The 84th William Lowell Putnam Mathematical Competition
Saturday, December 2, 2023

A1 For a positive integer n, let $f_n(x) = \cos(x)\cos(2x)\cos(3x)\cdots\cos(nx)$. Find the smallest n such that $|f_n''(0)| > 2023$.

A2 Let n be an even positive integer. Let p be a monic, real polynomial of degree $2n$; that is to say, $p(x) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1 x + a_0$ for some real coefficients $a_0, \dots, a_{2n-1}$. Suppose that $p(1/k) = k^2$ for all integers k such that $1 \leq |k| \leq n$. Find all other real numbers x for which $p(1/x) = x^2$.

A3 Determine the smallest positive real number r such that there exist differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ satisfying

(a) $f(0) > 0$,

(b) $g(0) = 0$,

(c) $|f'(x)| \leq |g(x)|$ for all x,

(d) $|g'(x)| \leq |f(x)|$ for all x, and

(e) $f(r) = 0$.

A4 Let $v_1, \dots, v_{12}$ be unit vectors in $\mathbb{R}^3$ from the origin to the vertices of a regular icosahedron. Show that for every vector $v \in \mathbb{R}^3$ and every $\varepsilon > 0$, there exist integers $a_1, \dots, a_{12}$ such that $\|a_1 v_1 + \cdots + a_{12} v_{12} - v\| < \varepsilon$.

A5 For a nonnegative integer k, let $f(k)$ be the number of ones in the base 3 representation of k. Find all complex numbers z such that

$$\sum_{k=0}^{3^{1010}-1} (-2)^{f(k)} (z+k)^{2023} = 0.$$

A6 Alice and Bob play a game in which they take turns choosing integers from 1 to n. Before any integers are chosen, Bob selects a goal of "odd" or "even". On the first turn, Alice chooses one of the n integers. On the second turn, Bob chooses one of the remaining integers. They continue alternately choosing one of the integers that has not yet been chosen, until the nth turn, which is forced and ends the game. Bob wins if the parity of $\{k\colon \text{the number } k \text{ was chosen on the } k\text{th turn}\}$ matches his goal. For which values of n does Bob have a winning strategy?

B1 Consider an m-by-n grid of unit squares, indexed by (i,j) with $1 \leq i \leq m$ and $1 \leq j \leq n$. There are $(m-1)(n-1)$ coins, which are initially placed in the squares (i,j) with $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$. If a coin occupies the square (i,j) with $i \leq m-1$ and $j \leq n-1$ and the squares $(i+1,j), (i,j+1)$, and $(i+1,j+1)$ are unoccupied, then a legal move is to slide the coin from (i,j) to $(i+1,j+1)$. How many distinct configurations of coins can be reached starting from the initial configuration by a (possibly empty) sequence of legal moves?

B2 For each positive integer n, let $k(n)$ be the number of ones in the binary representation of $2023 \cdot n$. What is the minimum value of $k(n)$?

B3 A sequence $y_1, y_2, \dots, y_k$ of real numbers is called *zigzag* if $k = 1$, or if $y_2 - y_1, y_3 - y_2, \dots, y_k - y_{k-1}$ are nonzero and alternate in sign. Let $X_1, X_2, \dots, X_n$ be chosen independently from the uniform distribution on $[0,1]$. Let $a(X_1, X_2, \dots, X_n)$ be the largest value of k for which there exists an increasing sequence of integers $i_1, i_2, \dots, i_k$ such that $X_{i_1}, X_{i_2}, \dots, X_{i_k}$ is zigzag. Find the expected value of $a(X_1, X_2, \dots, X_n)$ for $n \geq 2$.

B4 For a nonnegative integer n and a strictly increasing sequence of real numbers $t_0, t_1, \dots, t_n$, let $f(t)$ be the corresponding real-valued function defined for $t \geq t_0$ by the following properties:

(a) $f(t)$ is continuous for $t \geq t_0$, and is twice differentiable for all $t > t_0$ other than $t_1, \dots, t_n$;

(b) $f(t_0) = 1/2$;

(c) $\lim_{t \to t_k^+} f'(t) = 0$ for $0 \leq k \leq n$;

(d) For $0 \leq k \leq n-1$, we have $f''(t) = k+1$ when $t_k < t < t_{k+1}$, and $f''(t) = n+1$ when $t > t_n$.

Considering all choices of n and $t_0, t_1, \dots, t_n$ such that $t_k \geq t_{k-1} + 1$ for $1 \leq k \leq n$, what is the least possible value of T for which $f(t_0 + T) = 2023$?

B5 Determine which positive integers n have the following property: For all integers m that are relatively prime to n, there exists a permutation $\pi\colon \{1,2,\dots,n\} \to \{1,2,\dots,n\}$ such that $\pi(\pi(k)) \equiv mk \pmod{n}$ for all $k \in \{1,2,\dots,n\}$.

B6 Let n be a positive integer. For i and j in $\{1,2,\dots,n\}$, let $s(i,j)$ be the number of pairs (a,b) of nonnegative integers satisfying $ai + bj = n$. Let S be the n-by-n matrix whose (i,j) entry is $s(i,j)$. For example, when $n = 5$, we have $S = \begin{bmatrix} 6 & 3 & 2 & 2 & 2 \\ 3 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 & 2 \end{bmatrix}$. Compute the determinant of S.