Math 362: Mathematical Statistics II

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Chapter 11. Regression

§ 11.1 Introduction

§ 11.4 Covariance and Correlation

§ 11.2 The Method of Least Squares

§ 11.3 The Linear Model

§ 11.A Appendix Multiple/Multivariate Linear Regression

§ 11.5 The Bivariate Normal Distribution

Plan

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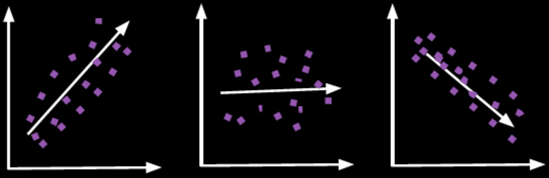
CORRELATION



Positive Correlation

Zero Correlation

Negative Correlation

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_x \sigma_y}$$

Covarianced normalized by Standard Deviation

Correlation between X and Y

Standard deviation of X

Standard deviation of Y

Notation: $Corr(X, Y) = \rho(X, Y) = \rho_{XY}$

Computing: $\mathrm{Var}(X) = \sigma_X^2$, $\mathrm{Var}(Y) = \sigma_Y^2$, $\mathrm{Cov}(X, Y) = \sigma_{XY}$

$$\Downarrow$$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

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$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

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$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Thm. For any two random variables X and Y,

a. $|\rho(X,Y)| \leq 1$

b. $\rho(X,Y) = 1$ if and only if $Y = aX + b$ for some $a > 0$ and $b \in \mathbb{R}$;
$\rho(X,Y) = -1$ if and only if $Y = aX + b$ for some $a < 0$ and $b \in \mathbb{R}$.

Proof. (a)

$$|\rho(X,Y)| \leq 1$$

$$\Updownarrow$$

$$\begin{aligned} |\mathbb{E}\left((X-\mathbb{E}(X))(Y-\mathbb{E}(Y))\right)| &\leq \sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)} \\ &= \sqrt{\mathbb{E}\left((X-\mathbb{E}(X))^2\right)}\sqrt{\mathbb{E}\left((Y-\mathbb{E}(Y))^2\right)} \end{aligned}$$

which is nothing but the Cauchy-Schwartz inequality.

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which is nothing but the Cauchy-Schwartz inequality.

(b) In the Cauchy-Schwartz inequality, the equality holds if and only if for some $a \neq 0$,

$$X - \mathbb{E}(X) = a[Y - E(Y)]$$

namely,

$$X = aY + b, \qquad \text{with} \quad b = \mathbb{E}(X) - a\mathbb{E}(Y).$$

In particular, $a > 0$ corresponds to the case $\rho(X, Y) = 1$ and $a < 0$ to $\rho(X, Y) = -1$. $\square$

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Estimating $\rho(X, Y)$ – Sample correlation coefficient

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}}$$

$$= \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\sqrt{\mathbb{E}[X^2] - \mathbb{E}[X]^2}\sqrt{\mathbb{E}[Y^2] - \mathbb{E}[Y]^2}}$$

$$\Downarrow$$

$$R = \frac{n\sum_{i=1}^n X_i Y_i - \left(\sum_{i=1}^n X_i\right)\left(\sum_{i=1}^n Y_i\right)}{\sqrt{n\sum_{i=1}^n X_i^2 - \left(\sum_{i=1}^n X_i\right)^2}\sqrt{n\sum_{i=1}^n Y_i^2 - \left(\sum_{i=1}^n Y_i\right)^2}}$$

Pearson product-moment correlation coefficient

or

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Thm.

$$R^2 = 1 - \frac{SSE}{SST} = \frac{SST - SSE}{SST} = \frac{SSTR}{SST}$$

where

$$SSE = \sum_{i=1}^{n} \left(Y_i - \widehat{Y}_i \right)^2, \quad \widehat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$$

$$SST = \sum_{i=1}^{n} \left(Y_i - \overline{Y}_i \right)^2, \quad \text{and} \quad SSTR = SST - SSE.$$

Remark SSE: sum of square errors $\sim$ the variation in y_i's not explained by L.M.

SST: Total sum of squares $\sim$ total variability.

SSTR: Treatment sum of sqrs. $\sim$ the variation in y_i's explained by L.M.

R^2 (or r^2 when X_i and Y_i are replaced by x_i and y_i) $\sim$ proportion of total variation in the y_i's that can be attributed to L.M.

Coefficient of determination or simply *R squared*

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Coefficient of determination or simply *R squared*

Proof

□

Adjusted R-squared

Def. The adjusted R-squareed:

$$R^2_{adj} := 1 - \frac{MSE}{MST}$$

where

$$MSE = \frac{SSE}{n-q} \qquad \text{and} \qquad MST = \frac{SST}{n-1}$$

and q is number of parameters in the model.

Relation:

$$R^2_{adj} = 1 - \left(1 - R^2\right)\frac{n-1}{n-q}$$

MSE: Mean squared error.

MST: Mean squared total.

MSR = MSTR: Mean square for treatment (or regresssion).

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