Math 362: Mathematical Statistics II

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Chapter 5. Estimation

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Rationale. Point estimate doesn't provide precision information.

By using the variance of the estimator, one can construct an interval such that with a high probability that interval will contain the unknown parameter.

- The interval is called **confidence interval**.
- The high probability is **confidence level**.

E.g. 1. A random sample of size 4, $(Y_1 = 6.5,\ Y_2 = 9.2,\ Y_3 = 9.9,\ Y_4 = 12.4)$, from a normal population:

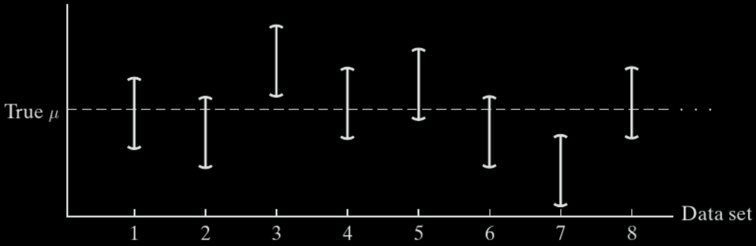
$$f_Y(y;\mu) = \frac{1}{\sqrt{2\pi}\,0.8} e^{-\frac{1}{2}\left(\frac{y-\mu}{0.8}\right)^2}.$$

Both MLE and MME give $\mu_e = \bar{y} = \frac{1}{4}(6.5 + 9.2 + 9.9 + 12.4) = 9.5$.
The estimator $\widehat{\mu} = \overline{Y}$ follows normal distribution.

Construct 95%-confidence interval for μ ...

"The parameter is an unknown constant and no probability statement concerning its value may be made."

–Jerzy Neyman, original developer of confidence intervals.



Possible 95% confidence intervals for μ

In general, for a normal population with σ known, the $100(1-\alpha)\%$ **confidence interval** for μ is

$$\left(\bar{y} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{y} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

Comment: There are many variations

1. One-sided interval such as

$$\left(\bar{y} - z_{\alpha}\frac{\sigma}{\sqrt{n}}, \bar{y}\right) \quad \text{or} \quad \left(\bar{y}, \bar{y} + z_{\alpha}\frac{\sigma}{\sqrt{n}}\right)$$

2. σ is unknown and sample size is small: z-score $\rightarrow$ t-score
3. σ is unknown and sample size is large: z-score by CLT
4. Non-Gaussian population but sample size is large: z-score by CLT

Theorem. Let k be the number of successes in n independent trials, where n is large and $p = \mathbb{P}(\textit{success})$ is unknown. An approximate $100(1-\alpha)\%$ confidence interval for p is the set of numbers

$$\left(\frac{k}{n} - z_{\alpha/2}\sqrt{\frac{(k/n)(1-k/n)}{n}},\ \frac{k}{n} + z_{\alpha/2}\sqrt{\frac{(k/n)(1-k/n)}{n}}\right).$$

Proof: It follows the following facts:

- $X \sim$binomial(n,p) iff $X = Y_1 + \cdots + Y_n$, while Y_i are i.i.d. Bernoulli(p):

$$\mathbb{E}[Y_i] = p \quad \text{and} \quad \mathrm{Var}(Y_i) = p(1-p).$$

- **Central Limit Theorem**: Let $W_1, W_2, \cdots, W_n$ be an sequence of i.i.d. random variables, whose distribution has mean μ and variance σ^2, then

$$\frac{\sum_{i=1}^n W_i - n\mu}{\sqrt{n\sigma^2}} \quad \textit{approximately} \text{ follows } \quad N(0,1), \quad \text{when } n \text{ is large.}$$

- When the sample size n is large, by the central limit theorem,

$$\frac{\sum_{i=1}^{n} Y_i - np}{\sqrt{np(1-p)}} \quad \overset{\text{ap.}}{\sim} \quad N(0,1)$$

$$\|$$

$$\frac{X - np}{\sqrt{np(1-p)}} = \frac{\frac{X}{n} - p}{\sqrt{\frac{p(1-p)}{n}}} \approx \frac{\frac{X}{n} - p}{\sqrt{\frac{p_e(1-p_e)}{n}}}$$

- Since $p_e = \frac{k}{n}$, we see that

$$\mathbb{P}\left(-z_{\alpha/2} \leq \frac{\frac{X}{n} - p}{\sqrt{\frac{\frac{k}{n}\left(1-\frac{k}{n}\right)}{n}}} \leq z_{\alpha/2}\right) \approx 1 - \alpha$$

i.e., the $100(1-\alpha)\%$ confidence interval for p is

$$\left(\frac{k}{n} - z_{\alpha/2}\sqrt{\frac{(k/n)(1-k/n)}{n}},\ \frac{k}{n} + z_{\alpha/2}\sqrt{\frac{(k/n)(1-k/n)}{n}}\right).$$

$\square$

E.g. 1. Use *median test* to check the randomness of a random generator.

Suppose $y_1, \cdots, y_n$ denote measurements presumed to have come from a continuous pdf $f_Y(y)$. Let k denote the number of y_i's that are less than the median of $f_Y(y)$. If the sample is random, we would expect the difference between $\frac{k}{n}$ and $\frac{1}{2}$ to be small. More specifically, a 95% confidence interval based on k should contain the value 0.5.

Let $f_Y(y) = e^{-y}$. The median is $m = 0.69315$.

```r
#! /usr/bin/Rscript
main <- function() {
  args <- commandArgs(trailingOnly = TRUE)
  n <- 100 # Number of random samples.
  r <- as.numeric(args[1]) # Rate of the exponential
  # Check if the rate argument is given.
  if (is.na(r)) return("Please provide the rate and try again.")

  # Now start computing ...
  f <- function (y) pexp(y, rate = r)-0.5
  m <- uniroot(f, lower = 0, upper = 100, tol = 1e-9)$root
  print(paste("For rate ", r, "exponential distribution,",
              "the median is equal to ", round(m,3)))
  data <- rexp(n,r) # Generate n random samples
  data <- round(data,3) # Round to 3 digits after decimal
  data <- matrix(data, nrow = 10,ncol = 10) # Turn the data to a matrix
  prmatrix(data) # Show data on terminal
  k <- sum(data > m) # Count how many entries is bigger than m
  lowerbd = k/n - 1.96 * sqrt((k/n)*(1-k/n)/n);
  upperbd = k/n + 1.96 *sqrt((k/n)*(1-k/n)/n);
  print(paste("The 95% confidence interval is (",
              round(lowerbd,3), ",",
              round(upperbd,3), ")"))
}
main()
```

Try commandline ...

```
Math362:./Example-5-3-2.R 1
[1] "For rate  1 exponential distribution, the median is equal to  0.693"
       [,1]  [,2]  [,3]  [,4]  [,5]  [,6]  [,7]  [,8]  [,9] [,10]
 [1,] 1.324 1.211 0.561 0.640 2.816 2.348 0.788 2.243 1.759 0.103
 [2,] 0.476 2.288 0.106 0.079 0.636 1.941 0.801 3.838 0.612 0.030
 [3,] 1.085 0.305 0.354 1.013 0.687 1.656 1.043 0.389 1.476 2.158
 [4,] 1.267 1.031 0.917 0.681 0.912 0.236 0.054 0.862 0.065 0.402
 [5,] 0.957 1.003 1.665 1.137 0.378 1.182 0.659 1.923 1.127 0.364
 [6,] 0.307 0.127 0.203 0.394 1.392 2.378 4.192 0.365 3.227 0.337
 [7,] 0.707 0.049 0.391 1.967 1.220 2.605 0.887 1.749 1.479 1.526
 [8,] 0.662 0.141 0.318 0.523 0.646 1.202 0.442 0.174 1.178 0.177
 [9,] 0.397 0.493 0.214 0.522 2.024 4.109 1.268 1.041 0.948 0.382
[10,] 2.260 0.292 0.437 0.962 0.224 4.221 0.594 0.218 0.601 0.941
[1] "The 95% confidence interval is ( 0.422 , 0.618 )"
Math362:./Example-5-3-2.R 10
[1] "For rate  10 exponential distribution, the median is equal to  0.069"
       [,1]  [,2]  [,3]  [,4]  [,5]  [,6]  [,7]  [,8]  [,9] [,10]
 [1,] 0.199 0.069 0.013 0.025 0.000 0.107 0.068 0.116 0.066 0.146
 [2,] 0.027 0.076 0.044 0.458 0.052 0.127 0.100 0.100 0.014 0.061
 [3,] 0.014 0.078 0.044 0.072 0.028 0.141 0.038 0.022 0.037 0.093
 [4,] 0.042 0.015 0.250 0.132 0.292 0.072 0.105 0.244 0.046 0.054
 [5,] 0.134 0.074 0.182 0.057 0.021 0.038 0.095 0.196 0.004 0.048
 [6,] 0.016 0.021 0.163 0.030 0.139 0.063 0.054 0.006 0.023 0.051
 [7,] 0.227 0.055 0.091 0.121 0.066 0.114 0.004 0.021 0.035 0.211
 [8,] 0.113 0.083 0.129 0.338 0.160 0.008 0.014 0.167 0.050 0.127
 [9,] 0.053 0.073 0.054 0.098 0.004 0.036 0.274 0.276 0.004 0.159
[10,] 0.045 0.469 0.152 0.003 0.129 0.017 0.084 0.072 0.162 0.007
[1] "The 95% confidence interval is ( 0.392 , 0.588 )"
Math362:
```

Instead of the C.I. $\left(\frac{k}{n} - z_{\alpha/2}\sqrt{\frac{(k/n)(1-k/n)}{n}},\ \frac{k}{n} + z_{\alpha/2}\sqrt{\frac{(k/n)(1-k/n)}{n}}\right)$.

One can simply specify the mean $\dfrac{k}{n}$ and

the **margin of error**: $$d := z_{\alpha/2}\sqrt{\frac{(k/n)(1-k/n)}{n}}.$$

$$\max_{p\in(0,1)} p(1-p) = p(1-p)\Big|_{p=1/2} = 1/4 \quad\Longrightarrow\quad d \le \frac{z_{\alpha/2}}{2\sqrt{n}} =: d_m.$$

Comment:

1. When p is close to $1/2$, $d \approx \frac{z_{\alpha/2}}{2\sqrt{n}}$, which is equivalent to $\sigma_p \approx \frac{1}{2\sqrt{n}}$.

   E.g., $n = 1000$, $k/n = 0.48$, and $\alpha = 5\%$, then

$$d = 1.96\sqrt{\frac{0.48 \times 0.52}{1000}} = 0.0309\underline{7} \quad \text{and} \quad d_m = \frac{1.96}{2\sqrt{1000}} = 0.0309\underline{9}$$

$$\sigma_p = \sqrt{\frac{0.48 \times 0.52}{1000}} = 0.015\underline{7}9873 \quad \text{and} \quad \sigma_p \approx \frac{1}{2\sqrt{1000}} = 0.015\underline{8}1139.$$

2. When p is away from $1/2$, the discrepancy between d and d_m becomes big....

E.g. Running for presidency. Max and Sirius obtained 480 and 520 votes, respectively. What is probability that Max will win?

What if the sample size is $n = 5000$, and Max obtained 2400 votes.

Choosing sample sizes

$$d \leq z_{\alpha/2}\sqrt{p(1-p)/n} \quad \Longleftrightarrow \quad n \geq \frac{z_{\alpha/2}^2 p(1-p)}{d^2} \qquad \text{(When } p \text{ is known)}$$

$$d \leq \frac{z_{\alpha/2}}{2\sqrt{n}} \quad \Longleftrightarrow \quad n \geq \frac{z_{\alpha/2}^2}{4d^2} \qquad \text{(When } p \text{ is unknown)}$$

E.g. Anti-smoking campaign. Need to find an 95% C.I. with a margin of error equal to 1%. Determine the sample size?

Answer: $n \geq \frac{1.96^2}{4 \times 0.01^2} = 9640.$

E.g.' In order to reduce the sample size, a small sample is used to determine p. One finds that $p \approx 0.22$. Determine the sample size again.

Answer: $n \geq \frac{1.96^2 \times 0.22 \times 0.78}{\times 0.01^2} = 6592.2.$