

# Math 362: Mathematical Statistics II

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# Chapter 10. Goodness-of-fit Tests

§ 10.1 Introduction

§ 10.2 The Multinomial Distribution

§ 10.3 Goodness-of-Fit Tests: All Parameters Known

§ 10.4 Goodness-of-Fit Tests: Parameters Unknown

§ 10.5 Contingency Tables

# Chapter 10. Goodness-of-fit Tests

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§ 10.5 Contingency Tables

E.g. 1 Whether are the two ratings independent?

		Ebert Ratings			Total
		Down	Sideways	Up	
Siskel Ratings	Down	24	8	13	45
	Sideways	8	13	11	32
	Up	10	9	64	83
	Total	<u>42</u>	<u>30</u>	<u>88</u>	<u>160</u>

E.g. 2 Whether is the suicide rate independent of the mobility factor?

**Table 10.5.7**

City	Suicides per 100,000, $x_i$	Mobility Index, $y_i$	City	Suicides per 100,000, $x_i$	Mobility Index, $y_i$
New York	19.3	54.3	Washington	22.5	37.1
Chicago	17.0	51.5	Minneapolis	23.8	56.3
Philadelphia	17.5	64.6	New Orleans	17.2	82.9
Detroit	16.5	42.5	Cincinnati	23.9	62.2
Los Angeles	23.8	20.3	Newark	21.4	51.9
Cleveland	20.1	52.2	Kansas City	24.5	49.4
St. Louis	24.8	62.4	Seattle	31.7	30.7
Baltimore	18.0	72.0	Indianapolis	21.0	66.1
Boston	14.8	59.4	Rochester	17.2	68.0
Pittsburgh	14.9	70.0	Jersey City	10.1	56.5
San Francisco	40.0	43.8	Louisville	16.6	78.7
Milwaukee	19.3	66.2	Portland	29.3	33.2
Buffalo	13.8	67.6			

$$\bar{x} = 20.8 \quad \text{and} \quad \bar{y} = 56.0$$

**Table 10.5.8**

		Mobility Index	
		Low (<56.0)	High ( $\geq$ 56.0)
Suicide Rate	High ( $\geq$ 20.8)	7	4
	Low (<20.8)	3	11

**Thm 10.4.1** Suppose that  $n$  observations are taken on a sample space partitioned by the events  $A_1, \dots, A_r$  and  $B_1, \dots, B_c$ .

Let  $p_i = \mathbb{P}(A_i)$ ,  $q_j = \mathbb{P}(B_j)$ ,  $p_{ij} = \mathbb{P}(A_i \cap B_j)$ .

Let  $X_{ij}$  be the number of observations belonging to  $A_i \cap B_j$ .

a) Provided that  $np_{ij} \geq 5$  for all  $i, j$ , the r.v.

$$D_2 = \sum_{i=1}^r \sum_{j=1}^c \frac{(X_{ij} - np_{ij})^2}{np_{ij}} \sim \text{Chi square of f.d. } rc - 1$$

b) To test  $H_0 : A_i$ 's are independent of  $B_j$ 's, calculate the test statistic

$$d_2 = \sum_{i=1}^r \sum_{j=1}^c \frac{(k_{ij} - n\hat{p}_i\hat{q}_j)^2}{n\hat{p}_i\hat{q}_j}$$

where  $\hat{p}_i$  and  $\hat{q}_j$  are MLE's for  $p_i$  and  $q_j$ , respectively.

Provided that  $n\hat{p}_i\hat{q}_j \geq 5$  for all  $i, j$ , the critical region is

$$(\chi_{1-\alpha, (r-1)(c-1)}^2, +\infty)$$

E.g. 1 Sol: Compute the expected frequencies:

		Ebert Ratings			Total
		Down	Sideways	Up	
Siskel Ratings	Down	24 (11.8)	8 (8.4)	13 (24.8)	45
	Sideways	8 (8.4)	13 (6.0)	11 (17.6)	32
	Up	10 (21.8)	9 (15.6)	64 (45.6)	83
	Total	42	30	88	160

$$\implies d_2 = \dots = 45.37$$

Critical region is

$$\left(\chi_{0.99, (3-1) \times (3-1)}^2, +\infty\right) = (13.277, +\infty)$$

Alternatively  $P$ -value =  $\mathbb{P}(\chi_4^2 \geq 45.37) = 3.33 \times 10^{-9}$ .

Rejection at  $\alpha = 0.01$ .

□

E.g. 2 Sol: Compute the expected frequencies:

<b>Table 10.5.9</b>			
		Mobility Index	
		Low (<56.0)	High (≥56.0)
Suicide	High (≥20.8)	4.4*	6.6
Rate	Low (<20.8)	5.6	8.4

\* $\hat{E}(X_{11}) = 4.4$  does not quite satisfy the " $n\hat{p}_i\hat{q}_j \geq 5$ " restriction stated in Theorem 10.5.1, but 4.4 is close enough to 5 to maintain the integrity of the  $\chi^2$  approximation.

$$\implies d_2 = \dots = 4.57$$

Critical region is

$$(\chi_{0.95, (2-1) \times (2-1)}^2, +\infty) = (3.41, +\infty)$$

Alternatively  $P$ -value =  $\mathbb{P}(\chi_1^2 \geq 4.57) = 0.033$

Rejection at  $\alpha = 0.05$ .

