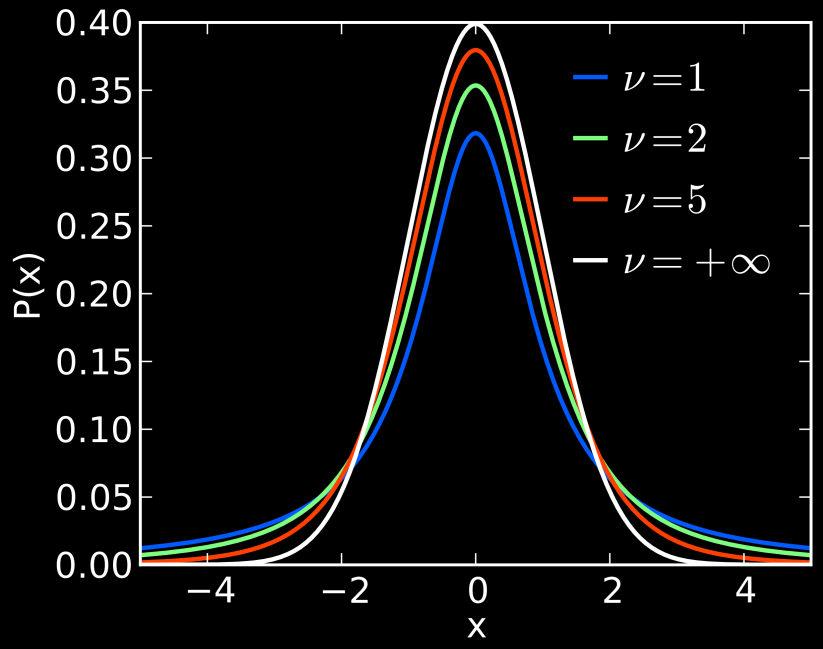
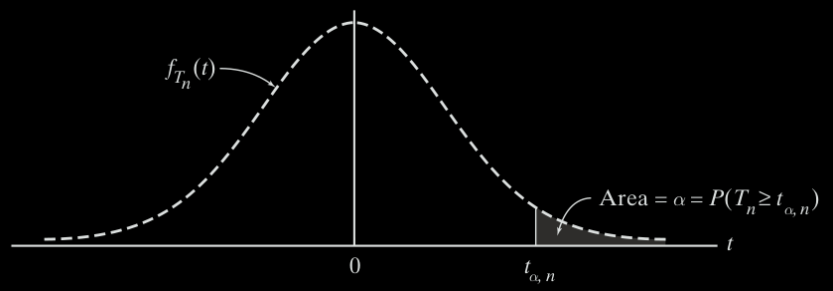
□




```
# Draw Student t−density
x=seq(−5,5,0.01)
pdf= cbind(dt(x, df = 1),
           dt(x, df = 2),
           dt(x, df = 5),
           dt(x, df = 100))
matplot(x,pdf, type = "l")
title("Student's t−distributions")
```

# t Table

| | $\alpha$ | | | | | | |
|---|---|---|---|---|---|---|---|
| df | .20 | .15 | .10 | .05 | .025 | .01 | .005 |
| 1 | 1.376 | 1.963 | 3.078 | 6.3138 | 12.706 | 31.821 | 63.657 |
| 2 | 1.061 | 1.386 | 1.886 | 2.9200 | 4.3027 | 6.965 | 9.9248 |
| 3 | 0.978 | 1.250 | 1.638 | 2.3534 | 3.1825 | 4.541 | 5.8409 |
| 4 | 0.941 | 1.190 | 1.533 | 2.1318 | 2.7764 | 3.747 | 4.6041 |
| 5 | 0.920 | 1.156 | 1.476 | 2.0150 | 2.5706 | 3.365 | 4.0321 |
| 6 | 0.906 | 1.134 | 1.440 | 1.9432 | 2.4469 | 3.143 | 3.7074 |
| ⋮ | | | ⋮ | | | | |
| 30 | 0.854 | 1.055 | 1.310 | 1.6973 | 2.0423 | 2.457 | 2.7500 |
| ∞ | 0.84 | 1.04 | 1.28 | 1.64 | 1.96 | 2.33 | 2.58 |




$$\mathbb{P}(T_3 > 4.541) = 0.01 \iff t_{0.01,3} = 4.541$$

```
> 1−pt(4.541, df =3)
[1] 0.009998238
```

```
> alpha = 0.01
> qt(1−alpha, df = 3)
[1] 4.540703
```

```
> 1 − scipy.stats.t.cdf(4.541, 3)
[1] 0.009998238064494 07
```

```
> scipy.stats.t.ppf(1−0.01, 3)
[1] 4.540702858698419
```

**Thm 7.3.5.** Let $Y_1, \cdots, Y_n$ be a random sample from $N(\mu, \sigma^2)$. Then

$$T_{n-1} = \frac{\overline{Y} - \mu}{S/\sqrt{n}} \quad \sim \quad \text{Student's t of degree } n-1.$$

Proof.

$$\frac{\overline{Y} - \mu}{S/\sqrt{n}} = \frac{\dfrac{\overline{Y} - \mu}{\sigma/\sqrt{n}}}{\sqrt{\dfrac{(n-1)S^2}{\sigma^2(n-1)}}}$$

$$\frac{\overline{Y} - \mu}{\sigma/\sqrt{n}} \sim N(0,1) \quad \perp \quad \frac{(n-1)S^2}{\sigma^2} \sim \text{Chi Square}(n-1)$$

By Def. 7.3.3 ... □

**Thm 7.3.5.** Let $Y_1, \cdots, Y_n$ be a random sample from $N(\mu, \sigma^2)$. Then

$$T_{n-1} = \frac{\overline{Y} - \mu}{S/\sqrt{n}} \quad \sim \quad \text{Student's t of degree } n-1.$$

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As $n \to \infty$, Students' t distribution will converge to $N(0,1)$:

Thm 7.3.6. $f_{T_n}(x) \to f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ as $n \to \infty$, where $Z \sim N(0,1)$.

Proof By Stirling's formula:

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z (1 + O(1/z)) \qquad \text{as } z \to \infty$$

$$\Longrightarrow \quad \lim_{n\to\infty} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)} = \frac{1}{\sqrt{2\pi}}$$

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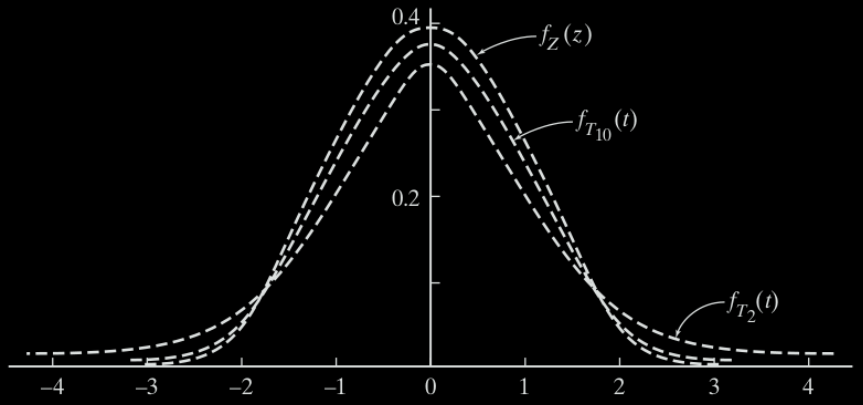


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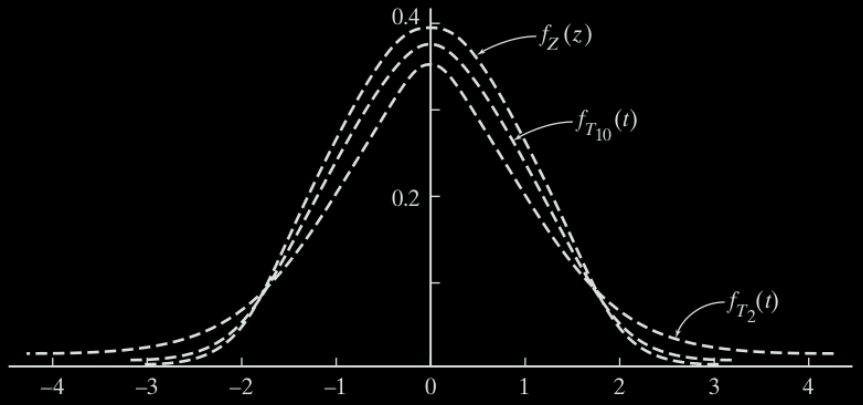


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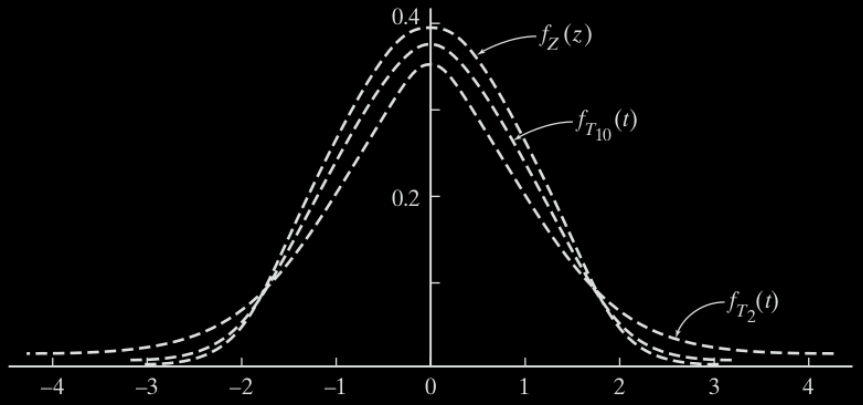


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§ 7.1 Introduction

§ 7.2 Comparing $\frac{\overline{Y}-\mu}{\sigma/\sqrt{n}}$ and $\frac{\overline{Y}-\mu}{S/\sqrt{n}}$

§ 7.3 Deriving the Distribution of $\frac{\overline{Y}-\mu}{S/\sqrt{n}}$

§ 7.4 Drawing Inferences About $\mu$

§ 7.5 Drawing Inferences About $\sigma^2$

# Chapter 7. Inference Based on The Normal Distribution

§ 7.1 Introduction

§ 7.2 Comparing $\frac{\overline{Y}-\mu}{\sigma/\sqrt{n}}$ and $\frac{\overline{Y}-\mu}{S/\sqrt{n}}$

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Let $Y_1, \cdots, Y_n$ be a random sample from $N(\mu, \sigma^2)$.

**Question** Find a test statistic $\Lambda$ in order to test $H_0 : \mu = \mu_0$ v.s. $H_1 : \mu \neq \mu_0$.

**Case I.** $\sigma^2$ is known: $\Lambda = \dfrac{\overline{Y} - \mu_0}{\sigma/\sqrt{n}}$

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# Summary

A random sample of size $n$ from
a normal distribution $N(\mu, \sigma^2)$

| | $\sigma^2$ known | $\sigma^2$ unknown |
|---|---|---|
| Statistic | $Z = \frac{\overline{Y}-\mu}{\sigma/\sqrt{n}}$ | $T_{n-1} = \frac{\overline{Y}-\mu}{S/\sqrt{n}}$ |
| Score | $z = \frac{\bar{y}-\mu}{\sigma/\sqrt{n}}$ | $t = \frac{\bar{y}-\mu}{s/\sqrt{n}}$ |
| Table | $z_\alpha$ | $t_{\alpha,n-1}$ |
| $100(1-\alpha)\%$ C.I. | $\left(\bar{y} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{y} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$ | $\left(\bar{y} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}, \bar{y} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}\right)$ |
| Test $H_0 : \mu = \mu_0$ | | |
| $H_1 : \mu > \mu_0$ | Reject $H_0$ if $z \geq z_\alpha$ | Reject $H_0$ if $t \geq t_{\alpha,n-1}$ |
| $H_1 : \mu < \mu_0$ | Reject $H_0$ if $z \leq z_\alpha$ | Reject $H_0$ if $t \leq t_{\alpha,n-1}$ |
| $H_1 : \mu \neq \mu_0$ | Reject $H_0$ if $\lvert z\rvert \geq z_{\alpha/2}$ | Reject $H_0$ if $\lvert t\rvert \geq t_{\alpha/2,n-1}$ |

# Computing $s$ from data

Step 1 $a = \sum_{i=1}^{n} y_i$

Step 2. $b = \sum_{i=1}^{n} y_i^2$

Step 3. $s = \sqrt{\frac{nb - a^2}{n(n-1)}}$

Proof.

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2 = \frac{n\left(\sum_{i=1}^{n} y_i^2\right) - \left(\sum_{i=1}^{n} y_i\right)^2}{n(n-1)}$$

□

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**Case 7.4.1** How far apart are the bat and the insect when the bat first senses that insect is there?

Or, what is the effective range of a bat's echolocation system?

Answer the question by contruct a 95% C.I.

Sol. ... □

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**Table 7.4.1**

| Catch Number | Detection Distance (cm) |
|---|---|
| 1 | 62 |
| 2 | 52 |
| 3 | 68 |
| 4 | 23 |
| 5 | 34 |
| 6 | 45 |
| 7 | 27 |
| 8 | 42 |
| 9 | 83 |
| 10 | 56 |
| 11 | 40 |

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**Sol.** ... □

```python
# Case7_4_1.py
import numpy as np
import scipy.stats as st


# returns confidence interval of mean
def confIntMean(a, conf=0.95):
    mean, sem, m = np.mean(a), st.sem(a), st.t.ppf((1+conf)/2., len(a)-1)
    return mean - m*sem, mean + m*sem


def main():
    alpha = 5
    data = np.array([62, 52, 68, 23, 34, 45, 27, 42, 83, 56, 40])
    lower, upper = confIntMean(data, 1-alpha/100)
    print("""\

    The {alpha}% confidence interval is ({lower:.2f},{upper:.2f})

        """.format(**locals()))


if __name__ == "__main__":
    main()
```

```
In [83]: run Case7_4_1.py

    The 95% confidence interval is (36.21,60.51)
```

Eg. 7.4.2 Bank approval rates for inner-city residents v.s. rural ones.

Approval rate for rural residents is 62%.

Do bank treat two groups equally? $\alpha = 0.05$

Sol.

$$H_0 : \mu = 62 \qquad v.s. \qquad H_1 : \mu \neq 62.$$

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Approval rate for rural residents is 62%.

Do bank treat two groups equally? $\alpha = 0.05$

**Table 7.4.3**

| Bank | Location | Affiliation | Percent Approved |
|---|---|---|---|
| 1 | 3rd & Morgan | AU | 59 |
| 2 | Jefferson Pike | TU | 65 |
| 3 | East 150th & Clark | TU | 69 |
| 4 | Midway Mall | FT | 53 |
| 5 | N. Charter Highway | FT | 60 |
| 6 | Lewis & Abbot | AU | 53 |
| 7 | West 10th & Lorain | FT | 58 |
| 8 | Highway 70 | FT | 64 |
| 9 | Parkway Northwest | AU | 46 |
| 10 | Lanier & Tower | TU | 67 |
| 11 | King & Tara Court | AU | 51 |
| 12 | Bluedot Corners | FT | 59 |

Sol.

$$H_0 : \mu = 62 \qquad v.s. \qquad H_1 : \mu \neq 62.$$

**Table 7.4.4**

| Banks | $n$ | $\overline{y}$ | $s$ | $t$ Ratio | Critical Value | Reject $H_0$? |
|---|---|---|---|---|---|---|
| All | 12 | 58.667 | 6.946 | $-1.66$ | $\pm 2.2010$ | No |

**Table 7.4.5**

| Banks | $n$ | $\overline{y}$ | $s$ | $t$ Ratio | Critical Value | Reject $H_0$? |
|---|---|---|---|---|---|---|
| American United | 4 | 52.25 | 5.38 | $-3.63$ | $\pm 3.1825$ | Yes |
| Federal Trust | 5 | 58.80 | 3.96 | $-1.81$ | $\pm 2.7764$ | No |
| Third Union | 3 | 67.00 | 2.00 | $+4.33$ | $\pm 4.3027$ | Yes |

```
# Eg7_4_2.py
import numpy as np
import scipy.stats as st

data = np.array([59, 65, 69, 53, 60, 53, 58, 64, 46, 67, 51, 59])
alpha = 5
mean, sem = np.mean(data), st.sem(data)
n = len(data)
s = sem * np.sqrt(n)
cv = st.t.ppf(1-alpha/200., len(data)-1)
tRatio = (mean-62)/sem


print("""\

      n={n}, sample mean={mean:.3f}, s={s:.3f}, t Ratio={tRatio:.2f}, Critical values
          ={cv:.4f}
      """.format(**locals()))
```

```
In [113]: run Eg7_4_2.py

      n=12, sample mean=58.667, s=6.946, t Ratio=-1.66, Critical values=2.2010
```

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$$\frac{(n-1)S^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(Y_i - \overline{Y}\right)^2 \sim \text{Chi Square}(n-1)$$

$$P\left(\chi^2_{\alpha/2,n-1} \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi^2_{1-\alpha/2,n-1}\right) = 1-\alpha$$

$100(1-\alpha)\%$ C.I. for $\sigma^2$:

$$\left(\frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}, \frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}}\right)$$

$100(1-\alpha)\%$ C.I. for $\sigma$:

$$\left(\sqrt{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}}, \sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}}}\right)$$

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$$\left(\frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}, \frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}}\right)$$

$100(1-\alpha)\%$ C.I. for $\sigma$:

$$\left(\sqrt{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}}, \sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}}}\right)$$

For a random sample of size $n$ from $N(\mu, \sigma^2)$:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left( Y_i - \overline{Y} \right)^2$$

$$\Downarrow$$

$$\frac{(n-1)S^2}{\sigma^2} = \frac{1}{\sigma^2} \sum_{i=1}^{n} \left( Y_i - \overline{Y} \right)^2 \sim \text{Chi Square}(n-1)$$

$$\mathbb{P}\left( \chi^2_{\alpha/2,n-1} \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi^2_{1-\alpha/2,n-1} \right) = 1 - \alpha.$$

$100(1-\alpha)\%$ C.I. for $\sigma^2$:

$$\left( \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}, \frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}} \right)$$

$100(1-\alpha)\%$ C.I. for $\sigma$:

$$\left( \sqrt{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}}, \sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}}} \right)$$

Testing $H_0 : \sigma^2 = \sigma_0^2$

v.s.

(at the $\alpha$ level of significance)

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

$H_1 : \sigma^2 < \sigma_0^2$:

Reject $H_0$ if

$\chi^2 \leq \chi^2_{\alpha,n-1}$

$H_1 : \sigma^2 \neq \sigma_0^2$:

Reject $H_0$ if

$\chi^2 \leq \chi^2_{\alpha/2,n-1}$ or

$\chi^2 \geq \chi^2_{1-\alpha/2,n-1}$

$H_1 : \sigma^2 > \sigma_0^2$:

Reject $H_0$ if

$\chi^2 \geq \chi^2_{1-\alpha,n-1}$

E.g. 1. The width of a confidence interval for $\sigma^2$ is a function of $n$ and $S^2$:

$$W = \frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}} - \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}$$

Find the smallest $n$ such that the average width of a 95% C.I. for $\sigma^2$ is no greater than $0.8\sigma^2$.

Sol. Notice that $\mathbb{E}[S^2] = \sigma^2$. Hence, we need to find $n$ s.t.

$$(n-1)\left(\frac{1}{\chi^2_{0.025,n-1}} - \frac{1}{\chi^2_{0.975,n-1}}\right) \leq 0.8.$$

Trial and error (numerics on R) gives $n = 57$.

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Trial and error (numerics on R) gives $n = 57$.

```
> # Example 7.5.1
> n=seq(45,60,1)
> l=qchisq(0.025,n-1)
> u=qchisq(0.975,n-1)
> e=(n-1)* (1/l-1/u)
> m=cbind(n,l,u,e)
> colnames(m) = c("n",
+                 "chi(0.025,n-1)",
+                 "chi(0.975,n-1)",
+                 "error")
> m
       n chi(0.025,n-1) chi(0.975,n-1) error
 [1,] 45       27.57457       64.20146 0.9103307
 [2,] 46       28.36615       65.41016 0.8984312
 [3,] 47       29.16005       66.61653 0.8869812
 [4,] 48       29.95620       67.82065 0.8759533
 [5,] 49       30.75451       69.02259 0.8653224
 [6,] 50       31.55492       70.22241 0.8550654
 [7,] 51       32.35736       71.42020 0.8451612
 [8,] 52       33.16179       72.61599 0.8355901
 [9,] 53       33.96813       73.80986 0.8263340
[10,] 54       34.77633       75.00186 0.8173761
[11,] 55       35.58634       76.19205 0.8087008
[12,] 56       36.39811       77.38047 0.8002937

[13,] 57       37.21159       78.56716 0.7921414
[14,] 58       38.02674       79.75219 0.7842313
[15,] 59       38.84351       80.93559 0.7765517
[16,] 60       39.66186       82.11741 0.7690918
```

## Case Study 7.5.2

Mutual funds are investment vehicles consisting of a portfolio of various types of investments. If such an investment is to meet annual spending needs, the owner of shares in the fund is interested in the average of the annual returns of the fund. Investors are also concerned with the volatility of the annual returns, measured by the variance or standard deviation. One common method of evaluating a mutual fund is to compare it to a benchmark, the Lipper Average being one of these. This index number is the average of returns from a universe of mutual funds.

The Global Rock Fund is a typical mutual fund, with heavy investments in international funds. It claimed to best the Lipper Average in terms of volatility over the period from 1989 through 2007. Its returns are given in the table below.

| Year | Investment Return % | Year | Investment Return % |
|---|---|---|---|
| 1989 | 15.32 | 1999 | 27.43 |
| 1990 | 1.62 | 2000 | 8.57 |
| 1991 | 28.43 | 2001 | 1.88 |
| 1992 | 11.91 | 2002 | −7.96 |
| 1993 | 20.71 | 2003 | 35.98 |
| 1994 | −2.15 | 2004 | 14.27 |
| 1995 | 23.29 | 2005 | 10.33 |
| 1996 | 15.96 | 2006 | 15.94 |
| 1997 | 11.12 | 2007 | 16.71 |
| 1998 | 0.37 | | |

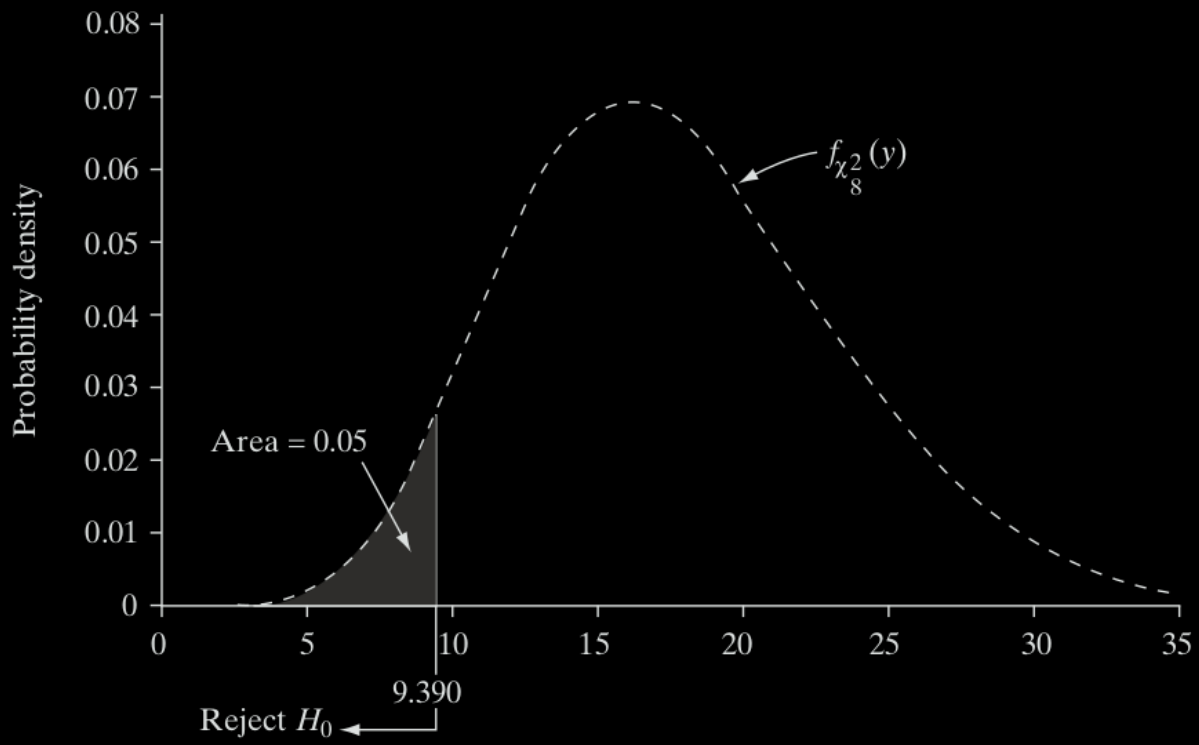
The standard deviation for these returns is 11.28%, while the corresponding figure for the Lipper Average is 11.67%. Now, clearly, the Global Rock Fund has a smaller standard deviation than the Lipper Average, but is this small difference due just to random variation? The hypothesis test is meant to answer such questions.

$$H_0 : \sigma^2 = (11.67)^2$$

versus

$$H_1 : \sigma^2 < (11.67)^2$$

Let $\alpha = 0.05$. With $n = 19$, the critical value for the chi square ratio [from part (b) of Theorem 7.5.2] is $\chi^2_{1-\alpha,n-1} = \chi^2_{.05,18} = 9.390$ (see Figure 7.5.3). But

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{(19-1)(11.28)^2}{(11.67)^2} = 16.82$$

so our decision is clear: Do not reject $H_0$.