

Math 362: Mathematical Statistics II

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Chapter 5. Estimation

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§ 5.3 Interval Estimation

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§ 5.6 Sufficient Estimators

Rationale: Let $\hat{\theta}$ be an estimator to the unknown parameter θ . Whether does $\hat{\theta}$ contain all information about θ ?

Equivalently, how can one reduce the random sample of size n , denoted by (X_1, \dots, X_n) , to a function without losing any information about θ ?

E.g., let's choose the function $h(X_1, \dots, X_n) := \frac{1}{n} \sum_{i=1}^n X_i$, the sample mean. In many cases, $h(X_1, \dots, X_n)$ contains all relevant information about the true mean $\mathbb{E}(X)$. In that case, $h(X_1, \dots, X_n)$, as an estimator, is sufficient.

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Definition. Let (X_1, \dots, X_n) be a random sample of size n from a discrete population with a unknown parameter θ , of which $\hat{\theta}$ (resp. θ_e) be an estimator (resp. estimate). We call $\hat{\theta}$ and θ_e **sufficient** if

$$\mathbb{P} \left(X_1 = k_1, \dots, X_n = k_n \mid \hat{\theta} = \theta_e \right) = b(k_1, \dots, k_n) \quad (\text{Sufficiency-1})$$

is a function that does not depend on θ .

In case for random sample (Y_1, \dots, Y_n) from the continuous population, (Sufficiency-1) should be

$$f_{Y_1, \dots, Y_n \mid \hat{\theta} = \theta_e} \left(y_1, \dots, y_n \mid \hat{\theta} = \theta_e \right) = b(y_1, \dots, y_n)$$

Note: $\hat{\theta} = h(X_1, \dots, X_n)$ and $\theta_e = h(k_1, \dots, k_n)$.
or $\hat{\theta} = h(Y_1, \dots, Y_n)$ and $\theta_e = h(y_1, \dots, y_n)$.

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Definition. ... $\hat{\theta}$ (or θ_e) is **sufficient** if the likelihood function can be factorized as:

$$L(\theta) = \begin{cases} \prod_{i=1}^n p_X(k_i; \theta) = g(\theta_e, \theta) b(k_1, \dots, k_n) & \text{Discrete} \\ \prod_{i=1}^n f_Y(y_i; \theta) = g(\theta_e, \theta) b(y_1, \dots, y_n) & \text{Continuous} \end{cases} \quad (\text{Sufficiency-2})$$

where g is a function of two arguments only and b is a function that does not depend on θ .

E.g. 1. A random sample of size n from Bernoulli(P). $\hat{p} = \sum_{i=1}^n X_i$. Check sufficiency of \hat{p} for p by (Sufficiency-1):

Case I: If $k_1, \dots, k_n \in \{0, 1\}$ such that $\sum_{i=1}^n k_i \neq c$, then

$$\mathbb{P}(X_1 = k_1, \dots, X_n = k_n \mid \hat{p} = c) = 0.$$

Case II: If $k_1, \dots, k_n \in \{0, 1\}$ such that $\sum_{i=1}^n k_i = c$, then

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&= \frac{\mathbb{P}(X_1 = k_1, \dots, X_{n-1} = k_{n-1}, X_n = c - \sum_{i=1}^{n-1} k_i)}{\mathbb{P}(\sum_{i=1}^n X_i = c)} \\
&= \frac{(\prod_{i=1}^{n-1} p^{k_i} (1-p)^{1-k_i}) \times p^{c - \sum_{i=1}^{n-1} k_i} (1-p)^{1-c + \sum_{i=1}^{n-1} k_i}}{\binom{n}{c} p^c (1-p)^{n-c}} \\
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&= \frac{1}{\binom{n}{c}}.
\end{aligned}$$

In summary,

$$\mathbb{P}(X_1 = k_1, \dots, X_n = k_n \mid \hat{p} = c) = \begin{cases} \frac{1}{\binom{n}{c}} & \text{if } k_i \in \{0, 1\} \text{ s.t. } \sum_{i=1}^n k_i = c, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, by (Sufficiency-1), $\hat{p} = \sum_{i=1}^n X_i$ is a sufficient estimator for p .

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$$\mathbb{P}(X_1 = k_1, \dots, X_n = k_n \mid \hat{p} = c) = \begin{cases} \frac{1}{\binom{n}{c}} & \text{if } k_i \in \{0, 1\} \text{ s.t. } \sum_{i=1}^n k_i = c, \\ 0 & \text{otherwise.} \end{cases}$$

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E.g. 1'. As in E.g. 1, check sufficiency of \hat{p} for p by (Sufficiency-2):

Notice that $p_e = \sum_{i=1}^n k_i$. Then

$$\begin{aligned}L(p) &= \prod_{i=1}^n p_{X_i}(k_i; p) = \prod_{i=1}^n p^{k_i} (1-p)^{1-k_i} \\ &= p^{\sum_{i=1}^n k_i} (1-p)^{n-\sum_{i=1}^n k_i} \\ &= p^{p_e} (1-p)^{n-p_e}\end{aligned}$$

Therefore, p_e (or \hat{p}) is sufficient since (Sufficiency-2) is satisfied with

$$g(p_e, p) = p^{p_e} (1-p)^{n-p_e} \quad \text{and} \quad b(k_1, \dots, k_n) = 1.$$

Comment 1: The sufficiency of p_e can also be proved by using the Factorization Theorem.

Comment 2: The sufficiency of \hat{p} can also be proved by using the Factorization Theorem. In fact, $L(p) = p^{p_e} (1-p)^{n-p_e}$ can be written as $L(p) = g(\hat{p}, p) b(k_1, \dots, k_n)$ with $g(\hat{p}, p) = p^{n\hat{p}} (1-p)^{n-n\hat{p}}$ and $b(k_1, \dots, k_n) = 1$.

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Comment 1. The sufficiency of p_e (or \hat{p}) can be checked by (Sufficiency-1) as well. In fact, $L(p) = g(p_e, p) b(k_1, \dots, k_n)$ with $b(k_1, \dots, k_n) = 1$. The sufficiency of p_e (or \hat{p}) can also be checked by (Sufficiency-3) as well. In fact, $L(p) = g(p_e, p) b(k_1, \dots, k_n)$ with $b(k_1, \dots, k_n) = 1$.

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Comment 1. The above example shows that the sample mean \bar{X} is sufficient for p in the Bernoulli case. In fact, \bar{X} is sufficient for p in the binomial case as well. The reason is that the binomial distribution is a special case of the Bernoulli distribution. In fact, if X_1, \dots, X_n are independent Bernoulli random variables with parameter p , then $Y = \sum_{i=1}^n X_i$ is a binomial random variable with parameters n and p . The likelihood function for p based on Y is $L(p) = \binom{n}{y} p^y (1-p)^{n-y}$, which is the same as the likelihood function based on X_1, \dots, X_n . Therefore, \bar{X} is sufficient for p in the binomial case as well.

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$$g(p_e, p) = p^{p_e} (1-p)^{n-p_e} \quad \text{and} \quad b(k_1, \dots, k_n) = 1.$$

Comment

Notice that p_e is a function of p and p is a function of p_e . In fact, p_e is a one-to-one function of p and p is a one-to-one function of p_e . This is why we can write p_e as a function of p and p as a function of p_e .

Therefore, p_e is a sufficient statistic for p .

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Let X_1, \dots, X_n be independent Bernoulli random variables with parameter p . Then $p_e = \sum_{i=1}^n X_i$ is the sum of X_1, \dots, X_n . The joint pmf of X_1, \dots, X_n is $L(p) = \prod_{i=1}^n p^{X_i} (1-p)^{1-X_i} = p^{\sum_{i=1}^n X_i} (1-p)^{n-\sum_{i=1}^n X_i} = p^{p_e} (1-p)^{n-p_e}$. The pmf of p_e is $g(p_e, p) = p^{p_e} (1-p)^{n-p_e}$. The joint pmf of X_1, \dots, X_n can be written as $L(p) = g(p_e, p) b(k_1, \dots, k_n)$ where $b(k_1, \dots, k_n) = 1$. This shows that p_e is a sufficient statistic for p .

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1. The estimator \widehat{p} is sufficient but not unbiased since $\mathbb{E}(\widehat{p}) = np \neq p$.
 2. Any one-to-one function of a sufficient estimator is again a sufficient estimator. E.g., $\widehat{p}_2 := \frac{1}{n}\widehat{p}$, which is a unbiased, sufficient, and MVE.
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- Comment**
1. The estimator \widehat{p} is sufficient but not unbiased since $\mathbb{E}(\widehat{p}) = np \neq p$.
 2. Any one-to-one function of a sufficient estimator is again a sufficient estimator. E.g., $\widehat{p}_2 := \frac{1}{n}\widehat{p}$, which is a unbiased, sufficient, and MVE.
 3. $\widehat{p}_3 := X_1$ is not sufficient!

E.g. 2. Poisson(λ), $p_X(k; \lambda) = e^{-\lambda} \lambda^k / k!$, $k = 0, 1, \dots$. Show that $\hat{\lambda} = (\sum_{i=1}^n X_i)^2$ is sufficient for λ for a sample of size n .

Sol: The Corresponding estimate is $\lambda_e = (\sum_{i=1}^n k_i)^2$.

$$\begin{aligned}
 L(\lambda) &= \prod_{i=1}^n \frac{e^{-\lambda} \lambda^{k_i}}{k_i!} \\
 &= e^{-n\lambda} \lambda^{\sum_{i=1}^n k_i} \left(\prod_{i=1}^n k_i! \right)^{-1} \\
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Hence, $\hat{\lambda}$ is sufficient estimator for λ .

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E.g. 3. Let Y_1, \dots, Y_n be a random sample from $f_Y(y; \theta) = \frac{2y}{\theta^2}$ for $y \in [0, \theta]$.
 Whether is the MLE $\hat{\theta} = Y_{max}$ sufficient for θ ?

Sol: The corresponding estimate is $\theta_e = y_{max}$.

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 L(\theta) &= \prod_{i=1}^n \frac{2y_i}{\theta^2} I_{[0, \theta]}(y_i) = 2^n \theta^{-2n} \left(\prod_{i=1}^n y_i \right) \times \prod_{i=1}^n I_{[0, \theta]}(y_i) \\
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