Math 362: Mathematical Statistics II

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Last updated on April 24, 2021

2021 Spring

Chapter 12. The Analysis of Variance

Plan

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1. John Wilder Tukey (June 16, 1915 – July 26, 2000) was an American mathematician best known for development of the Fast Fourier Transform (FFT) algorithm and box plot.
2. The Tukey range test, the Tukey lambda distribution, the Tukey test of additivity, and the Teichmüller-Tukey lemma all bear his name.
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$N(\mu_1, \sigma^2)$	$N(\mu_2, \sigma^2)$	$\cdots$	$N(\mu_2, \sigma^2)$
Y_{11}	Y_{12}	$\cdots$	Y_{1k}
Y_{21}	Y_{22}	$\cdots$	Y_{2k}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
Y_{r1}	Y_{r2}	$\cdots$	Y_{rk}

Goal For any $i \neq j$, test

$$H_0 : \mu_i = \mu_j \qquad v.s. \qquad H_1 : \mu_i \neq \mu_j$$

at the α level of significance defined as

$$\mathbb{P}\left(\bigcup_{j=1}^{\binom{k}{2}} E_j\right) = \alpha$$

where there are $\binom{k}{2}$ pairs, and E_j is the event of making a type I error for the j-th pair.

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Goal' Simultaneous C.I.'s for $\binom{k}{2}$ pairs of means:

Given α, find I_{ij}, the C.I. for $\mu_i - \mu_j$ (with $i, j = 1, \cdots, k$ and $i \neq j$), s.t.

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??? Why not the standard pair-wise two-sample t-test?

Suppose $\mathbb{P}(E_j) = \alpha_*$. Then

$$\alpha = \mathbb{P}\left(\bigcup_{j=1}^{\binom{k}{2}} E_j\right) = 1 - \mathbb{P}\left(\bigcap_{j=1}^{\binom{k}{2}} E_j^c\right) \approx 1 - \prod_{j=1}^{\binom{k}{2}} \mathbb{P}(E_j^c) = 1 - (1-\alpha_*)^{\binom{k}{2}}$$

Hence,

$$\alpha_* \approx 1 - (1-\alpha)^{1/\binom{k}{2}}$$

E.g., $\alpha = 0.05$

k	5	8	100
α_*	0.0051162	0.001830	0.00001036

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Bonferroni's method

— A straightforward method

$$\mathbb{P}\left(\mu_i - \mu_j \in I_{ij},\ \forall i \neq j\right)$$

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Remark This is an approximation. The resulting C.I. are in general too wide.

The exact, and much more precise, solution is given by J.W. Turkey.

One can also construct simultaneous C.I. for all possible linear combinations of the parameters $\sum_{j=1}^{k} c_j \mu_j$, this can be acchieved by **Scheffé's method**. A simple verson is given in §12.4.

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Tukey's HSD (honestly significant difference) test

Let's construct $(1-\alpha)100\%$ C.I.'s simultaneously for all pairs.

$$\mathbb{P}\left(\left|(\overline{Y}_{\cdot i}-\mu_i)-(\overline{Y}_{\cdot j}-\mu_j)\right| \le \mathcal{E}, \quad \forall i \ne j\right) = 1-\alpha$$

$$||$$

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Def. Let $W_1, \cdots, W_k$ be k i.i.d. r.v.'s from $N(\mu, \sigma^2)$. Let R denote their range:

$$R = \max_i W_i - \min_i W_i.$$

Let S^2 be an unbiased estimator for σ^2 independent of the W_i's and based on ν df. Define the **Studentized range,** $Q_{k,\nu}$, to be the ratio:

$$Q_{k,\nu} := \frac{R}{S}.$$

Remark
0.1 We need $R \perp S$ to mimic Student's t-distribution.
0.2 In the following $\nu = n - k = rk - k = r(k - 1)$.

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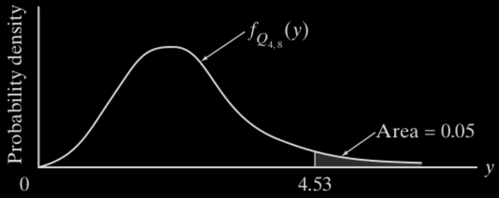

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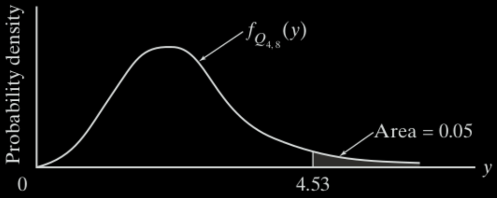

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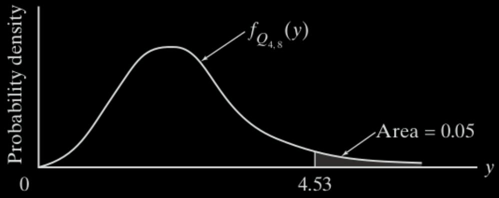

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$Q_{k,\nu} \sim$ **Studentized range distribution** with parameters k and ν.

k: number of groups.

ν: degrees of freedom.

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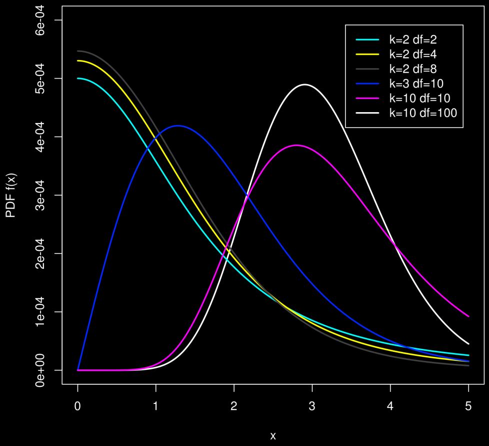

Let's find one example that all requirements of the $Q_{k,\nu}$ are satisfied.

1. Take $W_j = \overline{Y}_{\cdot j} - \mu_j$, $j = 1, \cdots, k \quad \Longrightarrow \quad W_j \sim N(0, \sigma^2/r)$.

2. MSE or the pooled variance S_p^2 — MSE/r
   is an unbiased estimator for σ^2 — σ^2/r
   is $\perp \{\overline{Y}_{\cdot j}\}_{j=1,\cdots,k}$, hence $\perp \{W_j\}_{j=1,\cdots,k}$

3. df of MSE is equal to $n - k = kr - k = k(r-1)$.

$$\Longrightarrow \quad \frac{\max_i W_i - \min_j W_j}{\sqrt{MSE/r}} \sim \text{Studentized range distribution}(k,\ rk-k)$$

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$$\mathbb{P}\left(\frac{\max_i W_i - \min_j W_j}{\sqrt{MSE/r}} \leq Q_{\alpha,k,rk-k}\right) = 1 - \alpha$$

$$||$$

$$\mathbb{P}\left(\max_i W_i - \min_j W_j \leq \frac{Q_{\alpha,k,rk-k}}{\sqrt{r}}\sqrt{MSE}\right)$$

$$||$$

$$\mathbb{P}\left(|W_i - W_j| \leq \frac{Q_{\alpha,k,rk-k}}{\sqrt{r}}\sqrt{MSE},\ \forall i \neq j\right)$$

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$$\mathbb{P}\left(\left|\left(\overline{Y}_{\cdot i} - \overline{Y}_{\cdot j}\right) - (\mu_i - \mu_j)\right| \leq \frac{Q_{\alpha,k,rk-k}}{\sqrt{r}}\sqrt{MSE},\ \forall i \neq j\right)$$

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Therefore, for all $i \neq j$, the $100(1-\alpha)\%$ C.I. for $\mu_i - \mu_j$ is

$$\overline{Y}_{\cdot i} - \overline{Y}_{\cdot j} \pm \frac{Q_{\alpha,k,rk-k}}{\sqrt{2}}\sqrt{MSE}\sqrt{\frac{2}{r}}$$

To test $H_0 : \mu_i = \mu_j$ for specific $i \neq j$, reject H_0 in favor of $H_1 : \mu_i \neq \mu_j$ if the C.I. does NOT contain 0, at the α level of significance. □

Note: When sample sizes are not equal, use the **Tukey-Kramer method**:

$$\overline{Y}_{\cdot i} - \overline{Y}_{\cdot j} \pm \frac{Q_{\alpha,k,rk-k}}{\sqrt{2}}\sqrt{MSE}\sqrt{\frac{1}{r_i} + \frac{1}{r_j}}$$

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E.g. 2 A certain fraction of antibiotics injected into the bloodstream are "bound" to serum proteins. This phenomenon bears directly on the effectiveness of the medication, because the binding decreases the systemic uptake of the drug. Table below lists the binding percentages in bovine serum measured for five widely prescribed antibiotics. Which antibiotics have similar binding properties, and which are different?

E.g. 2 A certain fraction of antibiotics injected into the bloodstream are "bound" to serum proteins. This phenomenon bears directly on the effectiveness of the medication, because the binding decreases the systemic uptake of the drug. Table below lists the binding percentages in bovine serum measured for five widely prescribed antibiotics. Which antibiotics have similar binding properties, and which are different?

Table 12.3.1

	Penicillin G	Tetra-cycline	Strepto-mycin	Erythro-mycin	Chloram-phenicol
	29.6	27.3	5.8	21.6	29.2
	24.3	32.6	6.2	17.4	32.8
	28.5	30.8	11.0	18.3	25.0
	32.0	34.8	8.3	19.0	24.2
$T_{.j}$	114.4	125.5	31.3	76.3	111.2
$\overline{Y}_{.j}$	28.6	31.4	7.8	19.1	27.8

To answer that question requires that we make all $\binom{5}{2} = 10$ pairwise comparisons of μ_i versus μ_j. First, MSE must be computed. From the entries in Table 12.3.1,

$$SSE = \sum_{j=1}^{5} \sum_{i=1}^{4} \left(Y_{ij} - \overline{Y}_{.j}\right)^2 = 135.83$$

so $MSE = 135.83/(20 - 5) = 9.06$. Let $\alpha = 0.05$. Since $n - k = 20 - 5 = 15$, the appropriate cutoff from the studentized range distribution is $Q_{.05,5,15} = 4.37$. Therefore, $D = 4.37/\sqrt{4} = 2.185$ and $D\sqrt{MSE} = 6.58$.

Table 12.3.2

Pairwise Difference	$\overline{Y}_{.i} - \overline{Y}_{.j}$	Tukey Interval	Conclusion
$\mu_1 - \mu_2$	−2.8	(−9.38, 3.78)	NS
$\mu_1 - \mu_3$	20.8	(14.22, 27.38)	Reject
$\mu_1 - \mu_4$	9.5	(2.92, 16.08)	Reject
$\mu_1 - \mu_5$	0.8	(−5.78, 7.38)	NS
$\mu_2 - \mu_3$	23.6	(17.02, 30.18)	Reject
$\mu_2 - \mu_4$	12.3	(5.72, 18.88)	Reject
$\mu_2 - \mu_5$	3.6	(−2.98, 10.18)	NS
$\mu_3 - \mu_4$	−11.3	(−17.88, −4.72)	Reject
$\mu_3 - \mu_5$	−20.0	(−26.58, −13.42)	Reject
$\mu_4 - \mu_5$	−8.7	(−15.28, −2.12)	Reject

```
> # Case Study 12.3.1
> # Input data first
> Input <- c("
+ rates group
+ 29.6 M1
+ 24.3 M1
+ 28.5 M1
+ 32.0 M1
+ 27.3 M2
+ 32.6 M2
+ 30.8 M2
+ 34.8 M2
+ 5.8  M3
+ 6.2  M3
+ 11.0 M3
+ 8.3  M3
+ 21.6 M4
+ 17.4 M4
+ 18.3 M4
+ 19.0 M4
+ 29.2 M5
+ 32.8 M5
+ 25.0 M5
+ 24.2 M5
+ ")
> Data = read.table(
  textConnection(Input),
+        header=TRUE)
```

```
> # Compute one-way ANOVA test
> res.aov <- aov(rates ~ group, data = Data)
> # Summary of the analysis
> summary(res.aov)
            Df Sum Sq Mean Sq F value Pr(>F)
group        4 1480.8   370.2   40.88 6.74e-08 ***
Residuals   15  135.8     9.1
---
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' '
        1
```

```
> # Tukey multiple pairwise-comparisons
> TukeyHSD(res.aov)
  Tukey multiple comparisons of means
    95% family-wise confidence level

Fit: aov(formula = rates ~ group, data = Data)

$group
          diff        lwr        upr     p adj
M2-M1 2.775 -3.795401 9.345401 0.6928357
M3-M1 -20.775 -27.345401 -14.204599 0.0000006
M4-M1 -9.525 -16.095401 -2.954599 0.0034588
M5-M1 -0.800 -7.370401 5.770401 0.9952758
M3-M2 -23.550 -30.120401 -16.979599 0.0000001
M4-M2 -12.300 -18.870401 -5.729599 0.0003007
M5-M2 -3.575 -10.145401 2.995401 0.4737713
M4-M3 11.250 4.679599 17.820401 0.0007429
M5-M3 19.975 13.404599 26.545401 0.0000010
M5-M4 8.725 2.154599 15.295401 0.0071611
```

```
> round(TukeyHSD(res.aov)$group,2)
           diff     lwr     upr  p adj
M2-M1    2.78   -3.80    9.35   0.69
M3-M1  -20.77  -27.35  -14.20   0.00
M4-M1   -9.52  -16.10   -2.95   0.00
M5-M1   -0.80   -7.37    5.77   1.00
M3-M2  -23.55  -30.12  -16.98   0.00
M4-M2  -12.30  -18.87   -5.73   0.00
M5-M2   -3.58  -10.15    3.00   0.47
M4-M3   11.25    4.68   17.82   0.00
M5-M3   19.97   13.40   26.55   0.00
M5-M4    8.73    2.15   15.30   0.01
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
(Adjusted p values reported -- single-step method)
```

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$\mu_3 - \mu_5$	−20.0	(−26.58, −13.42)	Reject

```
> # Or one may use multcomp package or multiple
      comparisons
> library(multcomp)
> summary(glht(res.aov, linfct = mcp(group = "Tukey")))

   Simultaneous Tests for General Linear Hypotheses

Multiple Comparisons of Means: Tukey Contrasts


Fit: aov(formula = rates ~ group, data = Data)

Linear Hypotheses:
              Estimate Std. Error t value Pr(>|t|)
M2 - M1 == 0 2.775   2.128   1.304  0.69283
M3 - M1 == 0 -20.775 2.128 -9.764 < 0.001 ***
M4 - M1 == 0 -9.525 2.128 -4.477 0.00348 **
M5 - M1 == 0 -0.800 2.128 -0.376 0.99528
M3 - M2 == 0 -23.550 2.128 -11.068 < 0.001 ***
M4 - M2 == 0 -12.300 2.128 -5.781 < 0.001 ***
M5 - M2 == 0 -3.575 2.128 -1.680 0.47374
M4 - M3 == 0 11.250  2.128   5.287  < 0.001 ***
M5 - M3 == 0 19.975  2.128   9.388  < 0.001 ***
M5 - M4 == 0 8.725   2.128   4.101  0.00717 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
(Adjusted p values reported -- single-step method)
```

```
                 Estimate Std. Error t value Pr(>|t|)
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M3 - M2 == 0 -23.550 2.128 -11.068 < 0.001 ***
M4 - M2 == 0 -12.300 2.128 -5.781 < 0.001 ***
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Two more examples of ANOVA using R

E.g. 1 http://www.sthda.com/english/wiki/one-way-anova-test-in-r

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