

Math 362: Mathematical Statistics II

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Chapter 5. Estimation

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§ 5.2 Estimating parameters: MLE and MME

§ 5.3 Interval Estimation

§ 5.4 Properties of Estimators

§ 5.5 Minimum-Variance Estimators: The Cramér-Rao Lower Bound

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Plan

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§ 5.4 Properties of Estimators

Question: Estimators are not in general unique (MLE or MME ...). How to select one estimator?

Recall: For a random sample of size n from the population with given pdf, we have X_1, \dots, X_n , which are i.i.d. r.v.'s. The estimator $\hat{\theta}$ is a function of X_i 's:

$$\hat{\theta} = \hat{\theta}(X_1, \dots, X_n).$$

Criteria:

1. Unbiased. (Mean)
2. Efficiency, the minimum-variance estimator. (Variance)
3. Sufficiency.
4. Consistency. (Asymptotic behavior)

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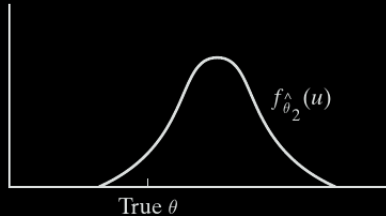
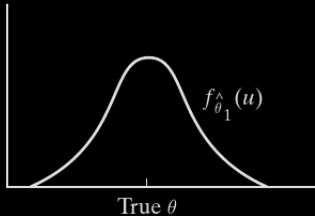
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Unbiasedness



Definition 5.4.1. Given a random sample of size n whose population distribution depends on an unknown parameter θ , let $\hat{\theta}$ be an estimator of θ .

Then $\hat{\theta}$ is called **unbiased** if $\mathbb{E}(\hat{\theta}) = \theta$;

and $\hat{\theta}$ is called **asymptotically unbiased** if $\lim_{n \rightarrow \infty} \mathbb{E}(\hat{\theta}) = \theta$.

E.g. 1. $f_Y(y; \theta) = \frac{2y}{\theta^2}$ if $y \in [0, \theta]$.

$$- \hat{\theta}_1 = \frac{3}{2} \bar{Y}$$

$$- \hat{\theta}_2 = Y_{\max}$$

$$- \hat{\theta}_3 = \frac{2n+1}{2n} Y_{\max}$$

E.g. 2. Let X_1, \dots, X_n be a random sample of size n with the unknown parameter $\theta = \mathbb{E}(X)$. Show that for any constants a_i 's,

$$\hat{\theta} = \sum_{i=1}^n a_i X_i \text{ is unbiased} \iff \sum_{i=1}^n a_i = 1.$$

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E.g. 3. Let X_1, \dots, X_n be a random sample of size n with the unknown parameter $\sigma^2 = \text{Var}(X)$.

$$- \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2$$

$$- S^2 = \text{Sample Variance} = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

$$- S = \text{Sample Standard Deviation} = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2}. \quad (\text{Biased for } \sigma!)$$

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$n\bar{Y} = \sum_{i=1}^n Y_i \sim \text{Gamma distribution}(n, \lambda)$. Hence,

$$\begin{aligned}\mathbb{E}(\hat{\lambda}) &= \mathbb{E}\left(\frac{1}{\bar{Y}}\right) = n \int_0^{\infty} \frac{1}{y} \frac{\lambda^n}{\Gamma(n)} y^{n-1} e^{-\lambda y} dy \\ &= \frac{n\lambda}{n-1} \int_0^{\infty} \underbrace{\frac{\lambda^{n-1}}{\Gamma(n-1)} y^{(n-1)-1} e^{-\lambda y}}_{\text{pdf for Gamma distr. } (n-1, \lambda)} dy \\ &= \frac{n}{n-1} \lambda.\end{aligned}$$

Biased! But $\mathbb{E}(\hat{\lambda}) = \frac{n}{n-1} \lambda \rightarrow \lambda$ as $n \rightarrow \infty$. (Asymptotically unbiased.)

Note: $\hat{\lambda}^* = \frac{n-1}{n\bar{Y}}$ is unbiased.

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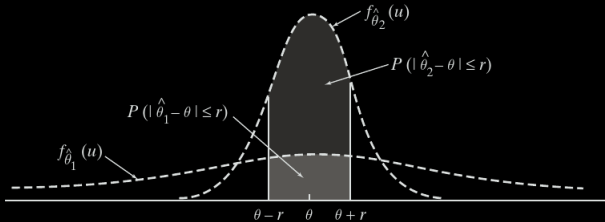
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Efficiency



Definition 5.4.2. Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two unbiased estimators for a parameter θ . If $\text{Var}(\hat{\theta}_1) < \text{Var}(\hat{\theta}_2)$, then we say that $\hat{\theta}_1$ is **more efficient** than $\hat{\theta}_2$. The **relative efficiency** of $\hat{\theta}_1$ w.r.t. $\hat{\theta}_2$ is the ratio $\text{Var}(\hat{\theta}_2)/\text{Var}(\hat{\theta}_1)$.

E.g. 1. $f_Y(y; \theta) = \frac{2y}{\theta^2}$ if $y \in [0, \theta]$. Which is more efficient? Find the relative efficiency of $\hat{\theta}_1$ w.r.t. $\hat{\theta}_3$.

$$- \hat{\theta}_1 = \frac{3}{2} \bar{Y}$$

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E.g. 2. Let X_1, \dots, X_n be a random sample of size n with the unknown parameter $\theta = \mathbb{E}(X)$ (suppose $\sigma^2 = \text{Var}(X) < \infty$).

Consider the following two unbiased estimators of θ :

$$\hat{\theta}_1 = \frac{1}{n} \sum_{i=1}^n X_i \quad \text{and} \quad \hat{\theta}_2 = \frac{1}{n} \sum_{i=1}^n (X_i - X_{(1)})$$

$$\text{Var}(\hat{\theta}_1) = \frac{\sigma^2}{n} \quad \text{and} \quad \text{Var}(\hat{\theta}_2) = \frac{\sigma^2}{n} \left(\frac{n-1}{n} \right)$$

$$\text{Eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{\text{Var}(\hat{\theta}_2)}{\text{Var}(\hat{\theta}_1)} = \frac{\sigma^2}{n} \left(\frac{n-1}{n} \right) \cdot \frac{n}{\sigma^2} = \frac{n-1}{n} < 1$$

Therefore, $\hat{\theta}_1$ is more efficient than $\hat{\theta}_2$.

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$$\text{Var}(\hat{\theta}_1) = \frac{\sigma^2}{n} \quad \text{and} \quad \text{Var}(\hat{\theta}_2) = \frac{2\sigma^4}{n}$$

Which estimator is more efficient? (Assume $\sigma^2 > 0$)

How many observations are needed to estimate θ with a relative error of 10%?

E.g. 1. $f_Y(y; \theta) = \frac{2y}{\theta^2}$ if $y \in [0, \theta]$. Which is more efficient? Find the relative efficiency of $\hat{\theta}_1$ w.r.t. $\hat{\theta}_3$.

$$- \hat{\theta}_1 = \frac{3}{2} \bar{Y}$$

$$- \hat{\theta}_3 = \frac{2n+1}{2n} Y_{\max}.$$

E.g. 2. Let X_1, \dots, X_n be a random sample of size n with the unknown parameter $\theta = \mathbb{E}(X)$ (suppose $\sigma^2 = \text{Var}(X) < \infty$).

Let $\hat{\theta}_1 = \bar{X}$ and $\hat{\theta}_2 = \frac{1}{n} \sum_{i=1}^n X_i^2 - \frac{1}{n} \left(\sum_{i=1}^n X_i \right)^2$.

Let $\hat{\theta}_3 = \frac{1}{n} \sum_{i=1}^n X_i^2 - \frac{1}{n} \left(\sum_{i=1}^n X_i \right)^2$.

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Let $\hat{\theta}_6 = \frac{1}{n} \sum_{i=1}^n X_i^2 - \frac{1}{n} \left(\sum_{i=1}^n X_i \right)^2$.

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E.g. 2. Let X_1, \dots, X_n be a random sample of size n with the unknown parameter $\theta = \mathbb{E}(X)$ (suppose $\sigma^2 = \text{Var}(X) < \infty$).

Among all possible unbiased estimators $\hat{\theta} = \sum_{i=1}^n a_i X_i$ with $\sum_{i=1}^n a_i = 1$. Find the most efficient one.

Sol:

$$\text{Var}(\hat{\theta}) = \sum_{i=1}^n a_i^2 \text{Var}(X) = \sigma^2 \sum_{i=1}^n a_i^2 \geq \sigma^2 \frac{1}{n} \left(\sum_{i=1}^n a_i \right)^2 = \frac{1}{n} \sigma^2,$$

with equality iff $a_1 = \dots = a_n = 1/n$.

Hence, the most efficient one is the sample mean $\hat{\theta} = \bar{X}$. □

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