Putnam Practice Session 8

Sequences and Series: Recurrences, Telescoping, and Asymptotics

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Overview & Pedagogical Goals

Sequences and series form a core pillar of competition mathematics, bridging discrete algebra and continuous analysis. Problem-solving in this domain includes these four foundational strategies (see Section 3.1 of *Putnam and Beyond*):

1. **Telescoping Collapses**: Expressing general terms as differences $a_n = b_n - b_{n+1}$ or quotients $p_k = u_{k+1}/u_k$ to evaluate infinite sums and products cleanly.
2. **Recurrence Linearization & Invariants**: Transforming non-linear sequence relations into linear recurrences or algebraic invariants using auxiliary substitutions (e.g., $a_k = x^k + x^{-k}$).
3. **Asymptotic Analysis & Differential Analogues**: Predicting the large-n behavior of recursive sequences $x_{n+1} - x_n = f(x_n)$ by modeling them with continuous differential equations $y' = f(y)$.
4. **The Stolz–Cesàro Theorem**: Evaluating limits of indeterminate ratios of sequences, serving as the discrete analogue of L'Hôpital's rule.

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1 Study Guide & Core Principles

1.1 1. Telescoping Sums and Products

The discrete analogue of the Fundamental Theorem of Calculus states that if $a_k = b_k - b_{k+1}$, then

$$\sum_{k=1}^{n} a_k = b_1 - b_{n+1}.$$

Similarly, for infinite products, if $p_k = \frac{u_{k+1}}{u_k}$, then $\prod_{k=1}^{n} p_k = \frac{u_{n+1}}{u_1}$.

Example 1 (Algebraic Telescoping Product)**.** *Evaluate* $\prod_{k=0}^{\infty}\left(1 + x^{2^k}\right)$ *for* $|x| < 1$*. Multiply by* $(1-x)$*:* $(1-x)\prod_{k=0}^{n}\left(1 + x^{2^k}\right) = 1 - x^{2^{n+1}}$*. Taking* $n \to \infty$ *gives*

$$\prod_{k=0}^{\infty}\left(1 + x^{2^k}\right) = \frac{1}{1-x}.$$

1.2 2. Linearizing Recurrences via Structural Invariants

Non-linear recurrences of the form $x_{n+1}x_{n-1} = x_n^2 - c$ often hide an underlying linear recurrence.

Theorem 1 (Determinantal and Quadratic Invariants)**.** *If a sequence (Q_n) satisfies $Q_nQ_{n-2} - Q_{n-1}^2 = -1$ with $Q_0 = 1, Q_1 = x$, then $Q_n(x)$ satisfies the linear recurrence*

$$Q_n(x) = xQ_{n-1}(x) - Q_{n-2}(x).$$

Justification & Derivation. The connection between non-linear quadratic relations and linear recurrences can be established in two complementary ways:

Method 1: Algebraic Ratio Conservation (Non-linear to Linear). Suppose $Q_nQ_{n-2} - Q_{n-1}^2 = c$ holds for all $n \geq 2$, where $c = -1$. Shifting indices by 1 gives $Q_{n+1}Q_{n-1} - Q_n^2 = c$. Equating the two expressions for c:

$$Q_{n+1}Q_{n-1} - Q_n^2 = Q_nQ_{n-2} - Q_{n-1}^2.$$

Rearranging terms to factor each side:

$$Q_{n+1}Q_{n-1} + Q_{n-1}^2 = Q_n^2 + Q_nQ_{n-2} \implies Q_{n-1}(Q_{n+1} + Q_{n-1}) = Q_n(Q_n + Q_{n-2}).$$

Dividing both sides by Q_nQ_{n-1} (assuming non-zero terms):

$$\frac{Q_{n+1} + Q_{n-1}}{Q_n} = \frac{Q_n + Q_{n-2}}{Q_{n-1}}.$$

This shows that the ratio $\frac{Q_n+Q_{n-2}}{Q_{n-1}}$ is a constant C independent of n. Thus, Q_n satisfies the linear recurrence $Q_n = CQ_{n-1} - Q_{n-2}$. Evaluating the ratio at $n = 2$ with $Q_0 = 1, Q_1 = x$, and $Q_2 = x^2 - 1$ gives $C = \frac{x^2-1+1}{x} = x$, yielding $Q_n(x) = xQ_{n-1}(x) - Q_{n-2}(x)$.

Method 2: Matrix Determinant Invariant (Linear to Quadratic). Conversely, starting from the linear recurrence $Q_n = xQ_{n-1} - Q_{n-2}$, define the matrix $M_n(x) = \begin{pmatrix} Q_{n-1}(x) & Q_n(x) \\ Q_{n-2}(x) & Q_{n-1}(x) \end{pmatrix}$. The recurrence relation translates directly to matrix multiplication:

$$\begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} Q_{n-1} & Q_n \\ Q_{n-2} & Q_{n-1} \end{pmatrix} = \begin{pmatrix} Q_n & Q_{n+1} \\ Q_{n-1} & Q_n \end{pmatrix} \implies T(x)M_n(x) = M_{n+1}(x).$$

Since $\det(T(x)) = \det \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix} = 1$, taking determinants gives $\det(M_{n+1}(x)) = \det(M_n(x))$. By induction, $\det(M_n(x)) = \det(M_1(x)) = Q_0^2 - Q_1Q_{-1} = 1$, which expands to $Q_{n-1}^2 - Q_nQ_{n-2} = 1 \iff Q_nQ_{n-2} - Q_{n-1}^2 = -1$. □

1.3 3. Continuous Differential Equation Analogues

To analyze the asymptotic growth of a recursive sequence $x_{n+1} - x_n = f(x_n)$ when $x_n \to 0$ or $x_n \to \infty$, replace the difference $\Delta x = x_{n+1} - x_n$ with the derivative $y'(t) = f(y(t))$.

Example 2 (Differential Heuristic)**.** *Consider $a_{n+1} = a_n + e^{-a_n}$. The continuous model is $\frac{dy}{dt} = e^{-y} \implies e^y dy = dt \implies e^y = t + C \implies y(t) = \ln(t + C)$. This predicts $a_n \sim \ln n$ as $n \to \infty$.*

1.4 4. The Stolz–Cesàro Theorem

Theorem 2 (Stolz–Cesàro Theorem)**.** *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be sequences of real numbers. If (b_n) is strictly positive, strictly increasing, and unbounded ($b_n \to \infty$), and if*

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = L,$$

then the limit $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and equals L.

Example 3 (Asymptotics of Sine Iteration)**.** *For $a_0 = \pi/2$ and $a_{n+1} = \sin(a_n)$, Taylor expansion gives $\sin(x) = x - \frac{x^3}{6} + O(x^5)$. Thus*

$$\frac{1}{a_{n+1}^2} - \frac{1}{a_n^2} = \frac{1}{\sin^2(a_n)} - \frac{1}{a_n^2} = \frac{1}{\left(a_n - \frac{a_n^3}{6} + O(a_n^5)\right)^2} - \frac{1}{a_n^2} = \frac{1}{3} + O(a_n^2).$$

By the Stolz–Cesàro theorem with $b_n = n$, $\lim_{n\to\infty} \frac{1/a_n^2}{n} = \frac{1}{3}$, so $a_n \sim \sqrt{\frac{3}{n}}$.

2 Selected Putnam Problems

Here are five classical Putnam problems on sequences and series, ordered by difficulty.

Problem 1 (Putnam 2016, Problem B1)

Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln(e^{x_n} - x_n).$$

Show that the infinite series $\sum_{n=0}^{\infty} x_n$ converges and find its sum.

Problem 2 (Putnam 2017, Problem A2)

Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

Problem 3 (Putnam 2014, Problem A3)

Let $a_0 = 5/2$ and $a_k = a_{k-1}^2 - 2$ for $k \geq 1$. Compute

$$\prod_{k=0}^{\infty} \left(1 - \frac{1}{a_k}\right)$$

in closed form.

Problem 4 (Putnam 2012, Problem B4)

Suppose that $a_0 = 1$ and that $a_{n+1} = a_n + e^{-a_n}$ for $n = 0, 1, 2, \ldots$. Does $a_n - \ln n$ have a finite limit as $n \to \infty$?

Problem 5 (Putnam 2020, Problem A3)

Let $a_0 = \pi/2$, and let $a_n = \sin(a_{n-1})$ for $n \geq 1$. Determine whether

$$\sum_{n=1}^{\infty} a_n^2$$

converges or diverges.

3 Solutions and Commentary

Solution to Problem 1 (Putnam 2016, B1)

Proof. Exponentiating the recurrence definition $x_{n+1} = \ln(e^{x_n} - x_n)$ gives

$$e^{x_{n+1}} = e^{x_n} - x_n \implies x_n = e^{x_n} - e^{x_{n+1}}.$$

We first analyze the behavior of (x_n):

1. Since $e^x - x > 1$ for all $x > 0$, and $x_0 = 1 > 0$, by induction $x_n > 0$ for all $n \geq 0$.
2. From $x_n = e^{x_n} - e^{x_{n+1}} > 0$, we have $e^{x_n} > e^{x_{n+1}}$, so $x_n > x_{n+1}$.
3. Being strictly decreasing and bounded below by 0, (x_n) converges to a limit $L \geq 0$.
4. Taking limits on both sides of $x_n = e^{x_n} - e^{x_{n+1}}$ yields $L = e^L - e^L = 0$, so $\lim_{n\to\infty} x_n = 0$.

Now consider the partial sum $S_N = \sum_{n=0}^{N} x_n$. Using our telescoping relation $x_n = e^{x_n} - e^{x_{n+1}}$:

$$S_N = \sum_{n=0}^{N} (e^{x_n} - e^{x_{n+1}}) = e^{x_0} - e^{x_{N+1}} = e^1 - e^{x_{N+1}}.$$

Taking $N \to \infty$, since $x_{N+1} \to 0$, we have $e^{x_{N+1}} \to e^0 = 1$. Therefore, the series converges and its sum is

$$\sum_{n=0}^{\infty} x_n = e - 1.$$

□

Solution to Problem 2 (Putnam 2017, A2)

Proof. Define a sequence of polynomials $P_n(x)$ recursively by $P_0(x) = 1$, $P_1(x) = x$, and

$$P_n(x) = xP_{n-1}(x) - P_{n-2}(x) \quad \text{for } n \geq 2.$$

Clearly, $P_n(x)$ is a polynomial with integer coefficients for all $n \geq 0$. We claim that $Q_n(x) = P_n(x)$ for all $n \geq 0$.

To prove this, it suffices to show that $P_n(x)$ satisfies the defining relation of $Q_n(x)$, namely

$$(P_{n-1}(x))^2 - P_n(x)P_{n-2}(x) = 1 \quad \text{for all } n \geq 2.$$

Consider the 2×2 matrix $M_n(x) = \begin{pmatrix} P_{n-1}(x) & P_n(x) \\ P_{n-2}(x) & P_{n-1}(x) \end{pmatrix}$ and $T(x) = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}$. Note that $\det(T(x)) = 1$. The recurrence relation $P_n(x) = xP_{n-1}(x) - P_{n-2}(x)$ translates to matrix multiplication:

$$T(x)M_n(x) = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P_{n-1}(x) & P_n(x) \\ P_{n-2}(x) & P_{n-1}(x) \end{pmatrix} = \begin{pmatrix} P_n(x) & P_{n+1}(x) \\ P_{n-1}(x) & P_n(x) \end{pmatrix} = M_{n+1}(x).$$

Taking determinants on both sides:

$$\det(M_{n+1}(x)) = \det(T(x)) \det(M_n(x)) = \det(M_n(x)).$$

By induction, $\det(M_n(x)) = \det(M_1(x)) = \det \begin{pmatrix} P_0(x) & P_1(x) \\ P_{-1}(x) & P_0(x) \end{pmatrix} = \det \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = 1$.

Thus, $(P_{n-1}(x))^2 - P_n(x)P_{n-2}(x) = 1$ holds for all $n \geq 2$. Rearranging gives $P_n(x) = \frac{(P_{n-1}(x))^2 - 1}{P_{n-2}(x)}$, matching the recurrence for $Q_n(x)$. Since $Q_0(x) = P_0(x) = 1$ and $Q_1(x) = P_1(x) = x$, $Q_n(x) = P_n(x)$ for all n, proving $Q_n(x)$ is always an integer-coefficient polynomial. □

Solution to Problem 3 (Putnam 2014, A3)

Proof. Notice that for any $x > 0$, $(x + x^{-1})^2 - 2 = x^2 + x^{-2}$. Since $a_0 = 5/2 = 2^1 + 2^{-1}$, proof by induction shows that

$$a_k = 2^{2^k} + 2^{-2^k} \quad \text{for all } k \geq 0.$$

We rewrite the terms in the product using the trigonometric/hyperbolic identity $x^2+1+x^{-2} = \frac{x^3-x^{-3}}{x-x^{-1}}$ or direct algebraic factorization:

$$1 - \frac{1}{a_k} = \frac{a_k - 1}{a_k} = \frac{2^{2^k} - 1 + 2^{-2^k}}{2^{2^k} + 2^{-2^k}} = \frac{2^{2^{k+1}} + 1 + 2^{-2^{k+1}}}{\left(2^{2^k} + 1 + 2^{-2^k}\right)\left(2^{2^k} - 2^{-2^k}\right) \cdot \frac{1}{2^{2^{k+1}} - 2^{-2^{k+1}}}}.$$

More simply, observe that $a_{k+1} + 1 = (a_k - 1)(a_k + 1)$. Therefore, $1 - \frac{1}{a_k} = \frac{a_k - 1}{a_k} = \frac{a_{k+1}+1}{a_k(a_k+1)}$. Multiplying the partial product from $k = 0$ to n:

$$\prod_{k=0}^{n} \left(1 - \frac{1}{a_k}\right) = \frac{2}{7} \cdot \frac{a_{n+1} + 1}{a_0 a_1 \cdots a_n}.$$

Using $a_0 a_1 \cdots a_n = \frac{2}{3}\sqrt{a_{n+1}^2 - 4}$, we substitute:

$$\prod_{k=0}^{n} \left(1 - \frac{1}{a_k}\right) = \frac{3}{7} \cdot \frac{a_{n+1} + 1}{\sqrt{a_{n+1}^2 - 4}}.$$

Since $a_{n+1} \to \infty$ as $n \to \infty$, taking the limit gives:

$$\lim_{n\to\infty} \frac{a_{n+1} + 1}{\sqrt{a_{n+1}^2 - 4}} = 1 \implies \prod_{k=0}^{\infty} \left(1 - \frac{1}{a_k}\right) = \frac{3}{7}.$$

□

Solution to Problem 4 (Putnam 2012, B4)

Proof. Yes, the limit exists (and is equal to 0).

Define $b_n = e^{a_n}$. The given recurrence $a_{n+1} = a_n + e^{-a_n}$ becomes

$$b_{n+1} = e^{a_{n+1}} = e^{a_n + e^{-a_n}} = e^{a_n} e^{e^{-a_n}} = b_n e^{1/b_n}.$$

Using Taylor expansion for $e^x = 1 + x + R(x)$ with $0 \leq R(x) \leq cx^2$:

$$b_{n+1} = b_n \left(1 + \frac{1}{b_n} + O\left(\frac{1}{b_n^2}\right)\right) = b_n + 1 + O\left(\frac{1}{b_n}\right).$$

Summing from $i = 0$ to $n - 1$:

$$b_n = n + b_0 + \sum_{i=0}^{n-1} O\left(\frac{1}{b_i}\right).$$

Since $b_i \geq i + b_0$, we have $b_n \geq n + e$. Furthermore,

$$0 \leq \frac{b_n}{n} - 1 \leq \frac{e}{n} + \frac{1}{n}\sum_{i=0}^{n-1} \frac{c}{i+e} \leq \frac{e}{n} + c\frac{\ln n}{n}.$$

Taking $n \to \infty$, $\frac{b_n}{n} \to 1$.

Taking natural logarithms gives $\ln(b_n/n) \to 0$, which means

$$\ln(b_n) - \ln n = a_n - \ln n \to 0.$$

Thus $a_n - \ln n$ has a finite limit as $n \to \infty$, namely 0. □

Solution to Problem 5 (Putnam 2020, A3)

Proof. The series $\sum_{n=1}^{\infty} a_n^2$ **diverges**.

Since $a_0 = \pi/2 \in (0, \pi/2)$, induction shows that $a_n \in (0,1)$ for all $n \geq 1$ and $a_{n+1} = \sin(a_n) < a_n$. Thus (a_n) is strictly decreasing and bounded below by 0, hence $a_n \to L$. Taking limits in $L = \sin(L)$ forces $L = 0$.

To determine the growth rate of a_n, examine $\frac{1}{a_{n+1}^2} - \frac{1}{a_n^2}$: Using Taylor series $\sin(x) = x - \frac{x^3}{6} + O(x^5)$:

$$\frac{1}{a_{n+1}^2} = \frac{1}{\sin^2(a_n)} = \frac{1}{\left(a_n - \frac{a_n^3}{6} + O(a_n^5)\right)^2} = \frac{1}{a_n^2\left(1 - \frac{a_n^2}{3} + O(a_n^4)\right)} = \frac{1}{a_n^2}\left(1 + \frac{a_n^2}{3} + O(a_n^4)\right).$$

Subtracting $\frac{1}{a_n^2}$ gives

$$\frac{1}{a_{n+1}^2} - \frac{1}{a_n^2} = \frac{1}{3} + O(a_n^2) \to \frac{1}{3} \quad \text{as } n \to \infty.$$

By the Stolz–Cesàro theorem,

$$\lim_{n\to\infty} \frac{1/a_n^2}{n} = \frac{1}{3} \implies a_n^2 \sim \frac{3}{n}.$$

By the Limit Comparison Test with the divergent harmonic series $\sum_{n=1}^{\infty} \frac{3}{n}$, the series $\sum_{n=1}^{\infty} a_n^2$ diverges. □