Math 362: Mathematical Statistics II

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Chapter 12. The Analysis of Variance

1. John Wilder Tukey (June 16, 1915 – July 26, 2000) was an American mathematician best known for development of the Fast Fourier Transform (FFT) algorithm and box plot.

2. The Tukey range test, the Tukey lambda distribution, the Tukey test of additivity, and the Teichmüller-Tukey lemma all bear his name.

3. He is also credited with coining the term 'bit'.

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https://en.wikipedia.org/wiki/John_Tukey
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$N(\mu_1, \sigma^2)$	$N(\mu_2, \sigma^2)$	$\cdots$	$N(\mu_2, \sigma^2)$
Y_{11}	Y_{12}	$\cdots$	Y_{1k}
Y_{21}	Y_{22}	$\cdots$	Y_{2k}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
Y_{r1}	Y_{r2}	$\cdots$	Y_{rk}

Goal For any $i \neq j$, test

$$H_0 : \mu_i = \mu_j \qquad v.s. \qquad H_1 : \mu_i \neq \mu_j$$

at the α level of significance defined as

$$\mathbb{P}\left(\bigcup_{j=1}^{\binom{k}{2}} E_j\right) = \alpha$$

where there are $\binom{k}{2}$ pairs, and E_j is the event of making a type I error for the j-th pair.

Goal' Simultaneous C.I.'s for $\binom{k}{2}$ pairs of means:

Given α, find I_{ij}, the C.I. for $\mu_i - \mu_j$ (with $i, j = 1, \cdots, k$ and $i \neq j$), s.t.

$$\mathbb{P}\left(\mu_i - \mu_j \in I_{ij}, \ \forall i, j = 1, \cdots, k, i \neq j\right) = 1 - \alpha.$$

??? Why not the standard pair-wise two-sample t-test?

Suppose $\mathbb{P}(E_j) = \alpha_*$. Then

$$\alpha = \mathbb{P}\left(\bigcup_{j=1}^{\binom{k}{2}} E_j\right) = 1 - \mathbb{P}\left(\bigcap_{j=1}^{\binom{k}{2}} E_j^c\right) \approx 1 - \prod_{j=1}^{\binom{k}{2}} \mathbb{P}(E_j^c) = 1 - (1-\alpha_*)^{\binom{k}{2}}$$

Hence,

$$\alpha_* \approx 1 - (1-\alpha)^{1/\binom{k}{2}}$$

E.g., $\alpha = 0.05$

k	5	8	100
α_*	0.0051162	0.001830	0.00001036

Bonferroni's method
— A straightforward method

$$\mathbb{P}\left(\mu_i - \mu_j \in I_{ij}, \ \forall i \neq j\right)$$

$$\|$$

$$\mathbb{P}\left(\bigcap_{i \neq j} \mu_i - \mu_j \in I_{ij}\right)$$

$$\|$$

$$1 - \mathbb{P}\left(\bigcup_{i \neq j} \mu_i - \mu_j \notin I_{ij}\right)$$

$$\vee|$$

$$1 - \sum_{i \neq j} \mathbb{P}\left(\mu_i - \mu_j \notin I_{ij}\right)$$

$$\|$$

$$1 - \binom{k}{2}\alpha_*$$

1. If we choose $\alpha_* = \alpha / \binom{k}{2}$,

2. let I_{ij} be the $(1 - \alpha_*)100\%$ C.I. $i \neq j$

$$\Downarrow$$

$$\mathbb{P}\left(\mu_i - \mu_j \in I_{ij}, \ \forall i \neq j\right)$$

$$\vee|$$

$$1 - \binom{k}{2}\alpha_*$$

$$\|$$

$$1 - \alpha.$$

Remark This is an approximation. The resulting C.I. are in general too wide.

The exact, and much more precise, solution is given by J.W. Turkey.

One can also construct simultaneous C.I. for all possible linear combinations of the parameters $\sum_{j=1}^{k} c_j \mu_j$, this can be acchieved by **Scheffé's method**. A simple verson is given in §12.4.

Tukey's HSD (honestly significant difference) test

Let's construct $(1 - \alpha)100\%$ C.I.'s simultaneously for all pairs.

$$\mathbb{P}\left(\left|(\overline{Y}_{\cdot i} - \mu_i) - (\overline{Y}_{\cdot j} - \mu_j)\right| \leq \mathcal{E}, \quad \forall i \neq j\right) = 1 - \alpha$$

$$\|$$

$$\mathbb{P}\left(\max_i(\overline{Y}_{\cdot i} - \mu_i) - \min_j(\overline{Y}_{\cdot j} - \mu_j) \leq \mathcal{E}\right)$$

$$\|$$

$$\mathbb{P}\left(\max_i \overline{Y}_{\cdot i} - \min_j \overline{Y}_{\cdot j} \leq \mathcal{E}\right)$$
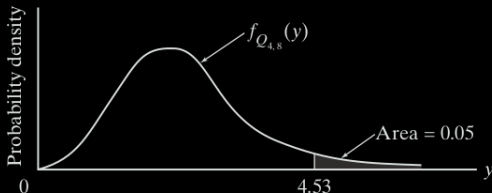
$\implies$ Needs to study ...

Def. Let $W_1, \cdots, W_k$ be k i.i.d. r.v.'s from $N(\mu, \sigma^2)$. Let R denote their range:

$$R = \max_i W_i - \min_i W_i.$$

Let S^2 be an unbiased estimator for σ^2 independent of the W_i's and based on ν df. Define the **Studentized range,** $Q_{k,\nu}$, to be the ratio:
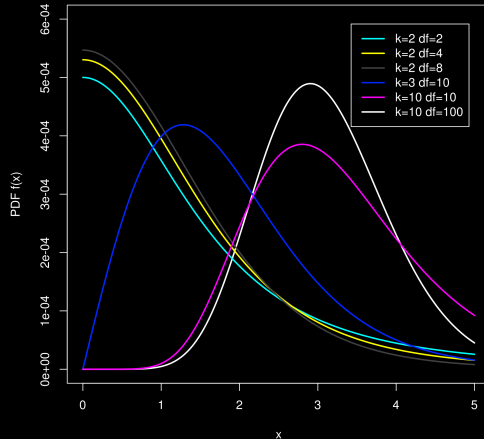
$$Q_{k,\nu} := \frac{R}{S}.$$

Remark 0.1 We need $R \perp S$ to mimic Student's t-distribution.
0.2 In the following $\nu = n - k = rk - k = r(k-1)$.

$Q_{k,\nu} \sim$ **Studentized range distribution** with parameters k and ν.

k: number of groups.

ν: degrees of freedom.

Let's find one example that all requirements of the $Q_{k,\nu}$ are satisfied.

1. Take $W_j = \overline{Y}_{\cdot j} - \mu_j, j = 1, \cdots, k \implies W_j \sim N(0, \sigma^2/r)$.

2. *MSE* or the pooled variance S_p^2 *MSE*/r

      is an unbiased estimator for σ^2 σ^2/r

      is $\perp \{\overline{Y}_{\cdot j}\}_{j=1,\cdots,k}$, hence $\perp \{W_j\}_{j=1,\cdots,k}$

3. *df* of *MSE* is equal to $n - k = kr - k = k(r-1)$.

$\implies \quad \dfrac{\max_i W_i - \min_j W_j}{\sqrt{MSE/r}} \sim$ Studentized range distribution($k, rk - k$)

$$\mathbb{P}\left(\frac{\max_i W_i - \min_j W_j}{\sqrt{MSE/r}} \le Q_{\alpha,k,rk-k}\right) = 1 - \alpha$$

$$\|$$

$$\mathbb{P}\left(\max_i W_i - \min_j W_j \le \frac{Q_{\alpha,k,rk-k}}{\sqrt{r}}\sqrt{MSE}\right)$$

$$\|$$

$$\mathbb{P}\left(|W_i - W_j| \le \frac{Q_{\alpha,k,rk-k}}{\sqrt{r}}\sqrt{MSE}, \ \forall i \ne j\right)$$

$$\|$$

$$\mathbb{P}\left(\left|\left(\overline{Y}_{\cdot i} - \overline{Y}_{\cdot j}\right) - (\mu_i - \mu_j)\right| \le \frac{Q_{\alpha,k,rk-k}}{\sqrt{r}}\sqrt{MSE}, \ \forall i \ne j\right)$$

$$\|$$

$$\mathbb{P}\left(\overline{Y}_{\cdot i} - \overline{Y}_{\cdot j} - \frac{Q_{\alpha,k,rk-k}}{\sqrt{r}}\sqrt{MSE} \le \mu_i - \mu_j \le \overline{Y}_{\cdot i} - \overline{Y}_{\cdot j} + \frac{Q_{\alpha,k,rk-k}}{\sqrt{r}}\sqrt{MSE}, \ \forall i \ne j\right)$$

Therefore, for all $i \neq j$, the $100(1 - \alpha)\%$ C.I. for $\mu_i - \mu_j$ is

$$\overline{Y}_{\cdot i} - \overline{Y}_{\cdot j} \pm \frac{Q_{\alpha, k, rk-k}}{\sqrt{2}} \sqrt{MSE} \sqrt{\frac{2}{r}}$$

To test $H_0 : \mu_i = \mu_j$ for specific $i \neq j$, reject H_0 in favor of $H_1 : \mu_i \neq \mu_j$ if the C.I. does NOT contain 0, at the α level of significance. $\qquad \square$

Note: When sample sizes are not equal, use the **Tukey-Kramer method**:

$$\overline{Y}_{\cdot i} - \overline{Y}_{\cdot j} \pm \frac{Q_{\alpha, k, rk-k}}{\sqrt{2}} \sqrt{MSE} \sqrt{\frac{1}{r_i} + \frac{1}{r_j}}$$

E.g. 2 A certain fraction of antibiotics injected into the bloodstream are "bound" to serum proteins. This phenomenon bears directly on the effectiveness of the medication, because the binding decreases the systemic uptake of the drug. Table below lists the binding percentages in bovine serum measured for five widely prescribed antibiotics. Which antibiotics have similar binding properties, and which are different?

Table 12.3.1

	Penicillin G	Tetra- cycline	Strepto- mycin	Erythro- mycin	Chloram- phenicol
	29.6	27.3	5.8	21.6	29.2
	24.3	32.6	6.2	17.4	32.8
	28.5	30.8	11.0	18.3	25.0
	32.0	34.8	8.3	19.0	24.2
$T_{.j}$	114.4	125.5	31.3	76.3	111.2
$\overline{Y}_{.j}$	28.6	31.4	7.8	19.1	27.8

To answer that question requires that we make all $\binom{5}{2} = 10$ pairwise comparisons of μ_i versus μ_j. First, MSE must be computed. From the entries in Table 12.3.1,

$$SSE = \sum_{j=1}^{5} \sum_{i=1}^{4} \left(Y_{ij} - \overline{Y}_{.j}\right)^2 = 135.83$$

so $MSE = 135.83/(20-5) = 9.06$. Let $\alpha = 0.05$. Since $n - k = 20 - 5 = 15$, the appropriate cutoff from the studentized range distribution is $Q_{.05,5,15} = 4.37$. Therefore, $D = 4.37/\sqrt{4} = 2.185$ and $D\sqrt{MSE} = 6.58$.

Table 12.3.2

Pairwise Difference	$\overline{Y}_{.i} - \overline{Y}_{.j}$	Tukey Interval	Conclusion
$\mu_1 - \mu_2$	-2.8	$(-9.38, 3.78)$	NS
$\mu_1 - \mu_3$	20.8	$(14.22, 27.38)$	Reject
$\mu_1 - \mu_4$	9.5	$(2.92, 16.08)$	Reject
$\mu_1 - \mu_5$	0.8	$(-5.78, 7.38)$	NS
$\mu_2 - \mu_3$	23.6	$(17.02, 30.18)$	Reject
$\mu_2 - \mu_4$	12.3	$(5.72, 18.88)$	Reject
$\mu_2 - \mu_5$	3.6	$(-2.98, 10.18)$	NS
$\mu_3 - \mu_4$	-11.3	$(-17.88, -4.72)$	Reject
$\mu_3 - \mu_5$	-20.0	$(-26.58, -13.42)$	Reject
$\mu_4 - \mu_5$	-8.7	$(-15.28, -2.12)$	Reject

```
> # Case Study 12.3.1
> # Input data first
> Input <- c("
+ rates group
+ 29.6 M1
+ 24.3 M1
+ 28.5 M1
+ 32.0 M1
+ 27.3 M2
+ 32.6 M2
+ 30.8 M2
+ 34.8 M2
+ 5.8  M3
+ 6.2  M3
+ 11.0 M3
+ 8.3  M3
+ 21.6 M4
+ 17.4 M4
+ 18.3 M4
+ 19.0 M4
+ 29.2 M5
+ 32.8 M5
+ 25.0 M5
+ 24.2 M5
+ ")
> Data = read.table(
  textConnection(Input),
+       header=TRUE)
```

```
> # Compute one-way ANOVA test
> res.aov <- aov(rates ~ group, data = Data)
> # Summary of the analysis
> summary(res.aov)
            Df Sum Sq Mean Sq F value Pr(>F)
group        4 1480.8   370.2   40.88 6.74e-08 ***
Residuals   15  135.8     9.1
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' '
     1
```

```
1  > # Tukey multiple pairwise-comparisons
2  > TukeyHSD(res.aov)
3    Tukey multiple comparisons of means
4      95% family-wise confidence level
5
6  Fit: aov(formula = rates ~ group, data = Data)
7
8  $group
9            diff        lwr         upr      p adj
10 M2-M1  2.775  -3.795401   9.345401 0.6928357
11 M3-M1 -20.775 -27.345401 -14.204599 0.0000006
12 M4-M1  -9.525 -16.095401  -2.954599 0.0034588
13 M5-M1  -0.800  -7.370401   5.770401 0.9952758
14 M3-M2 -23.550 -30.120401 -16.979599 0.0000001
15 M4-M2 -12.300 -18.870401  -5.729599 0.0003007
16 M5-M2  -3.575 -10.145401   2.995401 0.4737713
17 M4-M3  11.250   4.679599  17.820401 0.0007429
18 M5-M3  19.975  13.404599  26.545401 0.0000010
19 M5-M4   8.725   2.154599  15.295401 0.0071611
```

```
1  > round(TukeyHSD(res.aov)$group,2)
2           diff      lwr      upr   p adj
3  M2−M1   2.78    −3.80     9.35    0.69
4  M3−M1  −20.77  −27.35   −14.20    0.00
5  M4−M1   −9.52  −16.10    −2.95    0.00
6  M5−M1   −0.80   −7.37     5.77    1.00
7  M3−M2  −23.55  −30.12   −16.98    0.00
8  M4−M2  −12.30  −18.87    −5.73    0.00
9  M5−M2   −3.58  −10.15     3.00    0.47
10 M4−M3   11.25    4.68    17.82    0.00
11 M5−M3   19.97   13.40    26.55    0.00
12 M5−M4    8.73    2.15    15.30    0.01
13 − − −
14 Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 '
             ' 1
15 (Adjusted p values reported − − single−step
         method)
```

Table 12.3.2			
Pairwise Difference	$\overline{Y}_{.i} - \overline{Y}_{.j}$	Tukey Interval	Conclusion
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$\mu_2 - \mu_5$	3.6	(−2.98, 10.18)	NS
$\mu_3 - \mu_4$	−11.3	(−17.88, −4.72)	Reject
$\mu_3 - \mu_5$	−20.0	(−26.58, −13.42)	Reject
	8.7	(−15.38, −2.12)	Reject

```
> # Or one may use multcomp package or multiple
      comparisons
> library(multcomp)
> summary(glht(res.aov, linfct = mcp(group = "Tukey")))


  Simultaneous Tests for General Linear Hypotheses

Multiple Comparisons of Means: Tukey Contrasts


Fit: aov(formula = rates ~ group, data = Data)

Linear Hypotheses:
           Estimate Std. Error t value Pr(>|t|)
M2 − M1 == 0 2.775   2.128  1.304  0.69283
M3 − M1 == 0 −20.775 2.128 −9.764 < 0.001 ***
M4 − M1 == 0 −9.525 2.128 −4.477 0.00348 **
M5 − M1 == 0 −0.800 2.128 −0.376 0.99528
M3 − M2 == 0 −23.550 2.128 −11.068 < 0.001 ***
M4 − M2 == 0 −12.300 2.128 −5.781 < 0.001 ***
M5 − M2 == 0 −3.575 2.128 −1.680 0.47374
M4 − M3 == 0 11.250  2.128  5.287 < 0.001 ***
M5 − M3 == 0 19.975  2.128  9.388 < 0.001 ***
M5 − M4 == 0 8.725   2.128  4.101 0.00717 **
−−−
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
(Adjusted p values reported −− single−step method)
```

```
             Estimate Std. Error t value Pr(>|t|)
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M3 − M1 == 0 −20.775   2.128  −9.764  < 0.001 ***
M4 − M1 == 0 −9.525    2.128  −4.477  0.00348 **
M5 − M1 == 0 −0.800    2.128  −0.376  0.99527
M3 − M2 == 0 −23.550   2.128 −11.068  < 0.001 ***
M4 − M2 == 0 −12.300   2.128  −5.781  < 0.001 ***
M5 − M2 == 0 −3.575    2.128  −1.680  0.47371
M4 − M3 == 0 11.250    2.128   5.287  < 0.001 ***
M5 − M3 == 0 19.975    2.128   9.388  < 0.001 ***
M5 − M4 == 0 8.725     2.128   4.101  0.00719 **
```

Table 12.3.2

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Two more examples of ANOVA using R

E.g. 1 `http://www.sthda.com/english/wiki/one-way-anova-test-in-r`

E.g. 2 `https://datascienceplus.com/one-way-anova-in-r/`