Math 362: Mathematical Statistics II

Le Chen
le.chen@emory.edu

Emory University
Atlanta, GA

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Chapter 9. Two-Sample Inferences

§ 9.1 Introduction

§ 9.2 Testing $H_0 : \mu_X = \mu_Y$

§ 9.3 Testing $H_0 : \sigma_X^2 = \sigma_Y^2$

§ 9.4 Binomial Data: Testing $H_0 : p_X = p_Y$

§ 9.5 Confidence Intervals for the Two-Sample Problem

Plan

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- Let $Y_1, \cdots, Y_m$ be a random sample of size m from $N(\mu_Y, \sigma_Y^2)$.

Prob. 1 Testing $H_0 : \mu_X = \mu_Y$ if $\sigma_X^2 = \sigma_Y^2$.

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- True means: μ_X, μ_Y
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When $\sigma_X^2 = \sigma_Y^2 = \sigma^2$

Def. The **pooled variance**: $S_p^2 = \frac{(n-1)S_X^2 + (m-1)S_Y^2}{n+m-2}$

$$= \frac{\sum_{i=1}^n (X_i - \overline{X})^2 + \sum_{j=1}^n (Y_j - \overline{Y})^2}{n+m-2}$$

Thm. $T_{n+m-2} = \frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{S_p\sqrt{\frac{1}{n} + \frac{1}{m}}} \sim$ Student t distr. of $n+m-2$ dgs of fd.

Proof. (See slides on Section 9.1) □

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When $\sigma_X^2 = \sigma_Y^2 = \sigma^2$

Testing $H_0 : \mu_X = \mu_Y$ v.s.

(at the α level of significance)

$$t = \frac{\bar{x} - \bar{y}}{s_p \sqrt{\frac{1}{n} + \frac{1}{m}}}$$

$H_1 : \mu_X < \mu_Y$:	$H_1 : \mu_X \neq \mu_Y$:	$H_1 : \mu_X > \mu_Y$:
Reject H_0 if	Reject H_0 if	Reject H_0 if
$t \leq -t_{\alpha,n+m-2}$	$\lvert t\rvert \geq t_{\alpha/2,n+m-2}$	$t \geq t_{\alpha,n+m-2}$

E.g. Test whether Mark Twain and Snodgrass are the same person by checking the proportion of three-letter words at the 99% level of significance.

Sol. We need to test

$$H_0 : \mu_X = \mu_Y \quad v.s. \quad H_1 : \mu_X \neq \mu_Y.$$

Since we are tesing whether they are the same person, one can assume that $\sigma_X^2 = \sigma_Y^2$.

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Table 9.2.1 Proportion of Three-Letter Words

Twain	Proportion	QCS	Proportion
Sergeant Fathom letter	0.225	Letter I	0.209
Madame Caprell letter	0.262	Letter II	0.205
Mark Twain letters in		Letter III	0.196
Territorial Enterprise		Letter IV	0.210
First letter	0.217	Letter V	0.202
Second letter	0.240	Letter VI	0.207
Third letter	0.230	Letter VII	0.224
Fourth letter	0.229	Letter VIII	0.223
First *Innocents Abroad* letter		Letter IX	0.220
First half	0.235	Letter X	0.201
Second half	0.217		

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1. $n = 8,\ m = 10,$

$$\sum_{i=1}^{n} x_i = 1.855, \quad \sum_{i=1}^{n} x_i^2 = 0.4316$$

$$\sum_{i=1}^{m} y_i = 2.097, \quad \sum_{i=1}^{m} y_i^2 = 0.4406$$

2. Hence,

$$\bar{x} = 1.855/8 = 02319 \quad \bar{y} = 2.097/10 = 0.2097$$

$$s_X^2 = \frac{8 \times 0.4316 - 1.855^2}{8 \times 7} = 0.0002103$$

$$s_Y^2 = \frac{10 \times 0.4406 - 2.097^2}{10 \times 9} = 0.0000955$$

$$s_p^2 = \frac{(n-1)s_X^2 + (m-1)s_Y^2}{n+m-2} = ... = 0.0001457$$

$$t = \frac{\bar{x} - \bar{y}}{s_p\sqrt{\frac{1}{n} + \frac{1}{m}}} = ... = 3.88$$

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3. Critical region: $|t| \geq t_{0.005,n+m-2} = t_{0.005,16} = 2.9208$.

4. Conclusion: Rejection! $\square$

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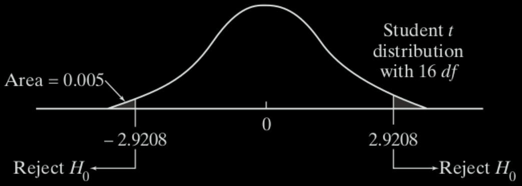

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Table 9.2.4

Large-Sales Companies	Return on Equity (%)	Small-Sales Companies	Return on Equity (%)
Deckers Outdoor	21	NVE	21
Jos. A. Bank Clothiers	23	Hi-Shear Technology	21
National Instruments	13	Bovie Medical	14
Dolby Laboratories	22	Rocky Mountain Chocolate Factory	31
Quest Software	7	Rochester Medical	19
Green Mountain Coffee Roasters	17	Anika Therapeutics	19
Lufkin Industries	19	Nathan's Famous	11
Red Hat	11	Somanetics	29
Matrix Service	2	Bolt Technology	20
DXP Enterprises	30	Energy Recovery	27
Franklin Electric	15	Transcend Services	27
LSB Industries	43	IEC Electronics	24

Sol. Let μ_X and μ_Y be the average returns. We are asked to test

$$H_0 : \mu_X = \mu_Y \qquad v.s. \qquad H_1 : \mu_X \neq \mu_Y.$$

1.

$$n = 12, \qquad \sum_{i=1}^{n} x_i = 223 \qquad \sum_{i=1}^{n} x_i^2 = 5421$$

$$m = 12, \qquad \sum_{i=1}^{m} y_i = 263 \qquad \sum_{i=1}^{m} y_i^2 = 6157$$

2.

$$\bar{x} = 18.5833, \qquad s_X^2 = 116.0833$$

$$\bar{y} = 21.9167, \qquad s_Y^2 = 35.7197$$

$$w = \frac{18.5833 - 21.9167}{\sqrt{\frac{116.0833}{12} + \frac{35.7197}{12}}} = -0.9371932.$$

$$\hat{\theta} = \frac{116.0833}{35.7179} = 3.250 \quad \Rightarrow \quad \nu = \left[\frac{(3.250+1)^2}{\frac{1}{11}3.250^2 + \frac{1}{11}1^2} \right] = [17.18403] = 17.$$

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