

Math 362: Mathematical Statistics II

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Last updated on April 13, 2021

2021 Spring

Chapter 5. Estimation

§ 5.1 Introduction

§ 5.2 Estimating parameters: MLE and MME

§ 5.3 Interval Estimation

§ 5.4 Properties of Estimators

§ 5.5 Minimum-Variance Estimators: The Cramér-Rao Lower Bound

§ 5.6 Sufficient Estimators

§ 5.7 Consistency

§ 5.8 Bayesian Estimation

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Rationale. Point estimate doesn't provide precision information.

By using the variance of the estimator, one can construct an interval such that with a high probability that interval will contain the unknown parameter.

- ▶ The interval is called **confidence interval**.
- ▶ The high probability is **confidence level**.

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E.g. 1. A random sample of size 4, ($Y_1 = 6.5$, $Y_2 = 9.2$, $Y_3 = 9.9$, $Y_4 = 12.4$), from a normal population:

$$f_Y(y; \mu) = \frac{1}{\sqrt{2\pi} \cdot 0.8} e^{-\frac{1}{2} \left(\frac{y-\mu}{0.8} \right)^2}.$$

Both MLE and MME give $\mu_e = \bar{y} = \frac{1}{4}(6.5 + 9.2 + 9.9 + 12.4) = 9.5$.
The estimator $\hat{\mu} = \bar{Y}$ follows normal distribution.

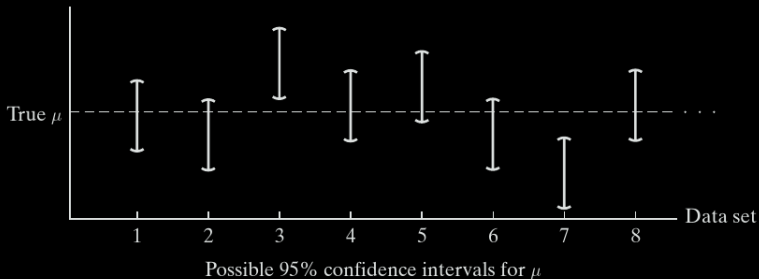
Construct 95%-confidence interval for μ ...

“The parameter is an unknown constant and no probability statement concerning its value may be made.”

–Jerzy Neyman, original developer of confidence intervals.

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In general, for a normal population with σ known, the $100(1 - \alpha)\%$ **confidence interval** for μ is

$$\left(\bar{y} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{y} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right)$$

Comment: There are many variations

1. Confidence interval for μ when σ is unknown
2. Confidence interval for σ^2 when μ is unknown
3. Confidence interval for σ^2 when μ is known
4. Confidence interval for μ when σ is unknown and $n < 30$

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1. One-sided interval such as

$$\left(\bar{y} - z_{\alpha} \frac{\sigma}{\sqrt{n}}, \bar{y} \right) \quad \text{or} \quad \left(\bar{y}, \bar{y} + z_{\alpha} \frac{\sigma}{\sqrt{n}} \right)$$

2. σ is unknown and sample size is small: z -score \rightarrow t -score
3. σ is unknown and sample size is large: z -score by CLT
4. Non-Gaussian population but sample size is large: z -score by CLT

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Theorem. Let k be the number of successes in n independent trials, where n is large and $p = \mathbb{P}(\text{success})$ is unknown. An approximate $100(1 - \alpha)\%$ confidence interval for p is the set of numbers

$$\left(\frac{k}{n} - z_{\alpha/2} \sqrt{\frac{(k/n)(1 - k/n)}{n}}, \frac{k}{n} + z_{\alpha/2} \sqrt{\frac{(k/n)(1 - k/n)}{n}} \right).$$

Proof: It follows the following facts:

- ▶ $\frac{k}{n} = \frac{1}{n} \sum_{i=1}^n X_i$ where $X_i = \begin{cases} 1 & \text{if trial } i \text{ is a success} \\ 0 & \text{if trial } i \text{ is a failure} \end{cases}$
- ▶ Central Limit Theorem

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Proof: It follows the following facts:

- ▶ $X \sim \text{binomial}(n, p)$ iff $X = Y_1 + \dots + Y_n$, while Y_i are i.i.d. Bernoulli(p):

$$\mathbb{E}[Y_i] = p \quad \text{and} \quad \text{Var}(Y_i) = p(1 - p).$$

- ▶ **Central Limit Theorem:** Let W_1, W_2, \dots, W_n be an sequence of i.i.d. random variables, whose distribution has mean μ and variance σ^2 , then

$$\frac{\sum_{i=1}^n W_i - n\mu}{\sqrt{n\sigma^2}} \quad \text{approximately follows} \quad N(0, 1), \quad \text{when } n \text{ is large.}$$

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- When the sample size n is large, by the central limit theorem,

$$\frac{\sum_{i=1}^n Y_i - np}{\sqrt{np(1-p)}} \stackrel{\text{ap.}}{\approx} N(0, 1)$$

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$$\frac{X - np}{\sqrt{np(1-p)}} = \frac{\frac{X}{n} - p}{\sqrt{\frac{p(1-p)}{n}}} \approx \frac{\frac{X}{n} - p}{\sqrt{\frac{p_e(1-p_e)}{n}}}$$

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$$\mathbb{P} \left(-z_{\alpha/2} \leq \frac{\frac{X}{n} - p}{\sqrt{\frac{\frac{k}{n}(1-\frac{k}{n})}{n}}} \leq z_{\alpha/2} \right) \approx 1 - \alpha$$

i.e., the $100(1 - \alpha)\%$ confidence interval for p is

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□

E.g. 1. Use *median test* to check the randomness of a random generator.

Suppose y_1, \dots, y_n denote measurements presumed to have come from a continuous pdf $f_Y(y)$. Let k denote the number of y_i 's that are less than the median of $f_Y(y)$. If the sample is random, we would expect the difference between $\frac{k}{n}$ and $\frac{1}{2}$ to be small. More specifically, a 95% confidence interval based on k should contain the value 0.5.

Let $f_Y(y) = e^{-y}$. The median is $m = 0.69315$.

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Let $f_Y(y) = e^{-y}$. The median is $m = 0.69315$.

```

1 #! /usr/bin/Rscript
2 main <- function() {
3   args <- commandArgs(trailingOnly = TRUE)
4   n <- 100 # Number of random samples.
5   r <- as.numeric(args[1]) # Rate of the exponential
6   # Check if the rate argument is given.
7   if (is.na(r)) return("Please provide the rate and try again.")
8
9   # Now start computing ...
10  f <- function (y) pexp(y, rate = r)-0.5
11  m <- uniroot(f, lower = 0, upper = 100, tol = 1e-9)$root
12  print(paste("For rate ", r, "exponential distribution,",
13            "the median is equal to ", round(m,3)))
14  data <- rexp(n,r) # Generate n random samples
15  data <- round(data,3) # Round to 3 digits after decimal
16  data <- matrix(data, nrow = 10,ncol = 10) # Turn the data to a matrix
17  prmatrix(data) # Show data on terminal
18  k <- sum(data > m) # Count how many entries is bigger than m
19  lowerbd = k/n - 1.96 * sqrt((k/n)*(1-k/n)/n);
20  upperbd = k/n + 1.96 *sqrt((k/n)*(1-k/n)/n);
21  print(paste("The 95% confidence interval is (",
22            round(lowerbd,3), ", ",
23            round(upperbd,3), ")"))
24 }
25 main()

```

Try commandline ...

Math362:./Example-5-3-2.R 1

```
[1] "For rate 1 exponential distribution, the median is equal to 0.693"  
      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10]  
[1,] 1.324 1.211 0.561 0.640 2.816 2.348 0.788 2.243 1.759 0.103  
[2,] 0.476 2.288 0.106 0.079 0.636 1.941 0.801 3.838 0.612 0.030  
[3,] 1.085 0.305 0.354 1.013 0.687 1.656 1.043 0.389 1.476 2.158  
[4,] 1.267 1.031 0.917 0.681 0.912 0.236 0.054 0.862 0.065 0.402  
[5,] 0.957 1.003 1.665 1.137 0.378 1.182 0.659 1.923 1.127 0.364  
[6,] 0.307 0.127 0.203 0.394 1.392 2.378 4.192 0.365 3.227 0.337  
[7,] 0.707 0.049 0.391 1.967 1.220 2.605 0.887 1.749 1.479 1.526  
[8,] 0.662 0.141 0.318 0.523 0.646 1.202 0.442 0.174 1.178 0.177  
[9,] 0.397 0.493 0.214 0.522 2.024 4.109 1.268 1.041 0.948 0.382  
[10,] 2.260 0.292 0.437 0.962 0.224 4.221 0.594 0.218 0.601 0.941  
[1] "The 95% confidence interval is ( 0.422 , 0.618 )"
```

Math362:./Example-5-3-2.R 10

```
[1] "For rate 10 exponential distribution, the median is equal to 0.069"  
      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10]  
[1,] 0.199 0.069 0.013 0.025 0.000 0.107 0.068 0.116 0.066 0.146  
[2,] 0.027 0.076 0.044 0.458 0.052 0.127 0.100 0.100 0.014 0.061  
[3,] 0.014 0.078 0.044 0.072 0.028 0.141 0.038 0.022 0.037 0.093  
[4,] 0.042 0.015 0.250 0.132 0.292 0.072 0.105 0.244 0.046 0.054  
[5,] 0.134 0.074 0.182 0.057 0.021 0.038 0.095 0.196 0.004 0.048  
[6,] 0.016 0.021 0.163 0.030 0.139 0.063 0.054 0.006 0.023 0.051  
[7,] 0.227 0.055 0.091 0.121 0.066 0.114 0.004 0.021 0.035 0.211  
[8,] 0.113 0.083 0.129 0.338 0.160 0.008 0.014 0.167 0.050 0.127  
[9,] 0.053 0.073 0.054 0.098 0.004 0.036 0.274 0.276 0.004 0.159  
[10,] 0.045 0.469 0.152 0.003 0.129 0.017 0.084 0.072 0.162 0.007  
[1] "The 95% confidence interval is ( 0.392 , 0.588 )"
```

Math362:█

Instead of the C.I. $\left(\frac{k}{n} - z_{\alpha/2} \sqrt{\frac{(k/n)(1-k/n)}{n}}, \frac{k}{n} + z_{\alpha/2} \sqrt{\frac{(k/n)(1-k/n)}{n}} \right)$.

One can simply specify the mean $\frac{k}{n}$ and

the **margin of error**: $d := z_{\alpha/2} \sqrt{\frac{(k/n)(1-k/n)}{n}}$.

$$\max_{p \in (0,1)} p(1-p) = p(1-p) \Big|_{p=1/2} = 1/4 \implies d \leq \frac{z_{\alpha/2}}{2\sqrt{n}} =: d_m.$$

Comment:

1. When p is close to $1/2$, $d \approx \frac{z_{\alpha/2}}{2\sqrt{n}}$, which is equivalent to $\sigma_p \approx \frac{1}{2\sqrt{n}}$.

E.g., $n = 1000$, $k/n = 0.48$, and $\alpha = 5\%$, then

$$d = 1.96\sqrt{\frac{0.48 \times 0.52}{1000}} = 0.03097 \quad \text{and} \quad d_m = \frac{1.96}{2\sqrt{1000}} = 0.03099$$

$$\sigma_p = \sqrt{\frac{0.48 \times 0.52}{1000}} = 0.01579873 \quad \text{and} \quad \sigma_p \approx \frac{1}{2\sqrt{1000}} = 0.01581139.$$

2. When p is away from $1/2$, the discrepancy between d and d_m becomes big....

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E.g. Running for presidency. Max and Sirius obtained 480 and 520 votes, respectively. What is probability that Max will win?

What if the sample size is $n = 5000$, and Max obtained 2400 votes.

Choosing sample sizes

$$d \leq z_{\alpha/2} \sqrt{p(1-p)/n} \iff n \geq \frac{z_{\alpha/2}^2 p(1-p)}{d^2} \quad (\text{When } p \text{ is known})$$

$$d \leq \frac{z_{\alpha/2}}{2\sqrt{n}} \iff n \geq \frac{z_{\alpha/2}^2}{4d^2} \quad (\text{When } p \text{ is unknown})$$

E.g. Anti-smoking campaign. Need to find an 95% C.I. with a margin of error equal to 1%. Determine the sample size?

$$\text{Answer: } n \geq \frac{1.96^2}{4 \times 0.01^2} = 9640.$$

E.g. In order to reduce the sample size, a small sample is used to determine p . One finds that $p \approx 0.22$. Determine the sample size again.

$$\text{Answer: } n \geq \frac{1.96^2 \times 0.22 \times 0.78}{\times 0.01^2} = 6592.2.$$

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