Math 362: Mathematical Statistics II

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Last updated on April 13, 2021

2021 Spring

Chapter 9. Two-Sample Inferences

§ 9.1 Introduction

§ 9.2 Testing $H_0 : \mu_X = \mu_Y$

§ 9.3 Testing $H_0 : \sigma_X^2 = \sigma_Y^2$

§ 9.4 Binomial Data: Testing $H_0 : p_X = p_Y$

§ 9.5 Confidence Intervals for the Two-Sample Problem

Plan

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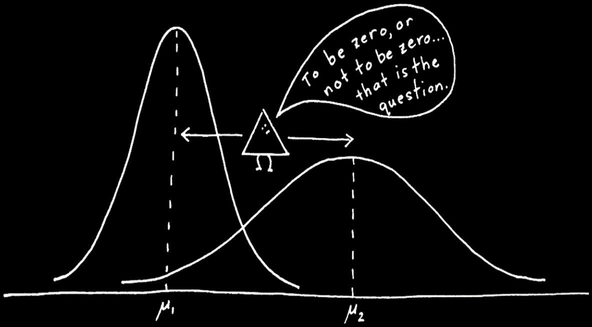


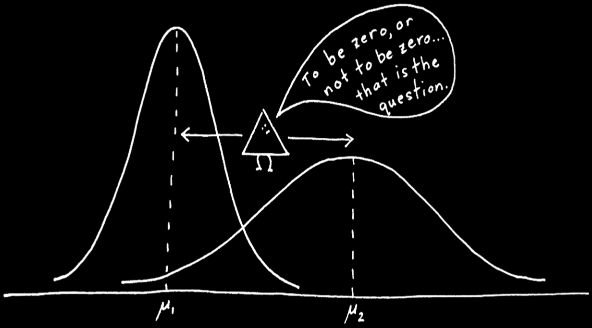
Multilevel designs:

1. Two methods applied to two independent sets of similar subjects.

   E.g., comparing two products.

2. Same method applied to two different kinds of subjects.

   E.g., comparing bones of European kids and American kids.



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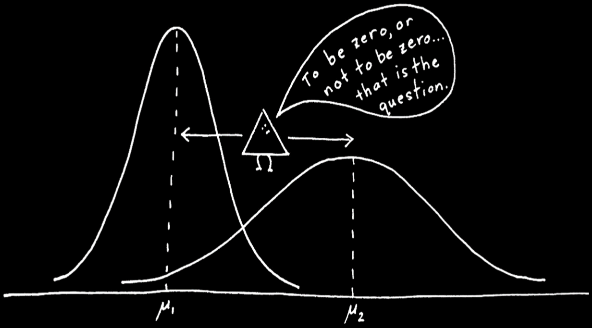

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Test for normal parameters (two sample test)

1. Let $X_1, \cdots, X_n$ be a random sample of size n from $N(\mu_X, \sigma_X^2)$.
2. Let $Y_1, \cdots, Y_m$ be a random sample of size m from $N(\mu_Y, \sigma_Y^2)$.

Prob. 1 Find a test statistic Λ in order to test $H_0 : \mu_X = \mu_Y$ v.s. $H_1 : \mu_X \neq \mu_Y$.

- 1-1 When σ_X^2 and σ_Y^2 are known
- 1-2 When $\sigma_X^2 = \sigma_Y^2$ is unknown
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Sol.

$$\frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} = \frac{\overline{X} - \overline{Y}}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \sim N(0,1)$$

Test statistics: $z = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}}$.

Critical region $|z| \geq z_{\alpha/2}$. □

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with $\sigma_X^2 = \sigma_Y^2 = \sigma^2$ but unknown.

Sol. Composite-vs-composite test with:

$$\omega = \{(\mu_X, \mu_Y, \sigma^2) : \mu_X = \mu_Y \in \mathbb{R}, \quad \sigma^2 > 0\}$$

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The likelihood function

$$L(\omega) = \prod_{i=1}^{n} f_X(x_i) \prod_{j=1}^{m} f_Y(y_j)$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^{m+n} \exp\left(-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i - \mu_X)^2 + \sum_{j=1}^{m}(y_j - \mu_Y)^2\right]\right)$$

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Under ω, the MLE $\omega_e = (\mu_{\omega_e}, \mu_{\omega_e}, \sigma^2_{\omega_e})$ is

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Hence,

$$L(\omega_e) = \left(\frac{e^{-1}}{2\pi\sigma^2_{\omega_e}}\right)^{\frac{n+m}{2}}$$

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$$\sigma^2_{\Omega_e} = \frac{\sum_{i=1}^{n}(x_i - \mu_{X_e})^2 + \sum_{j=1}^{m}(y_j - \mu_{Y_e})^2}{n+m}$$

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$$L(\Omega_e) = \left(\frac{e^{-1}}{2\pi\sigma^2_{\Omega_e}}\right)^{\frac{n+m}{2}}$$

$$\lambda = \frac{L(\omega_e)}{L(\Omega_e)} = \left(\frac{\sigma^2_{\Omega_e}}{\sigma^2_{\omega_e}}\right)^{\frac{m+n}{2}}$$

$$\lambda^{\frac{2}{n+m}} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{j=1}^{n}(y_j - \bar{y})^2}{\sum_{i=1}^{n}\left(x_i - \frac{n\bar{x}+m\bar{y}}{m+n}\right)^2 + \sum_{j=1}^{n}\left(y_j - \frac{n\bar{x}+m\bar{y}}{m+n}\right)^2}$$

$$\sum_{i=1}^{n}\left(x_i - \frac{n\bar{x} + m\bar{y}}{m+n}\right)^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2 + \frac{m^2 n}{(m+n)^2}(\bar{x} - \bar{y})^2$$

$$\sum_{j=1}^{m}\left(y_j - \frac{n\bar{x} + m\bar{y}}{m+n}\right)^2 = \sum_{j=1}^{m}(y_i - \bar{y})^2 + \frac{mn^2}{(m+n)^2}(\bar{x} - \bar{y})^2$$

$$\Downarrow$$

$$\sum_{i=1}^{n}\left(x_i - \frac{n\bar{x} + m\bar{y}}{m+n}\right)^2 + \sum_{j=1}^{n}\left(y_j - \frac{n\bar{x} + m\bar{y}}{m+n}\right)^2$$

$$\|$$

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{j=1}^{m}(y_i - \bar{y})^2 + \frac{mn}{m+n}(\bar{x} - \bar{y})^2$$

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$$||$$

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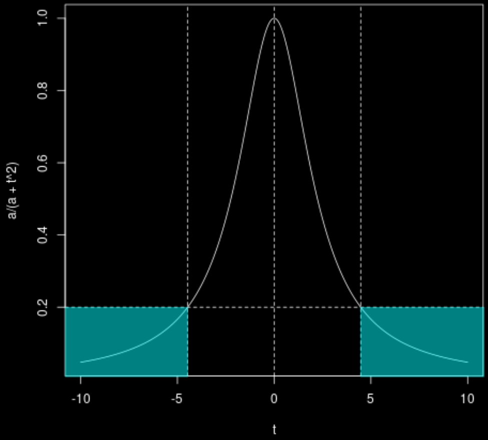
$$= \frac{1}{1 + \frac{(\bar{x} - \bar{y})^2}{\left[\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{j=1}^{m}(y_i - \bar{y})^2\right]\left(\frac{1}{m} + \frac{1}{n}\right)}}$$

$$= \frac{n + m - 2}{n + m - 2 + \frac{(\bar{x} - \bar{y})^2}{\frac{1}{n+m-2}\left[\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{j=1}^{m}(y_i - \bar{y})^2\right]\left(\frac{1}{m} + \frac{1}{n}\right)}}$$

$$= \frac{n + m - 2}{n + m - 2 + \frac{(\bar{x} - \bar{y})^2}{s_p^2\left(\frac{1}{m} + \frac{1}{n}\right)}} = \frac{n + m - 2}{n + m - 2 + t^2}.$$

$$\boxed{t := \frac{\bar{x} - \bar{y}}{s_p\sqrt{\frac{1}{m} + \frac{1}{n}}}}$$

$$t \mapsto \frac{a}{a + t^2}$$



One can use the following statistic

$$T = \frac{\overline{X} - \overline{Y}}{S_p\sqrt{\frac{1}{m} + \frac{1}{n}}}$$

where S_p^2 is called the *pooled sample variance*

$$S_p^2 = \frac{1}{n+m-2}\left[\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 + \sum_{i=1}^{m}\left(Y_j - \overline{Y}\right)^2\right]$$

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Three observations:

1. $\mathbb{E}[\overline{X} - \overline{Y}] = 0$ and

$$\mathrm{Var}(\overline{X} - \overline{Y}) = \mathrm{Var}(\overline{X}) + \mathrm{Var}(\overline{Y}) = \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m} = \sigma^2 \left(\frac{1}{n} + \frac{1}{m} \right)$$

   Hence, $\frac{\overline{X} - \overline{Y}}{\sigma\sqrt{\frac{1}{n} + \frac{1}{m}}} \sim N(0,1)$

2. $\frac{n+m-2}{\sigma^2} S_p^2 = \sum_{i=1}^{n} \left(\frac{X_i - \overline{X}}{\sigma} \right)^2 + \sum_{j=1}^{m} \left(\frac{Y_j - \overline{Y}}{\sigma} \right)^2 \sim$ Chi square$(n+m-2)$

3. $\frac{\overline{X} - \overline{Y}}{\sigma\sqrt{\frac{1}{n} + \frac{1}{m}}} \perp \frac{n+m-2}{\sigma^2} S_p^2$

$$\Longrightarrow T = \frac{\dfrac{\overline{X} - \overline{Y}}{\sigma\sqrt{\frac{1}{n} + \frac{1}{m}}}}{\sqrt{\dfrac{n+m-2}{\sigma^2} S_p^2 \times \dfrac{1}{n+m-2}}} = \frac{\overline{X} - \overline{Y}}{S_p\sqrt{\frac{1}{m} + \frac{1}{n}}} \sim \text{t distr.}(n+m-2)$$

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Hence, $\frac{\overline{X}-\overline{Y}}{\sigma\sqrt{\frac{1}{n}+\frac{1}{m}}} \sim N(0,1)$

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3. $\frac{\overline{X}-\overline{Y}}{\sigma\sqrt{\frac{1}{n}+\frac{1}{m}}} \perp \frac{n+m-2}{\sigma^2} S_p^2$

$$\implies T = \frac{\frac{\overline{X} - \overline{Y}}{\sigma\sqrt{\frac{1}{n} + \frac{1}{m}}}}{\sqrt{\frac{n+m-2}{\sigma^2} S_p^2 \times \frac{1}{n+m-2}}} = \frac{\overline{X}-\overline{Y}}{S_p\sqrt{\frac{1}{m}+\frac{1}{n}}} \sim \text{t distr.}(n+m-2)$$

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$$\mathrm{Var}(\overline{X} - \overline{Y}) = \mathrm{Var}(\overline{X}) + \mathrm{Var}(\overline{Y}) = \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m} = \sigma^2 \left(\frac{1}{n} + \frac{1}{m} \right)$$

Hence, $\frac{\overline{X}-\overline{Y}}{\sigma\sqrt{\frac{1}{n}+\frac{1}{m}}} \sim N(0,1)$

2. $\frac{n+m-2}{\sigma^2} S_p^2 = \sum_{i=1}^n \left(\frac{X_i - \overline{X}}{\sigma}\right)^2 + \sum_{j=1}^m \left(\frac{Y_j - \overline{Y}}{\sigma}\right)^2 \sim$ Chi square$(n+m-2)$

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Prob. 1-3 Find a test statistic for $H_0 : \mu_X = \mu_Y$ v.s. $H_1 : \mu_X \neq \mu_Y$, with $\sigma_X^2 \neq \sigma_Y^2$, both unknown.

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Assume that V follows Chi Square(ν) and assume that $V \perp U$.

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Proof.

$$\frac{V}{\nu}\left(\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}\right) = \frac{S_X^2}{n} + \frac{S_Y^2}{m}$$

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Second moments for Chi sqr(r) is $2r$. Hence, $\mathbb{E}(S_X^4) = \frac{\sigma_X^4}{n-1}$.

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$$\frac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2} \sim \text{F-disribution } (n-1, m-1)$$

Test statistic: $f = \frac{s_X^2/\sigma_X^2}{s_Y^2/\sigma_Y^2} = \frac{s_X^2}{s_Y^2}$

Critical regions: $f \leq F_{\alpha/2,n-1,m-1}$ or $f \geq F_{1-\alpha/2,n-1,m-1}$. □

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