Games Session

Below are some questions from recent Putnam exams regarding games, or (more generally) sequential processes. These problems typically involve combinatorics, and sometimes probability.

As usual, there are problems on the first page; you should think about them for a while before giving up or looking past the first page. Any partial progress is interesting, try to write down your ideas. There are hints on the second page, look at them if you like. Finally, there are solutions on the remaining pages. In a typical meeting we only have time to discuss our own ideas, but you can read the solutions later.

I. PROBLEMS

Look over the problems below. Try to identify one or more problems where you have some idea of how to start, For a real Putnam session, I recommend spending half an hour on this! If you make any progress, write it down.

2023 A6: Alice and Bob play a game in which they take turns choosing integers from 1 to n. Before any integers are chosen, Bob selects a goal of "odd" or "even". On the first turn, Alice chooses one of the n integers. On the second turn, Bob chooses one of the remaining integers. They continue alternately choosing one of the integers that has not yet been chosen, until the nth turn, which is forced and ends the game. Bob wins if the parity of $\{k\colon$ the number k was chosen on the kth turn$\}$ matches his goal. For which values of n does Bob have a winning strategy?

2023 B1: Consider an m-by-n grid of unit squares, indexed by (i,j) with $1 \leq i \leq m$ and $1 \leq j \leq n$. There are $(m-1)(n-1)$ coins, which are initially placed in the squares (i,j) with $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$. If a coin occupies the square (i,j) with $i \leq m-1$ and $j \leq n-1$ and the squares $(i+1,j),(i,j+1)$, and $(i+1,j+1)$ are unoccupied, then a legal move is to slide the coin from (i,j) to $(i+1,j+1)$. How many distinct configurations of coins can be reached starting from the initial configuration by a (possibly empty) sequence of legal moves?

2016 B4: Let A be a $2n \times 2n$ matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability $1/2$. Find the expected value of $\det(A-A^t)$ (as a function of n), where A^t is the transpose of A.

2012 B3: A round-robin tournament of $2n$ teams lasted for $2n-1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

2011 B4: In a tournament, 2011 players meet 2011 times to play a multiplayer game. Every game is played by all 2011 players together and ends with each of the players either winning or losing. The standings are kept in two 2011×2011 matrices, $T=(T_{hk})$ and $W=(W_{hk})$. Initially, $T=W=0$. After every game, for every (h,k) (including for $h=k$), if players h and k tied (that is, both won or both lost), the entry T_{hk} is increased by 1, while if player h won and player k lost, the entry W_{hk} is increased by 1 and W_{kh} is decreased by 1.

Prove that at the end of the tournament, $\det(T+iW)$ is a non-negative integer divisible by 2^{2010}.

II. HINTS

You don't get hints on a real exam, but these kinds of ideas may help with similar problems. Look these hints if you like, and see if you can make any further progress.

2023 A6: Even n is easier. Consider the numbers in pairs.

2023 B1: The complement looks like a path, and we can count paths.

2016 B4: A determinant is a sum over permutations, and a permutation is a product of orbits.

2012 B3: Construct a bipartite graph connecting teams and their "winning" days.

2011 B4: There is a complex matrix A so that $T + iW = \overline{A}^T A$.

The next page has solutions, don't continue until you want to see them!

III. SOLUTIONS

2023 A6: (Communicated by Kai Wang) For all n, Bob has a winning strategy. Note that we can interpret the game play as building a permutation of $\{1,\dots,n\}$, and the number of times an integer k is chosen on the k-th turn is exactly the number of fixed points of this permutation.

For n even, Bob selects the goal "even". Divide $\{1,\dots,n\}$ into the pairs $\{1,2\},\{3,4\},\dots$; each time Alice chooses an integer, Bob follows suit with the other integer in the same pair. For each pair $\{2k-1,2k\}$, we see that $2k-1$ is a fixed point if and only if $2k$ is, so the number of fixed points is even.

For n odd, Bob selects the goal "odd". On the first turn, if Alice chooses 1 or 2, then Bob chooses the other one to transpose into the strategy for $n-2$ (with no moves made). We may thus assume hereafter that Alice's first move is some $k>2$, which Bob counters with 2; at this point there is exactly one fixed point.

Thereafter, as long as Alice chooses j on the j-th turn (for $j \geq 3$ odd), either $j+1<k$, in which case Bob can choose $j+1$ to keep the number of fixed points odd; or $j+1=k$, in which case k is even and Bob can choose 1 to transpose into the strategy for $n-k$ (with no moves made).

Otherwise, at some odd turn j, Alice does not choose j. At this point, the number of fixed points is odd, and on each subsequent turn Bob can ensure that neither his own move nor Alice's next move does not create a fixed point: on any turn j for Bob, if $j+1$ is available Bob chooses it; otherwise, Bob has at least two choices available, so he can choose a value other than j.

2023 B1: The number of such configurations is $\binom{m+n-2}{m-1}$.

Initially the unoccupied squares form a path from $(1,n)$ to $(m,1)$ consisting of $m-1$ horizontal steps and $n-1$ vertical steps, and every move preserves this property. This yields an injective map from the set of reachable configurations to the set of paths of this form.

Since the number of such paths is evidently $\binom{m+n-2}{m-1}$ (as one can arrange the horizontal and vertical steps in any order), it will suffice to show that the map we just wrote down is also surjective; that is, that one can reach any path of this form by a sequence of moves.

This is easiest to see by working backwards. Ending at a given path, if this path is not the initial path, then it contains at least one sequence of squares of the form $(i,j)\to(i,j-1)\to(i+1,j-1)$. In this case the square $(i+1,j)$ must be occupied, so we can undo a move by replacing this sequence with $(i,j)\to(i+1,j)\to(i+1,j-1)$.

2016 B4: The expected value is $(2n)!/(4^n n!)$.

The determinant of $A-A^t$ is the sum over permutations σ of $\{1,\dots,2n\}$ of the product $\mathrm{sgn}(\sigma)\prod_{i=1}^{2n}(A-A^t)_{i\sigma(i)} = \mathrm{sgn}(\sigma)\prod_{i=1}^{2n}(A_{i\sigma(i)}-A_{\sigma(i)i})$. The expected value of the determinant is the sum over σ of the expected value of this product, which we denote E_σ.

Note that if we partition $\{1,\dots,2n\}$ into orbits for the action of σ, then partition the factors of the product accordingly, then no entry of A appears in more than one of these factors; consequently, these factors are independent random variables. This means that we can compute E_σ as the product of the expected values of the individual factors.

Any orbit of size 1 gives rise to the zero product, and hence the expected value of the corresponding factor is zero. For an orbit of size $m \geq 3$, the corresponding factor contains $2m$ distinct matrix entries, so again we may compute the expected value of the factor as the product of the expected values of the individual terms $A_{i\sigma(i)}-A_{\sigma(i)i}$. However, the distribution of this term is symmetric about 0, so its expected value is 0.

We conclude that $E_\sigma = 0$ unless σ acts with n orbits of size 2. To compute E_σ in this case, assume without loss of generality that the orbits of σ are $\{1,2\},\dots,\{2n-1,2n\}$; note that $\mathrm{sgn}(\sigma)=(-1)^n$. Then E_σ is the expected value of $\prod_{i=1}^n -(A_{(2i-1)2i}-A_{2i(2i-1)})^2$, which is $(-1)^n$ times the n-th power of the expected value of $(A_{12}-A_{21})^2$. Since $A_{12}-A_{21}$ takes the values $-1,0,1$ with probabilities $\frac{1}{4},\frac{1}{2},\frac{1}{4}$, its square takes the values $0,1$ with probabilities $\frac{1}{2},\frac{1}{2}$; we conclude that $E_\sigma = 2^{-n}$.

The permutations σ of this form correspond to unordered partitions of $\{1,\dots,2n\}$ into n sets of size 2, so there are $(2n)!/(n!(2!)^n)$ such permutations. Putting this together yields the result.

2012 B3: The answer is yes. We first note that for any collection of m days with $1 \leq m \leq 2n-1$, there are at least m distinct teams that won a game on at least one of those days. If not, then any of the teams that lost games on all of those days must in particular have lost to m other teams, a contradiction.

If we now construct a bipartite graph whose vertices are the $2n$ teams and the $2n-1$ days, with an edge linking a day to a team if that team won their game on that day, then any collection of m days is connected to a total of at least m teams.

It follows from Hall's Marriage Theorem that one can match the $2n-1$ days with $2n-1$ distinct teams that won on their respective days, as desired.

2011 B4: Number the games $1,\ldots,2011$, and let $A=(a_{jk})$ be the 2011×2011 matrix whose jk entry is 1 if player k wins game j and $i=\sqrt{-1}$ if player k loses game j. Then $\overline{a_{hj}}a_{jk}$ is 1 if players h and k tie in game j; i if player h wins and player k loses in game j; and $-i$ if h loses and k wins. It follows that $T+iW=\overline{A}^TA$.

Now the determinant of A is unchanged if we subtract the first row of A from each of the other rows, producing a matrix whose rows, besides the first one, are $(1-i)$ times a row of integers. Thus we can write $\det A=(1-i)^{2010}(a+bi)$ for some integers a,b. But then $\det(T+iW)=\det(\overline{A}^TA)=2^{2010}(a^2+b^2)$ is a nonnegative integer multiple of 2^{2010}, as desired.