Putnam Practice Session 8

Sequences and Series: Recurrences, Telescoping, and Asymptotics

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Overview & Pedagogical Goals

Sequences and series form a core pillar of competition mathematics, bridging discrete algebra and continuous analysis. Problem-solving in this domain includes these four foundational strategies (see Section 3.1 of *Putnam and Beyond*):

1. **Telescoping Collapses**: Expressing general terms as differences $a_n = b_n - b_{n+1}$ or quotients $p_k = u_{k+1}/u_k$ to evaluate infinite sums and products cleanly.
2. **Recurrence Linearization & Invariants**: Transforming non-linear sequence relations into linear recurrences or algebraic invariants using auxiliary substitutions (e.g., $a_k = x^k + x^{-k}$).
3. **Asymptotic Analysis & Differential Analogues**: Predicting the large-n behavior of recursive sequences $x_{n+1} - x_n = f(x_n)$ by modeling them with continuous differential equations $y' = f(y)$.
4. **The Stolz–Cesàro Theorem**: Evaluating limits of indeterminate ratios of sequences, serving as the discrete analogue of L'Hôpital's rule.

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1 Study Guide & Core Principles

1.1 1. Telescoping Sums and Products

The discrete analogue of the Fundamental Theorem of Calculus states that if $a_k = b_k - b_{k+1}$, then

$$\sum_{k=1}^{n} a_k = b_1 - b_{n+1}.$$

Similarly, for infinite products, if $p_k = \frac{u_{k+1}}{u_k}$, then $\prod_{k=1}^{n} p_k = \frac{u_{n+1}}{u_1}$.

Example 1 (Algebraic Telescoping Product)**.** *Evaluate* $\prod_{k=0}^{\infty}\left(1 + x^{2^k}\right)$ *for* $|x| < 1$*. Multiply by* $(1-x)$*:* $(1-x)\prod_{k=0}^{n}\left(1 + x^{2^k}\right) = 1 - x^{2^{n+1}}$*. Taking* $n \to \infty$ *gives*

$$\prod_{k=0}^{\infty}\left(1 + x^{2^k}\right) = \frac{1}{1-x}.$$

1.2 2. Linearizing Recurrences via Structural Invariants

Non-linear recurrences of the form $x_{n+1}x_{n-1} = x_n^2 - c$ often hide an underlying linear recurrence.

Theorem 1 (Determinantal and Quadratic Invariants)**.** *If a sequence (Q_n) satisfies $Q_nQ_{n-2} - Q_{n-1}^2 = -1$ with $Q_0 = 1, Q_1 = x$, then $Q_n(x)$ satisfies the linear recurrence*

$$Q_n(x) = xQ_{n-1}(x) - Q_{n-2}(x).$$

Justification & Derivation. The connection between non-linear quadratic relations and linear recurrences can be established in two complementary ways:

Method 1: Algebraic Ratio Conservation (Non-linear to Linear). Suppose $Q_nQ_{n-2} - Q_{n-1}^2 = c$ holds for all $n \geq 2$, where $c = -1$. Shifting indices by 1 gives $Q_{n+1}Q_{n-1} - Q_n^2 = c$. Equating the two expressions for c:

$$Q_{n+1}Q_{n-1} - Q_n^2 = Q_nQ_{n-2} - Q_{n-1}^2.$$

Rearranging terms to factor each side:

$$Q_{n+1}Q_{n-1} + Q_{n-1}^2 = Q_n^2 + Q_nQ_{n-2} \implies Q_{n-1}(Q_{n+1} + Q_{n-1}) = Q_n(Q_n + Q_{n-2}).$$

Dividing both sides by Q_nQ_{n-1} (assuming non-zero terms):

$$\frac{Q_{n+1} + Q_{n-1}}{Q_n} = \frac{Q_n + Q_{n-2}}{Q_{n-1}}.$$

This shows that the ratio $\frac{Q_n+Q_{n-2}}{Q_{n-1}}$ is a constant C independent of n. Thus, Q_n satisfies the linear recurrence $Q_n = CQ_{n-1} - Q_{n-2}$. Evaluating the ratio at $n = 2$ with $Q_0 = 1, Q_1 = x$, and $Q_2 = x^2 - 1$ gives $C = \frac{x^2-1+1}{x} = x$, yielding $Q_n(x) = xQ_{n-1}(x) - Q_{n-2}(x)$.

Method 2: Matrix Determinant Invariant (Linear to Quadratic). Conversely, starting from the linear recurrence $Q_n = xQ_{n-1} - Q_{n-2}$, define the matrix $M_n(x) = \begin{pmatrix} Q_{n-1}(x) & Q_n(x) \\ Q_{n-2}(x) & Q_{n-1}(x) \end{pmatrix}$. The recurrence relation translates directly to matrix multiplication:

$$\begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} Q_{n-1} & Q_n \\ Q_{n-2} & Q_{n-1} \end{pmatrix} = \begin{pmatrix} Q_n & Q_{n+1} \\ Q_{n-1} & Q_n \end{pmatrix} \implies T(x)M_n(x) = M_{n+1}(x).$$

Since $\det(T(x)) = \det \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix} = 1$, taking determinants gives $\det(M_{n+1}(x)) = \det(M_n(x))$. By induction, $\det(M_n(x)) = \det(M_1(x)) = Q_0^2 - Q_1Q_{-1} = 1$, which expands to $Q_{n-1}^2 - Q_nQ_{n-2} = 1 \iff Q_nQ_{n-2} - Q_{n-1}^2 = -1$. □

1.3 3. Continuous Differential Equation Analogues

To analyze the asymptotic growth of a recursive sequence $x_{n+1} - x_n = f(x_n)$ when $x_n \to 0$ or $x_n \to \infty$, replace the difference $\Delta x = x_{n+1} - x_n$ with the derivative $y'(t) = f(y(t))$.

Example 2 (Differential Heuristic)**.** *Consider $a_{n+1} = a_n + e^{-a_n}$. The continuous model is $\frac{dy}{dt} = e^{-y} \implies e^y dy = dt \implies e^y = t + C \implies y(t) = \ln(t + C)$. This predicts $a_n \sim \ln n$ as $n \to \infty$.*

1.4 4. The Stolz–Cesàro Theorem

Theorem 2 (Stolz–Cesàro Theorem)**.** *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be sequences of real numbers. If (b_n) is strictly positive, strictly increasing, and unbounded ($b_n \to \infty$), and if*

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = L,$$

then the limit $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and equals L.

Example 3 (Asymptotics of Sine Iteration)**.** *For $a_0 = \pi/2$ and $a_{n+1} = \sin(a_n)$, Taylor expansion gives $\sin(x) = x - \frac{x^3}{6} + O(x^5)$. Thus*

$$\frac{1}{a_{n+1}^2} - \frac{1}{a_n^2} = \frac{1}{\sin^2(a_n)} - \frac{1}{a_n^2} = \frac{1}{\left(a_n - \frac{a_n^3}{6} + O(a_n^5)\right)^2} - \frac{1}{a_n^2} = \frac{1}{3} + O(a_n^2).$$

By the Stolz–Cesàro theorem with $b_n = n$, $\lim_{n\to\infty} \frac{1/a_n^2}{n} = \frac{1}{3}$, so $a_n \sim \sqrt{\frac{3}{n}}$.

2 Selected Putnam Problems

Here are five classical Putnam problems on sequences and series, ordered by difficulty.

Problem 1 (Putnam 2016, Problem B1)

Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln(e^{x_n} - x_n).$$

Show that the infinite series $\sum_{n=0}^{\infty} x_n$ converges and find its sum.

Problem 2 (Putnam 2017, Problem A2)

Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

Problem 3 (Putnam 2014, Problem A3)

Let $a_0 = 5/2$ and $a_k = a_{k-1}^2 - 2$ for $k \geq 1$. Compute

$$\prod_{k=0}^{\infty} \left(1 - \frac{1}{a_k}\right)$$

in closed form.

SHORT VERSION ENDS HERE!

The "short" version of this document ends here, I will distribute it by email. The "full" version includes two more problems and full solutions. I'll release it during (or soon after) our meeting.