Games Session

Below are some questions from recent Putnam exams regarding games, or (more generally) sequential processes. These problems typically involve combinatorics, and sometimes probability.

As usual, there are problems on the first page; you should think about them for a while before giving up or looking past the first page. Any partial progress is interesting, try to write down your ideas. There are hints on the second page, look at them if you like. Finally, there are solutions on the remaining pages. In a typical meeting we only have time to discuss our own ideas, but you can read the solutions later.

I. PROBLEMS

Look over the problems below. Try to identify one or more problems where you have some idea of how to start, For a real Putnam session, I recommend spending half an hour on this! If you make any progress, write it down.

2023 A6: Alice and Bob play a game in which they take turns choosing integers from 1 to n. Before any integers are chosen, Bob selects a goal of "odd" or "even". On the first turn, Alice chooses one of the n integers. On the second turn, Bob chooses one of the remaining integers. They continue alternately choosing one of the integers that has not yet been chosen, until the nth turn, which is forced and ends the game. Bob wins if the parity of $\{k\colon$ the number k was chosen on the kth turn$\}$ matches his goal. For which values of n does Bob have a winning strategy?

2023 B1: Consider an m-by-n grid of unit squares, indexed by (i,j) with $1 \le i \le m$ and $1 \le j \le n$. There are $(m-1)(n-1)$ coins, which are initially placed in the squares (i,j) with $1 \le i \le m-1$ and $1 \le j \le n-1$. If a coin occupies the square (i,j) with $i \le m-1$ and $j \le n-1$ and the squares $(i+1,j),(i,j+1)$, and $(i+1,j+1)$ are unoccupied, then a legal move is to slide the coin from (i,j) to $(i+1,j+1)$. How many distinct configurations of coins can be reached starting from the initial configuration by a (possibly empty) sequence of legal moves?

2016 B4: Let A be a $2n \times 2n$ matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability $1/2$. Find the expected value of $\det(A - A^t)$ (as a function of n), where A^t is the transpose of A.

2012 B3: A round-robin tournament of $2n$ teams lasted for $2n-1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

2011 B4: In a tournament, 2011 players meet 2011 times to play a multiplayer game. Every game is played by all 2011 players together and ends with each of the players either winning or losing. The standings are kept in two 2011×2011 matrices, $T = (T_{hk})$ and $W = (W_{hk})$. Initially, $T = W = 0$. After every game, for every (h,k) (including for $h = k$), if players h and k tied (that is, both won or both lost), the entry T_{hk} is increased by 1, while if player h won and player k lost, the entry W_{hk} is increased by 1 and W_{kh} is decreased by 1.

Prove that at the end of the tournament, $\det(T + iW)$ is a non-negative integer divisible by 2^{2010}.

II. HINTS

You don't get hints on a real exam, but these kinds of ideas may help with similar problems. Look these hints if you like, and see if you can make any further progress.

2023 A6: Even n is easier. Consider the numbers in pairs.

2023 B1: The complement looks like a path, and we can count paths.

2016 B4: A determinant is a sum over permutations, and a permutation is a product of orbits.

2012 B3: Construct a bipartite graph connecting teams and their "winning" days.

2011 B4: There is a complex matrix A so that $T + iW = \overline{A}^T A$.

The next page has solutions, don't continue until you want to see them!