Math 362: Mathematical Statistics II

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Last updated on April 13, 2021

2021 Spring

Chapter 10. Goodness-of-fit Tests

Plan

§ 10.1 Introduction

§ 10.2 The Multinomial Distribution

§ 10.3 Goodness-of-Fit Tests: All Parameters Known

§ 10.4 Goodness-of-Fit Tests: Parameters Unknown

§ 10.5 Contingency Tables

E.g. 1 Whether are the two ratings independent?

Table 10.5.5

		Ebert Ratings			
		Down	Sideways	Up	Total
Siskel Ratings	Down	24	8	13	45
	Sideways	8	13	11	32
	Up	10	9	64	83
	Total	42	30	88	160

E.g. 2 Whether is the suicide rate independent of the mobility factor?

Table 10.5.7

City	Suicides per 100,000, x_i	Mobility Index, y_i	City	Suicides per 100,000, x_i	Mobility Index, y_i
New York	19.3	54.3	Washington	22.5	37.1
Chicago	17.0	51.5	Minneapolis	23.8	56.3
Philadelphia	17.5	64.6	New Orleans	17.2	82.9
Detroit	16.5	42.5	Cincinnati	23.9	62.2
Los Angeles	23.8	20.3	Newark	21.4	51.9
Cleveland	20.1	52.2	Kansas City	24.5	49.4
St. Louis	24.8	62.4	Seattle	31.7	30.7
Baltimore	18.0	72.0	Indianapolis	21.0	66.1
Boston	14.8	59.4	Rochester	17.2	68.0
Pittsburgh	14.9	70.0	Jersey City	10.1	56.5
San Francisco	40.0	43.8	Louisville	16.6	78.7
Milwaukee	19.3	66.2	Portland	29.3	33.2
Buffalo	13.8	67.6			

$$\bar{x} = 20.8 \quad \text{and} \quad \bar{y} = 56.0$$

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$$\bar{x} = 20.8 \quad \text{and} \quad \bar{y} = 56.0$$

Table 10.5.8

		Mobility Index	
		Low (<56.0)	High (≥56.0)
Suicide Rate	High (≥20.8)	7	4
	Low (<20.8)	3	11

Thm 10.4.1 Suppose that n observations are taken on a sample space partitioned by the events $A_1, \cdots, A_r$ and $B_1, \cdots, B_c$.

Let $p_i = \mathbb{P}(A_i)$, $q_j = \mathbb{P}(B_j)$, $p_{ij} = \mathbb{P}(A_i \cap B_j)$.

Let X_{ij} be the number of observations belonging to $A_i \cap B_j$.

a) Provided that $np_{ij} \geq 5$ for all i, j, the r.v.

$$D_2 = \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{(X_{ij} - np_{ij})^2}{np_{ij}} \sim \text{Chi square of f.d. } rc - 1$$

b) To test $H_0 : A_i$'s are independent of B_i's, calculate the test statistic

$$d_2 = \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{(k_{ij} - n\hat{p}_i\hat{q}_j)^2}{n\hat{p}_i\hat{q}_j}$$

where $\hat{p}_i$ and $\hat{q}_j$ are MLE's for p_i and q_j, respectively.

Provided that $n\hat{p}_i\hat{q}_j \geq 5$ for all i, j, the critical region is

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E.g. 1 Sol: Compute the expected frequencies:

Table 10.5.6

		Ebert Ratings			
		Down	Sideways	Up	Total
	Down	24 (11.8)	8 (8.4)	13 (24.8)	45
Siskel Ratings	Sideways	8 (8.4)	13 (6.0)	11 (17.6)	32
	Up	10 (21.8)	9 (15.6)	64 (45.6)	83
	Total	42	30	88	160

$$\Longrightarrow \quad d_2 = \cdots = 45.37$$

Critical region is

$$\left(\chi^2_{0.99,(3-1)\times(3-1)}, +\infty\right) = (13.277, +\infty)$$

Alternatively P-value $= \mathbb{P}(\chi^2_4 \geq 45.37) = 3.33 \times 10^{-9}$.

Rejection at $\alpha = 0.01$. □

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E.g. 2 Sol: Compute the expected frequencies:

Table 10.5.9

		Mobility Index	
		Low (<56.0)	High (≥56.0)
Suicide Rate	High (≥20.8)	4.4*	6.6
	Low (<20.8)	5.6	8.4

*$\hat{E}(X_{11}) = 4.4$ does not quite satisfy the "$n\hat{p}_i\hat{q}_j \geq 5$" restriction stated in Theorem 10.5.1, but 4.4 is close enough to 5 to maintain the integrity of the χ^2 approximation.

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