Math 362: Mathematical Statistics II

Le Chen
le.chen@emory.edu

Emory University
Atlanta, GA

Last updated on April 13, 2021

2021 Spring

Chapter 6. Hypothesis Testing

Chapter 6. Hypothesis Testing

§ 6.1 Introduction

§ 6.2 The Decision Rule

§ 6.3 Testing Binomial Data – $H_0 : p = p_0$

§ 6.4 Type I and Type II Errors

§ 6.5 A Notion of Optimality: The Generalized Likelihood Ratio

Setup: Let $X_1 = k_1, \cdots, X_n = k_n$ be a' random sample of size n from Bernoulli(p). $X = \sum_{i=1}^{n} X_i \sim$ Binomial(n, p). We want to test $H_0 : p = p_0$.

1. When n is large, use Z score. Large-sample test
2. Otherwise, use the exact binomial distribution. Small-sample test

$$n \text{ is large}$$
$$\Updownarrow$$
$$0 < np_0 - 3\sqrt{np_0(1-p_0)} < np_0 + 3\sqrt{np_0(1-p_0)} < n$$
$$\Updownarrow$$
$$n > 9 \times \max\left(\frac{1-p_0}{p_0}, \frac{p_0}{1-p_0}\right).$$

Large-sample test for p

Setup:

1. Let $X_1 = k_1, \cdots, X_n = k_n$ be a random sample of size n from Bernoulli(p).
2. Suppose $n > 9 \max\left(\frac{1-p_0}{p_0}, \frac{p_0}{1-p_0}\right)$.
3. Set $k = k_1 + \cdots + k_n$ and $z = \frac{k - np_0}{\sqrt{np_0(1-p_0)}}$.
4. The level of significance is α.

Test:

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p > p_0 \end{cases}$$

reject H_0 if $z \geq z_\alpha$.

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p < p_0 \end{cases}$$

reject H_0 if $z \leq -z_\alpha$.

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p \neq p_0 \end{cases}$$

reject H_0 if $|z| \geq z_{\alpha/2}$.

Small-sample test for p

E.g. $n = 19$, $p_0 = 0.85$, $\alpha = 0.10$. Find critical region for the two-sided test

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p \neq p_0 \end{cases}$$

Sol. $19 = n < 9 \times \max\left(\frac{0.85}{0.15}, \frac{0.15}{0.85}\right) = 51$, so small sample test.

By checking the table, the critical region is

$$C = \{k : k \leq 13 \quad \text{or} \quad k = 19\},$$

so that

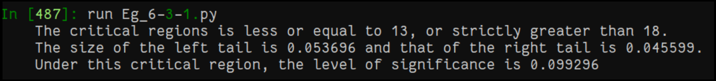
$$\begin{aligned} \alpha &= \mathbb{P}(X \in C | H_0 \text{ is true}) \\ &= \mathbb{P}(X \leq 13 | p = 0.85) + \mathbb{P}(X = 19 | p = 0.85) \\ &= 0.099295 \approx 0.10. \end{aligned}$$

$\square$

Binomial with n = 19 and p = 0.85

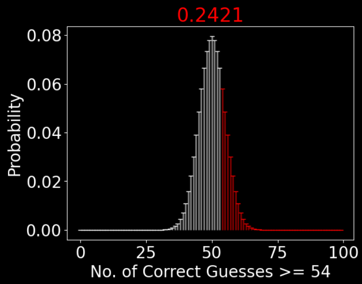
x	P(X = x)	
6	0.000000	$\rightarrow P(X \leq 13) = 0.053696$
7	0.000002	
8	0.000018	
9	0.000123	
10	0.000699	
11	0.003242	
12	0.012246	
13	0.037366	
14	0.090746	
15	0.171409	
16	0.242829	
17	0.242829	
18	0.152892	
19	0.045599	$\rightarrow P(X = 19) = 0.045599$

```
# Eg_6−3−1.py
from scipy.stats import binom
n = 19
p = 0.85
rv = binom(n, p)
low = rv.ppf(0.05)
upper = rv.ppf(0.95)
left  = round(rv.cdf(low), 6)
right = round(1−rv.cdf(upper), 6)
both = round(rv.cdf(low)+1−rv.cdf(upper), 6)
Results = ”””\
    The critical regions is less or equal to {low:.0f}, or strictly greater than {upper:.0f}.
    The size of the tail is {left:.6f} and that of the right tail is {right:.6f}.
    Under this critical region, the level of significance is {both:.6f}
”””.format(∗∗locals())
print(Results)
```



```
In [487]: run Eg_6-3-1.py
    The critical regions is less or equal to 13, or strictly greater than 18.
    The size of the left tail is 0.053696 and that of the right tail is 0.045599.
    Under this critical region, the level of significance is 0.099296
```

$X \sim \text{Binomial}(100, 1/2)$

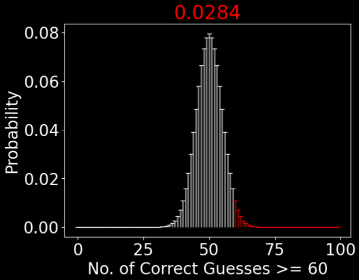


$$\mathbb{P}\left(X \geq 54\right) = \sum_{n=54}^{100} \binom{100}{n} \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{100-n} = 0.2421.$$

vs

$$\mathbb{P}\left(\frac{X - 50}{\sqrt{100 \times \frac{1}{2} \times \frac{1}{2}}} \geq \frac{54 - 50}{\sqrt{100 \times \frac{1}{2} \times \frac{1}{2}}}\right) \approx \mathbb{P}\left(Z \geq \frac{4}{5}\right) = 0.2119$$

$$X \sim \mathrm{Binomial}(100, 1/2)$$



$$\mathbb{P}\left(X \geq 60\right) = \sum_{n=60}^{100} \binom{100}{n} \left(\frac{1}{2}\right)^{n} \left(\frac{1}{2}\right)^{100-n} = 0.0284.$$

vs

$$\mathbb{P}\left(\frac{X-50}{\sqrt{100 \times \frac{1}{2} \times \frac{1}{2}}} \geq \frac{60-50}{\sqrt{100 \times \frac{1}{2} \times \frac{1}{2}}}\right) \approx \mathbb{P}\left(Z \geq 2\right) = 0.0228$$