Reals Session

Today we look at some accessible questions from recent Putnam exams, regarding real numbers and continuity. There is typically at least one such question each year. This may be my weakest area, but you may prefer it!

We'll take about half of our time just to look over the problems individually. Then we'll discuss any ideas you might have, and how to write them up. After that you might look at the hints (second page). We may not have time to go over the solutions, but you can read them on your own (pages 3–4).

I. PROBLEMS

Look over the problems below. Try to identify one or more problems where you have some idea of how to get started. For a real Putnam session, I recommend you spend at least half an hour just on this step!

2017 B3: Suppose that $f(x) = \sum_{i=0}^{\infty} c_i x^i$ is a power series for which each coefficient c_i is 0 or 1. Show that if $f(2/3) = 3/2$, then $f(1/2)$ must be irrational.

2016 B1: Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and $x_{n+1} = \ln(e^{x_n} - x_n)$ for $n \geq 0$. Show that $x_0 + x_1 + x_2 + \cdots$ converges, and find its sum.

2015 B1: Let f be a three times differentiable function (defined on $\mathbb{R}$ and real-valued) such that f has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros.

2014 B2: Suppose that f is a function on the interval $[1,3]$ such that $-1 \leq f(x) \leq 1$ for all x and $\int_1^3 f(x)\,dx = 0$. How large can $\int_1^3 \frac{f(x)}{x}\,dx$ be?

2013 B2: Let $C = \bigcup_{N=1}^{\infty} C_N$, where C_N denotes the set of those 'cosine polynomials' of the form

$$f(x) = 1 + \sum_{n=1}^{N} a_n \cos(2\pi n x)$$

for which:

(i) $f(x) \geq 0$ for all real x, and

(ii) $a_n = 0$ whenever n is a multiple of 3.

Determine the maximum value of $f(0)$ as f ranges through C, and prove that this maximum is attained.

2012 B1: Let S be a class of functions from $[0,\infty)$ to $[0,\infty)$ that satisfies:

(i) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x+1)$ are in S;

(ii) If $f(x)$ and $g(x)$ are in S, the functions $f(x) + g(x)$ and $f(g(x))$ are in S;

(iii) If $f(x)$ and $g(x)$ are in S and $f(x) \geq g(x)$ for all $x \geq 0$, then the function $f(x) - g(x)$ is in S.

Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

2011 A2: Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be sequences of positive real numbers such that $a_1 = b_1 = 1$ and $b_n = b_{n-1}a_n - 2$ for $n = 2, 3, \ldots$. Assume that the sequence (b_j) is bounded. Prove that

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 \ldots a_n}$$

converges, and evaluate S.

2011 B1: Let h and k be positive integers. Prove that for every $\varepsilon > 0$, there are positive integers m and n such that

$$\varepsilon < |h\sqrt{m} - k\sqrt{n}| < 2\varepsilon.$$

II. HINTS

You won't get hints on a real exam, but these ideas that may help you with similar problems. might help. Look these over, and see if you can make any further progress.

2017 B3: If a number is rational, then its binary expansion is eventually periodic (just like in decimal).

2016 B1: First show the sequence is positive and decreasing.

2015 B1: Apply Rolle's theorem, not to $f(x)$, but to $f(x)e^{cx}$ for some constant c.

2014 B2: Compare f to a step function with the same constraints.

2013 B2: Compare $f(0)$ and $f(1/3)$.

2012 B1: Put some other functions in S first.

2011 A2: Find an exact formula for the mth partial sum, using b's.

2011 B1: You can put a rational between any two reals.

The next page has solutions, don't continue until you want to see them!

III. SOLUTIONS

2017 B3: Suppose by way of contradiction that $f(1/2)$ is rational. Then $\sum_{i=0}^{\infty} c_i 2^{-i}$ is the binary expansion of a rational number, and hence must be eventually periodic; that is, there exist some integers m,n such that $c_i = c_{m+i}$ for all $i \geq n$. We may then write

$$f(x) = \sum_{i=0}^{n-1} c_i x^i + \frac{x^n}{1-x^m} \sum_{i=0}^{m-1} c_{n+i} x^i.$$

Evaluating at $x = 2/3$, we may equate $f(2/3) = 3/2$ with

$$\frac{1}{3^{n-1}} \sum_{i=0}^{n-1} c_i 2^i 3^{n-i} + \frac{2^n 3^m}{3^{n+m-1}(3^m - 2^m)} \sum_{i=0}^{m-1} c_{n+i} 2^i 3^{m-1-i};$$

since all terms on the right-hand side have odd denominator, the same must be true of the sum, a contradiction.

2016 B1: Note $e^x - x$ is strictly increasing for $x > 0$ (because its derivative $e^x - 1$ is positive), and its value at 0 is 1. By induction on n, we see that $x_n > 0$ for all n. Exponentiating, we see $x_n = e^{x_n} - e^{x_{n+1}}$. Since $x_n > 0$, we have $e^{x_n} > e^{x_{n+1}}$, so $x_n > x_{n+1}$. Since $\{x_n\}$ is decreasing and bounded below, it converges to a limit L. Taking limits in the equation yields $L = e^L - e^L$, so L is zero. Since $L = 0$, the sequence $\{e^{x_n}\}$ converges to 1.

We now have a telescoping sum: $x_0 + \cdots + x_n = (e^{x_0} - e^{x_1}) + \cdots + (e^{x_n} - e^{x_{n+1}}) = e^{x_0} - e^{x_{n+1}} = e - e^{x_{n+1}}$. Taking limits, we see that $x_0 + x_1 + \cdots$ converges to $e - 1$.

2015 B1: Let $g(x) = e^{x/2} f(x)$. Then g has at least 5 distinct real zeroes, and by repeated applications of Rolle's theorem, g', g'', g''' have at least $4,3,2$ distinct real zeroes, respectively. But

$$g'''(x) = \frac{1}{8} e^{x/2} (f(x) + 6f'(x) + 12f''(x) + 8f'''(x))$$

and $e^{x/2}$ is never zero, so we obtain the desired result.

2014 B2: Let $g(x)$ be 1 for $1 \leq x \leq 2$ and -1 for $2 < x \leq 3$, and define $h(x) = g(x) - f(x)$. Then $\int_1^3 h(x)\,dx = 0$ and $h(x) \geq 0$ for $1 \leq x \leq 2$, $h(x) \leq 0$ for $2 < x \leq 3$. Now

$$\begin{aligned} \int_1^3 \frac{h(x)}{x}\,dx &= \int_1^2 \frac{|h(x)|}{x}\,dx - \int_2^3 \frac{|h(x)|}{x}\,dx \\ &\geq \int_1^2 \frac{|h(x)|}{2}\,dx - \int_2^3 \frac{|h(x)|}{2}\,dx = 0, \end{aligned}$$

and thus $\int_1^3 \frac{f(x)}{x}\,dx \leq \int_1^3 \frac{g(x)}{x}\,dx = 2\log 2 - \log 3 = \log \frac{4}{3}$. Since $g(x)$ achieves the upper bound, the answer is $\log \frac{4}{3}$. (Some other solutions are available in the archive.)

2013 B2: We claim that the maximum value of $f(0)$ is 3. This is attained for $N = 2$, $a_1 = \frac{4}{3}$, $a_2 = \frac{2}{3}$: in this case $f(x) = 1 + \frac{4}{3}\cos(2\pi x) + \frac{2}{3}\cos(4\pi x) = 1 + \frac{4}{3}\cos(2\pi x) + \frac{2}{3}(2\cos^2(2\pi x) - 1) = \frac{1}{3}(2\cos(2\pi x) + 1)^2$ is always nonnegative.

Now suppose that $f = 1 + \sum_{n=1}^{N} a_n \cos(2\pi n x) \in C$. When n is an integer, $\cos(2\pi n/3)$ equals 0 if $3|n$ and $-1/2$ otherwise. Thus $a_n \cos(2\pi n/3) = -a_n/2$ for all n, and $f(1/3) = 1 - \sum_{n=1}^{N}(a_n/2)$. Since $f(1/3) \geq 0$, $\sum_{n=1}^{N} a_n \leq 2$, whence $f(0) = 1 + \sum_{n=1}^{N} a_n \leq 3$.

2012 B1: Each of the following functions belongs to S for the reasons indicated in th second column.

$f(x), g(x)$	given
$\ln(x+1)$	(i)
$\ln(f(x)+1), \ln(g(x)+1)$	(ii) plus two previous lines
$\ln(f(x)+1) + \ln(g(x)+1)$	(ii)
$e^x - 1$	(i)
$(f(x)+1)(g(x)+1) - 1$	(ii) plus two previous lines
$f(x)g(x) + f(x) + g(x)$	previous line
$f(x) + g(x)$	(ii) plus first line
$f(x)g(x)$	(iii) plus two previous lines

2011 A2: For $m \geq 1$, write

$$S_m = \frac{3}{2}\left(1 - \frac{b_1 \cdots b_m}{(b_1+2)\cdots(b_m+2)}\right).$$

Then $S_1 = 1 = 1/a_1$ and a quick calculation yields

$$S_m - S_{m-1} = \frac{b_1 \cdots b_{m-1}}{(b_2+2)\cdots(b_m+2)} = \frac{1}{a_1 \cdots a_m}$$

for $m \geq 2$, since $a_j = (b_j+2)/b_{j-1}$ for $j \geq 2$. It follows that $S_m = \sum_{n=1}^m 1/(a_1 \cdots a_n)$.

Now if (b_j) is bounded above by B, then $\frac{b_j}{b_j+2} \leq \frac{B}{B+2}$ for all j, and so $3/2 > S_m \geq 3/2(1-(\frac{B}{B+2})^m)$. Since $\frac{B}{B+2} < 1$, it follows that the sequence (S_m) converges to $S = 3/2$.

2011 B1: Since the rational numbers are dense in the reals, we can find positive integers a, b such that

$$\frac{3\varepsilon}{hk} < \frac{b}{a} < \frac{4\varepsilon}{hk}.$$

By multiplying a and b by a suitably large positive integer, we can also ensure that $3a^2 > b$. We then have

$$\frac{\varepsilon}{hk} < \frac{b}{3a} < \frac{b}{\sqrt{a^2+b}+a} = \sqrt{a^2+b} - a$$

and

$$\sqrt{a^2+b} - a = \frac{b}{\sqrt{a^2+b}+a} \leq \frac{b}{2a} < 2\frac{\varepsilon}{hk}.$$

We may then take $m = k^2(a^2+b), n = h^2a^2$.