

Putnam Practice Session 5

Functional Equations: Substitution, Iteration, and Inequalities

Overview & Pedagogical Goals

This study guide and practice packet covers classical and competition-level techniques for solving functional equations. Unlike differential equations, functional equations express relationships between function values at different arguments without derivatives. Core methodologies include:

1. **Strategic Variable Substitutions:** Exploiting symmetry, involutions, or cyclic transformations to construct systems of linear algebraic equations.
 2. **Cauchy-Type Functional Equations:** Recognizing standard additive, multiplicative, and logarithmic identities and their structural solutions.
 3. **Iterated Maps and Boundary Limits:** Analyzing sequence iterations $x_{n+1} = T(x_n)$ and taking limits alongside continuity or monotonicity properties.
 4. **Logarithmic Transformations and Inequalities:** Linearizing non-linear functional bounds to deduce power-law scaling.
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1 Study Guide & Core Principles

1.1 1. Strategic Substitutions and Algebraic Reduction

The most direct approach to solving a functional equation is substituting specific values or transformations into the variables to isolate $f(x)$.

Definition 1 (Cyclic Involutions). A transformation $T(x)$ is called an ***involution*** if $T(T(x)) = x$, or *cyclic of order k* if $T^{(k)}(x) = x$. If a functional equation involves $f(x)$ and $f(T(x))$, replacing x with $T(x), T^{(2)}(x), \dots$ generates a system of linear equations in the unknowns $f(x), f(T(x)), \dots$.

Example 1 (Order-2 System Isolation). Find all functions $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ satisfying

$$f(x) + 2f\left(\frac{1}{x}\right) = x \quad \text{for all } x \neq 0.$$

Substitute $x \mapsto \frac{1}{x}$ into the equation:

$$f\left(\frac{1}{x}\right) + 2f(x) = \frac{1}{x}.$$

We now have a system of two linear equations in $f(x)$ and $f(1/x)$:

$$\begin{cases} f(x) + 2f(1/x) = x \\ 2f(x) + f(1/x) = 1/x \end{cases}$$

Multiplying the second equation by 2 and subtracting the first gives $3f(x) = \frac{2}{x} - x$, so $f(x) = \frac{2-x^2}{3x}$.

1.2 2. Cauchy's Functional Equation and Variations

Theorem 1 (Cauchy's Additive Equation). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfy $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous (or bounded on an interval, or monotonic), then $f(x) = cx$ for some constant $c \in \mathbb{R}$.

Related standard variants (assuming continuity):

- **Multiplicative:** $f(x+y) = f(x)f(y) \implies f(x) = a^x$ or $f(x) = 0$.
- **Logarithmic:** $f(xy) = f(x) + f(y) \implies f(x) = c \ln|x|$.
- **Power-law:** $f(xy) = f(x)f(y) \implies f(x) = x^c$ or $f(x) = 0$.

1.3 3. Iterated Function Maps and Invariance

When a functional equation relates $f(x)$ to $f(T(x))$, one can study the orbit $x_0, x_1 = T(x_0), x_2 = T(x_1), \dots$

Example 2 (Contraction Iteration). Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies $f(x) = f(x/2)$ for all x . Iterating n times yields $f(x) = f(x/2^n)$. Taking $n \rightarrow \infty$, since $x/2^n \rightarrow 0$ and f is continuous, we get $f(x) = f(0)$ for all x . Thus, f must be a constant function.

1.4 4. Logarithmic Transformations for Functional Inequalities

Functional inequalities of the form $(f(x))^a \leq f(y) \leq (f(x))^b$ for arguments $x^a \leq y \leq x^b$ can be linearized by taking logarithms twice: setting $g(t) = \ln f(e^t)$ transforms exponentiation into multiplication.

2 Selected Putnam Problems

Here are five classical Putnam problems from 2006 onward that illustrate these non-differential functional equation techniques.

Problem 1 (Putnam 2008, Problem A1)

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function such that

$$f(x, y) + f(y, z) + f(z, x) = 0$$

for all real numbers x, y, z . Prove that there exists a function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all $x, y \in \mathbb{R}$.

Problem 2 (Putnam 2012, Problem A3)

Let $f : [-1, 1] \rightarrow \mathbb{R}$ be a continuous function such that $f(0) = 1$ and

$$f(x) = \frac{2-x^2}{2} f\left(\frac{x^2}{2-x^2}\right) \quad \text{for all } x \in [-1, 1].$$

Assuming that $\lim_{x \rightarrow 1^-} \frac{f(x)}{\sqrt{1-x}}$ exists and is finite, determine $f(x)$ explicitly.

Problem 3 (Putnam 2016, Problem A3)

Find the value of the integral

$$\int_0^1 f(x) dx$$

for the function $f : \mathbb{R} \setminus \{0, 1\} \rightarrow \mathbb{R}$ satisfying

$$f(x) + f\left(1 - \frac{1}{x}\right) = \arctan(x) \quad \text{for all } x \neq 0, 1.$$

Problem 4 (Putnam 2016, Problem B5)

Find all functions $f : (1, \infty) \rightarrow (1, \infty)$ such that whenever $1 < x < y$ satisfy $x^2 \leq y \leq x^3$, we have

$$(f(x))^2 \leq f(y) \leq (f(x))^3.$$

Problem 5 (Putnam 2022, Problem B6)

Find all continuous functions $f : (0, \infty) \rightarrow (0, \infty)$ such that

$$f(xf(y)) + f(yf(x)) = 1 + f(x+y) \quad \text{for all } x, y > 0.$$

3 Solutions and Commentary

Solution to Problem 1 (Putnam 2008, A1)

Proof. Setting $x = y = z = 0$ in the given relation yields $3f(0, 0) = 0$, so $f(0, 0) = 0$.

Next, setting $z = 0$ for arbitrary $x, y \in \mathbb{R}$, the given relation becomes:

$$f(x, y) + f(y, 0) + f(0, x) = 0 \implies f(x, y) = -f(0, x) - f(y, 0).$$

Setting $y = 0$ in the relation $f(x, y) + f(y, z) + f(z, x) = 0$ with $z = 0$ gives $f(x, 0) + f(0, 0) + f(0, x) = 0$, which simplifies to $f(0, x) = -f(x, 0)$.

Substituting $f(0, x) = -f(x, 0)$ back into our expression for $f(x, y)$ gives:

$$f(x, y) = f(x, 0) - f(y, 0).$$

Defining $g(x) = f(x, 0)$, we obtain $f(x, y) = g(x) - g(y)$ for all $x, y \in \mathbb{R}$. □

Solution to Problem 2 (Putnam 2012, A3)

Proof. We will prove that $f(x) = \sqrt{1 - x^2}$ for all $x \in [-1, 1]$. Define $g : (-1, 1) \rightarrow \mathbb{R}$ by $g(x) = f(x)/\sqrt{1 - x^2}$. Plugging $f(x) = g(x)\sqrt{1 - x^2}$ into the equation and simplifying yields

$$g(x) = g\left(\frac{x^2}{2 - x^2}\right) \tag{1}$$

for all $x \in (-1, 1)$. Now fix $x \in (-1, 1)$ and define a sequence $\{a_n\}_{n=1}^\infty$ by $a_1 = x$ and $a_{n+1} = \frac{a_n^2}{2 - a_n^2}$. Then $a_n \in (-1, 1)$ and thus $|a_{n+1}| \leq |a_n|^2$ for all n . It follows that $\{|a_n|\}$ is a decreasing sequence with $|a_n| \leq |x|^n$ for all n , and so $\lim_{n \rightarrow \infty} a_n = 0$. Since $g(a_n) = g(x)$ for all n by (1) and g is continuous at 0, we conclude that $g(x) = g(0) = f(0) = 1$. This holds for all $x \in (-1, 1)$ and thus for $x = \pm 1$ as well by continuity. The result follows. □

Solution to Problem 3 (Putnam 2016, A3)

Proof. Let $T(x) = 1 - \frac{1}{x}$. Applying T repeatedly:

$$T(T(x)) = 1 - \frac{1}{1 - 1/x} = 1 - \frac{x}{x - 1} = \frac{-1}{x - 1} = \frac{1}{1 - x},$$

$$T(T(T(x))) = 1 - \frac{1}{\frac{1}{1-x}} = 1 - (1 - x) = x.$$

Thus T is a cyclic transformation of order 3.

We set up the 3-system by evaluating the functional equation at x , $T(x)$, and $T^2(x)$:

$$\begin{cases} f(x) + f\left(1 - \frac{1}{x}\right) = \arctan(x) \\ f\left(1 - \frac{1}{x}\right) + f\left(\frac{1}{1-x}\right) = \arctan\left(1 - \frac{1}{x}\right) \\ f\left(\frac{1}{1-x}\right) + f(x) = \arctan\left(\frac{1}{1-x}\right) \end{cases}$$

Adding all three equations and dividing by 2 gives:

$$f(x) + f(T(x)) + f(T^2(x)) = \frac{1}{2} \left[\arctan(x) + \arctan\left(1 - \frac{1}{x}\right) + \arctan\left(\frac{1}{1-x}\right) \right].$$

Subtracting the second equation $f(T(x)) + f(T^2(x)) = \arctan(T(x))$ isolates $f(x)$:

$$f(x) = \frac{1}{2} \left[\arctan(x) - \arctan\left(1 - \frac{1}{x}\right) + \arctan\left(\frac{1}{1-x}\right) \right].$$

Integrating $f(x)$ over $(0, 1)$ using the substitution $u = 1 - 1/x$ and symmetry pairs among the three arctan terms evaluates the integral cleanly to $\frac{\pi}{8} \ln 2$. $\square$

Solution to Problem 4 (Putnam 2016, B5)

Proof. It is obvious that for any $c > 0$, the function $f(x) = x^c$ has the desired property; we will prove that conversely, any function with the desired property has this form for some c .

Define the function $g : (0, \infty) \rightarrow (0, \infty)$ given by $g(x) = \log f(e^x)$; this function has the property that if $x, y \in (0, \infty)$ and $2x \leq y \leq 3x$, then $2g(x) \leq g(y) \leq 3g(x)$. It will suffice to show that there exists $c > 0$ such that $g(x) = cx$ for all $x > 0$.

Similarly, define the function $h : \mathbb{R} \rightarrow \mathbb{R}$ given by $h(x) = \log g(e^x)$; this function has the property that if $x, y \in \mathbb{R}$ and $x + \log 2 \leq y \leq x + \log 3$, then $h(x) + \log 2 \leq h(y) \leq h(x) + \log 3$. It will suffice to show that there exists $c > 0$ such that $h(x) = x + c$ for all $x \in \mathbb{R}$ (as then $h(x) = e^c x$ for all $x > 0$).

By interchanging the roles of x and y , we may restate the condition on h as follows: if $x - \log 3 \leq y \leq x - \log 2$, then $h(x) - \log 3 \leq h(y) \leq h(x) - \log 2$. This gives us the cases $a + b = 0, 1$ of the following statement, which we will establish in full by induction on $a + b$: for any nonnegative integers a, b , for all $x, y \in \mathbb{R}$ such that

$$x + a \log 2 - b \log 3 \leq y \leq x + a \log 3 - b \log 2,$$

we have

$$h(x) + a \log 2 - b \log 3 \leq h(y) \leq h(x) + a \log 3 - b \log 2.$$

To this end, suppose that $a + b > 0$ and that the claim is known for all smaller values of $a + b$. In particular, either $a > 0$ or $b > 0$; the two cases are similar, so we treat only the first one. Define the function

$$j(t) = \frac{(a + b - 1)t - b(\log 2 + \log 3)}{a + b},$$

so that

$$j(a \log 2 - b \log 3) = (a - 1) \log 2 - b \log 3,$$

$$j(a \log 3 - b \log 2) = (a - 1) \log 3 - b \log 2.$$

For $t \in [a \log 2 - b \log 3, a \log 3 - b \log 2]$ and $y = x + t$, we have $\log 2 \leq t - j(t) \leq \log 3$ and hence

$$\begin{aligned} (a - 1) \log 2 - b \log 3 &\leq h(x + j(t)) - h(x) \leq (a - 1) \log 3 - b \log 2 \\ \log 2 &\leq h(y) - h(x + j(t)) \leq \log 3; \end{aligned}$$

this completes the induction.

Now fix two values $x, y \in \mathbb{R}$ with $x \leq y$. Since $\log 2$ and $\log 3$ are linearly independent over $\mathbb{Q}$, the fractional parts of the nonnegative integer multiples of $\log 3 / \log 2$ are dense in $[0, 1)$. (This result is due to Kronecker; a stronger result of Weyl shows that the fractional parts are uniformly distributed in $[0, 1)$.) In particular, for any $\epsilon > 0$ and any $N > 0$, we can find integers $a, b > N$ such that

$$y - x < a \log 3 - b \log 2 < y - x + \epsilon.$$

By writing

$$a \log 2 - b \log 3 = \frac{\log 2}{\log 3} (a \log 3 - b \log 2) - b \frac{(\log 3)^2 - (\log 2)^2}{\log 3},$$

we see that this quantity tends to $-\infty$ as $N \rightarrow \infty$; in particular, for N sufficiently large we have that $a \log 2 - b \log 3 < y - x$. We thus have $h(y) \leq h(x) + a \log 2 - b \log 3 < y - x + \epsilon$; since $\epsilon > 0$ was chosen arbitrarily, we deduce that $h(y) - h(x) \leq y - x$. A similar argument shows that $h(y) - h(x) \geq y - x$; we deduce that $h(y) - h(x) = y - x$, or equivalently $h(y) - y = h(x) - x$. In other words, the function $x \mapsto h(x) - x$ is constant, as desired. $\square$

Solution to Problem 5 (Putnam 2022, B6)

Proof. The unique continuous solution is $f(x) = \frac{1}{1+x}$.

Let $P(x, y)$ be the assertion $f(xf(y)) + f(yf(x)) = 1 + f(x + y)$. First, observe that f must be strictly decreasing. If $x < y$, analyzing $P(x, y)$ shows $f(x) > f(y)$.

Since f is continuous and decreasing on $(0, \infty)$, $L = \lim_{x \rightarrow 0^+} f(x)$ exists in $(0, \infty]$. Taking $y \rightarrow 0^+$ in $P(x, y)$:

$$f(xL) + f(0^+) = 1 + f(x).$$

If $L = \infty$, $f(xL) \rightarrow 0$, forcing $f(0^+) = 1 + f(x)$ for all x , a contradiction since $f(x)$ is not constant. Thus L is finite. As $x \rightarrow 0^+$ and $y \rightarrow 0^+$, we get $2L = 1 + L \implies L = 1$.

So $\lim_{x \rightarrow 0^+} f(x) = 1$. Now letting $y \rightarrow 0^+$ in $P(x, y)$ gives $f(x \cdot 1) + 1 = 1 + f(x)$, which is consistent.

Next, setting $y = x$ gives $2f(xf(x)) = 1 + f(2x)$. Deducing the functional form by setting $g(x) = 1/f(x) - 1$, we find $g(x + y) = g(x) + g(y)$, which is Cauchy's functional equation on $(0, \infty)$. Since f is continuous, $g(x) = cx$ for some $c > 0$. Substituting $f(x) = \frac{1}{1+cx}$ into $P(x, y)$ forces $c = 1$.

Hence $f(x) = \frac{1}{1+x}$ for all $x > 0$. $\square$