# Math 362: Mathematical Statistics II

Le Chen
`le.chen@emory.edu`

Emory University
Atlanta, GA

Last updated on April 13, 2021

2021 Spring

# Chapter 6. Hypothesis Testing

# Plan

§ 6.1 Introduction

§ 6.2 The Decision Rule

§ 6.3 Testing Binomial Data – $H_0 : p = p_0$

§ 6.4 Type I and Type II Errors

§ 6.5 A Notion of Optimality: The Generalized Likelihood Ratio

# Chapter 6. Hypothesis Testing

§ 6.1 Introduction

§ 6.2 The Decision Rule

§ 6.3 Testing Binomial Data – $H_0 : p = p_0$

§ 6.4 Type I and Type II Errors

§ 6.5 A Notion of Optimality: The Generalized Likelihood Ratio

**Setup:** Let $X_1 = k_1, \cdots, X_n = k_n$ be a‘ random sample of size $n$ from Bernoulli($p$). $X = \sum_{i=1}^{n} X_i \sim$ Binomial($n, p$). We want to test $H_0 : p = p_0$.

1. When $n$ is large, use $Z$ score. Large-sample test
2. Otherwise, use the exact binomial distribution. Small-sample test

$$n \text{ is large}$$

$$\Updownarrow$$

$$0 < np_0 - 3\sqrt{np_0(1-p_0)} < np_0 + 3\sqrt{np_0(1-p_0)} < n$$

$$\Updownarrow$$

$$n > 9 \times \max\left(\frac{1-p_0}{p_0}, \frac{p_0}{1-p_0}\right).$$

**Setup:** Let $X_1 = k_1, \cdots, X_n = k_n$ be a‘ random sample of size $n$ from Bernoulli($p$). $X = \sum_{i=1}^{n} X_i \sim$ Binomial($n, p$). We want to test $H_0 : p = p_0$.

1. When $n$ is large, use $Z$ score. Large-sample test
2. Otherwise, use the exact binomial distribution. Small-sample test

$$n \text{ is large}$$

$$\Updownarrow$$

$$0 < np_0 - 3\sqrt{np_0(1-p_0)} < np_0 + 3\sqrt{np_0(1-p_0)} < n$$

$$\Updownarrow$$

$$n > 9 \times \max\left(\frac{1-p_0}{p_0}, \frac{p_0}{1-p_0}\right).$$

**Setup:** Let $X_1 = k_1, \cdots, X_n = k_n$ be a‘ random sample of size $n$ from Bernoulli($p$). $X = \sum_{i=1}^{n} X_i \sim$ Binomial$(n, p)$. We want to test $H_0 : p = p_0$.

1. When $n$ is large, use $Z$ score. Large-sample test
2. Otherwise, use the exact binomial distribution. Small-sample test

$$n \text{ is large}$$

$$\Updownarrow$$

$$0 < np_0 - 3\sqrt{np_0(1-p_0)} < np_0 + 3\sqrt{np_0(1-p_0)} < n$$

$$\Updownarrow$$

$$n > 9 \times \max\left(\frac{1-p_0}{p_0}, \frac{p_0}{1-p_0}\right).$$

**Setup:** Let $X_1 = k_1, \cdots, X_n = k_n$ be a' random sample of size $n$ from Bernoulli($p$). $X = \sum_{i=1}^{n} X_i \sim$ Binomial($n, p$). We want to test $H_0 : p = p_0$.

1. When $n$ is large, use $Z$ score. Large-sample test
2. Otherwise, use the exact binomial distribution. Small-sample test

$$n \text{ is large}$$

$$\Updownarrow$$

$$0 < np_0 - 3\sqrt{np_0(1-p_0)} < np_0 + 3\sqrt{np_0(1-p_0)} < n$$

$$\Updownarrow$$

$$n > 9 \times \max\left(\frac{1-p_0}{p_0}, \frac{p_0}{1-p_0}\right).$$

**Setup:** Let $X_1 = k_1, \cdots, X_n = k_n$ be a‘ random sample of size $n$ from Bernoulli($p$). $X = \sum_{i=1}^{n} X_i \sim$ Binomial$(n, p)$. We want to test $H_0 : p = p_0$.

1. When $n$ is large, use $Z$ score. Large-sample test
2. Otherwise, use the exact binomial distribution. Small-sample test

$$\begin{gathered} n \text{ is large} \\ \Updownarrow \\ 0 < np_0 - 3\sqrt{np_0(1-p_0)} < np_0 + 3\sqrt{np_0(1-p_0)} < n \\ \Updownarrow \\ n > 9 \times \max\left(\frac{1-p_0}{p_0}, \frac{p_0}{1-p_0}\right). \end{gathered}$$

# Large-sample test for $p$

**Setup:**

1. Let $X_1 = k_1, \cdots, X_n = k_n$ be a random sample of size $n$ from Bernoulli($p$).
2. Suppose $n > 9\max\left(\frac{1-p_0}{p_0}, \frac{p_0}{1-p_0}\right)$.
3. Set $k = k_1 + \cdots + k_n$ and $z = \frac{k-np_0}{\sqrt{np_0(1-p_0)}}$.
4. The level of significance is $\alpha$.

**Test:**

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p > p_0 \end{cases}$$

reject $H_0$ if $z \geq z_\alpha$.

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p < p_0 \end{cases}$$

reject $H_0$ if $z \leq -z_\alpha$.

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p \neq p_0 \end{cases}$$

reject $H_0$ if $|z| \geq z_{\alpha/2}$.

# Large-sample test for $p$

**Setup:**

1. Let $X_1 = k_1, \cdots, X_n = k_n$ be a random sample of size $n$ from Bernoulli$(p)$.
2. Suppose $n > 9 \max\left(\frac{1-p_0}{p_0}, \frac{p_0}{1-p_0}\right)$.
3. Set $k = k_1 + \cdots + k_n$ and $z = \frac{k - np_0}{\sqrt{np_0(1-p_0)}}$.
4. The level of significance is $\alpha$.

**Test:**

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p > p_0 \end{cases} \qquad \begin{cases} H_0 : p = p_0 \\ H_1 : p < p_0 \end{cases} \qquad \begin{cases} H_0 : p = p_0 \\ H_1 : p \neq p_0 \end{cases}$$

reject $H_0$ if $z \geq z_\alpha$. reject $H_0$ if $z \leq -z_\alpha$. reject $H_0$ if $|z| \geq z_{\alpha/2}$.

# Large-sample test for $p$

**Setup:**

1. Let $X_1 = k_1, \cdots, X_n = k_n$ be a random sample of size $n$ from Bernoulli($p$).
2. Suppose $n > 9 \max\left(\frac{1-p_0}{p_0}, \frac{p_0}{1-p_0}\right)$.
3. Set $k = k_1 + \cdots + k_n$ and $z = \frac{k-np_0}{\sqrt{np_0(1-p_0)}}$.
4. The level of significance is $\alpha$.

**Test:**

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p > p_0 \end{cases}$$

reject $H_0$ if $z \geq z_\alpha$.

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p < p_0 \end{cases}$$

reject $H_0$ if $z \leq -z_\alpha$.

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p \neq p_0 \end{cases}$$

reject $H_0$ if $|z| \geq z_{\alpha/2}$.

# Large-sample test for $p$

**Setup:**

1. Let $X_1 = k_1, \cdots, X_n = k_n$ be a random sample of size $n$ from Bernoulli($p$).
2. Suppose $n > 9 \max\left(\frac{1-p_0}{p_0}, \frac{p_0}{1-p_0}\right)$.
3. Set $k = k_1 + \cdots + k_n$ and $z = \frac{k-np_0}{\sqrt{np_0(1-p_0)}}$.
4. The level of significance is $\alpha$.

**Test:**

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p > p_0 \end{cases}$$

reject $H_0$ if $z \geq z_\alpha$.

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p < p_0 \end{cases}$$

reject $H_0$ if $z \leq -z_\alpha$.

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p \neq p_0 \end{cases}$$

reject $H_0$ if $|z| \geq z_{\alpha/2}$.

# Large-sample test for $p$

**Setup:**

1. Let $X_1 = k_1, \cdots, X_n = k_n$ be a random sample of size $n$ from Bernoulli($p$).
2. Suppose $n > 9 \max\left(\frac{1-p_0}{p_0}, \frac{p_0}{1-p_0}\right)$.
3. Set $k = k_1 + \cdots + k_n$ and $z = \frac{k - np_0}{\sqrt{np_0(1-p_0)}}$.
4. The level of significance is $\alpha$.

**Test:**

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p > p_0 \end{cases}$$

reject $H_0$ if $z \geq z_\alpha$.

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p < p_0 \end{cases}$$

reject $H_0$ if $z \leq -z_\alpha$.

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p \neq p_0 \end{cases}$$

reject $H_0$ if $|z| \geq z_{\alpha/2}$.

# Large-sample test for $p$

**Setup:**

1. Let $X_1 = k_1, \cdots, X_n = k_n$ be a random sample of size $n$ from Bernoulli($p$).
2. Suppose $n > 9 \max\left(\frac{1-p_0}{p_0}, \frac{p_0}{1-p_0}\right)$.
3. Set $k = k_1 + \cdots + k_n$ and $z = \frac{k-np_0}{\sqrt{np_0(1-p_0)}}$.
4. The level of significance is $\alpha$.

**Test:**

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p > p_0 \end{cases}$$

reject $H_0$ if $z \geq z_\alpha$.

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p < p_0 \end{cases}$$

reject $H_0$ if $z \leq -z_\alpha$.

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p \neq p_0 \end{cases}$$

reject $H_0$ if $|z| \geq z_{\alpha/2}$.

# Small-sample test for $p$

**E.g.** $n = 19$, $p_0 = 0.85$, $\alpha = 0.10$. Find critical region for the two-sided test

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p \neq p_0 \end{cases}$$

Sol. $19 = n < 9 \times \max\left(\frac{0.85}{0.15}, \frac{0.15}{0.85}\right) = 51$, so small sample test.

By checking the table, the critical region is

$$C = \{k : k \leq 13 \quad \text{or} \quad k = 19\},$$

so that

$$\begin{aligned} \alpha &= \mathbb{P}(X \in C | H_0 \text{ is true}) \\ &= \mathbb{P}(X \leq 13 | p = 0.85) + \mathbb{P}(X = 19 | p = 0.85) \\ &= 0.099295 \approx 0.10. \end{aligned}$$

□

# Small-sample test for $p$

**E.g.** $n = 19$, $p_0 = 0.85$, $\alpha = 0.10$. Find critical region for the two-sided test

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p \neq p_0 \end{cases}$$

Sol. $19 = n < 9 \times \max\left(\frac{0.85}{0.15}, \frac{0.15}{0.85}\right) = 51$, so small sample test.

By checking the table, the critical region is

$$C = \{k : k \leq 13 \quad \text{or} \quad k = 19\},$$

so that

$$\begin{aligned} \alpha &= \mathbb{P}(X \in C | H_0 \text{ is true}) \\ &= \mathbb{P}(X \leq 13 | p = 0.85) + \mathbb{P}(X = 19 | p = 0.85) \\ &= 0.099295 \approx 0.10. \end{aligned}$$

☐

# Small-sample test for $p$

**E.g.** $n = 19$, $p_0 = 0.85$, $\alpha = 0.10$. Find critical region for the two-sided test

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p \neq p_0 \end{cases}$$

Sol. $19 = n < 9 \times \max\left(\frac{0.85}{0.15}, \frac{0.15}{0.85}\right) = 51$, so small sample test.

By checking the table, the critical region is

$$C = \{k : k \leq 13 \quad \text{or} \quad k = 19\},$$

so that

$$\begin{aligned} \alpha &= \mathbb{P}(X \in C | H_0 \text{ is true}) \\ &= \mathbb{P}(X \leq 13 | p = 0.85) + \mathbb{P}(X = 19 | p = 0.85) \\ &= 0.099295 \approx 0.10. \end{aligned}$$

$\square$

# Small-sample test for $p$

E.g. $n = 19$, $p_0 = 0.85$, $\alpha = 0.10$. Find critical region for the two-sided test

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p \neq p_0 \end{cases}$$

Sol. $19 = n < 9 \times \max\left(\frac{0.85}{0.15}, \frac{0.15}{0.85}\right) = 51$, so small sample test.

By checking the table, the critical region is

$$C = \{k : k \leq 13 \quad \text{or} \quad k = 19\},$$

so that

$$\begin{aligned} \alpha &= \mathbb{P}(X \in C | H_0 \text{ is true}) \\ &= \mathbb{P}(X \leq 13 | p = 0.85) + \mathbb{P}(X = 19 | p = 0.85) \\ &= 0.099295 \approx 0.10. \end{aligned}$$

□

# Small-sample test for $p$

**E.g.** $n = 19$, $p_0 = 0.85$, $\alpha = 0.10$. Find critical region for the two-sided test

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p \neq p_0 \end{cases}$$

Sol. $19 = n < 9 \times \max\left(\frac{0.85}{0.15}, \frac{0.15}{0.85}\right) = 51$, so small sample test.

By checking the table, the critical region is

$$C = \{k : k \leq 13 \quad \text{or} \quad k = 19\},$$

so that

$$\begin{aligned} \alpha &= \mathbb{P}(X \in C | H_0 \text{ is true}) \\ &= \mathbb{P}(X \leq 13 | p = 0.85) + \mathbb{P}(X = 19 | p = 0.85) \\ &= 0.099295 \approx 0.10. \end{aligned}$$

$\square$

# Small-sample test for $p$

**E.g.** $n = 19$, $p_0 = 0.85$, $\alpha = 0.10$. Find critical region for the two-sided test

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p \neq p_0 \end{cases}$$

Sol. $19 = n < 9 \times \max\left(\frac{0.85}{0.15}, \frac{0.15}{0.85}\right) = 51$, so small sample test.

By checking the table, the critical region is

$$C = \{k : k \leq 13 \quad \text{or} \quad k = 19\},$$

so that

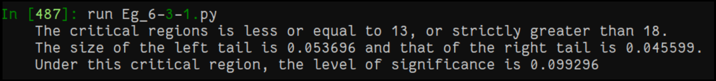
$$\begin{aligned} \alpha &= \mathbb{P}(X \in C | H_0 \ \text{is true}) \\ &= \mathbb{P}(X \leq 13 | p = 0.85) + \mathbb{P}(X = 19 | p = 0.85) \\ &= 0.099295 \approx 0.10. \end{aligned}$$

□

Binomial with n = 19 and p = 0.85

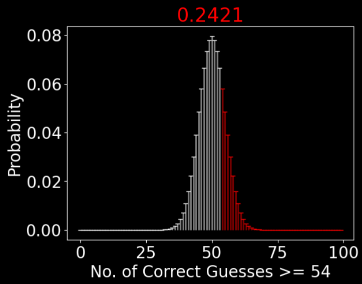
| x | P(X = x) | |
|---|---|---|
| 6 | 0.000000 | → $P(X \leq 13) = 0.053696$ |
| 7 | 0.000002 | |
| 8 | 0.000018 | |
| 9 | 0.000123 | |
| 10 | 0.000699 | |
| 11 | 0.003242 | |
| 12 | 0.012246 | |
| 13 | 0.037366 | |
| 14 | 0.090746 | |
| 15 | 0.171409 | |
| 16 | 0.242829 | |
| 17 | 0.242829 | |
| 18 | 0.152892 | |
| 19 | 0.045599 | → $P(X = 19) = 0.045599$ |

```python
# Eg_6-3-1.py
from scipy.stats import binom
n = 19
p = 0.85
rv = binom(n, p)
low = rv.ppf(0.05)
upper = rv.ppf(0.95)
left  = round(rv.cdf(low), 6)
right = round(1-rv.cdf(upper), 6)
both = round(rv.cdf(low)+1-rv.cdf(upper), 6)
Results = """\
    The critical regions is less or equal to {low:.0f}, or strictly greater than {upper:.0f}.
    The size of the tail is {left:.6f} and that of the right tail is {right:.6f}.
    Under this critical region, the level of significance is {both:.6f}
""".format(**locals())
print(Results)
```



```
In [487]: run Eg_6-3-1.py
   The critical regions is less or equal to 13, or strictly greater than 18.
   The size of the left tail is 0.053696 and that of the right tail is 0.045599.
   Under this critical region, the level of significance is 0.099296
```

$$X \sim \mathrm{Binomial}(100, 1/2)$$

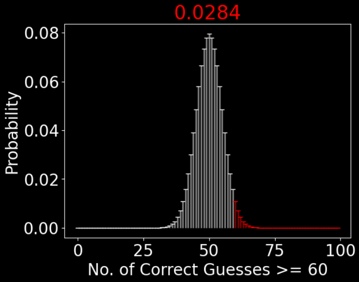


$$\mathbb{P}\left(X \geq 54\right) = \sum_{n=54}^{100} \binom{100}{n} \left(\frac{1}{2}\right)^{n} \left(\frac{1}{2}\right)^{100-n} = 0.2421.$$

vs

$$\mathbb{P}\left(\frac{X-50}{\sqrt{100 \times \frac{1}{2} \times \frac{1}{2}}} \geq \frac{54-50}{\sqrt{100 \times \frac{1}{2} \times \frac{1}{2}}}\right) \approx \mathbb{P}\left(Z \geq \frac{4}{5}\right) = 0.2119$$

$$X \sim \text{Binomial}(100, 1/2)$$



$$\mathbb{P}\left(X \geq 60\right) = \sum_{n=60}^{100} \binom{100}{n} \left(\frac{1}{2}\right)^{n} \left(\frac{1}{2}\right)^{100-n} = 0.0284.$$

vs

$$\mathbb{P}\left(\frac{X-50}{\sqrt{100 \times \frac{1}{2} \times \frac{1}{2}}} \geq \frac{60-50}{\sqrt{100 \times \frac{1}{2} \times \frac{1}{2}}}\right) \approx \mathbb{P}\left(Z \geq 2\right) = 0.0228$$