Putnam Practice Session 9

Single-Variable and Multivariable Integration

Overview & Pedagogical Goals

Integration is an essential cornerstone of university-level competition mathematics, bridging single-variable calculus, multivariable geometry, and real analysis (corresponding roughly to Section 3.2 of *Putnam and Beyond*). Mastery of integral calculus on the Putnam exam requires a blend of clever algebraic transformations and analytical rigor. Core methodologies include:

1. **Symmetric Substitutions and Interval Invariance**: Exploiting functional symmetries $x \mapsto a+b-x$ or $x \mapsto \frac{1-x}{1+x}$ to collapse non-trivial integrands into manageable constant multiples.
2. **Multivariable Domain Symmetrization**: Swapping variables in double integrals over circular or square domains to simplify complex algebraic expressions before converting to polar or spherical coordinates.
3. **Integral Bounding and Extremal Configurations**: Establishing upper and lower bounds using derivative bounds, step-function approximations, and classical integral inequalities (Cauchy–Schwarz, Hölder, Jensen).
4. **Improper Integrals and Divergence Analysis**: Decomposing unbounded domains into unit intervals to bound series of integrals via telescoping or squeeze arguments.

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1 Study Guide & Core Principles

1.1 1. Symmetry and Interval Involutions

A central tool in single-variable definite integration is the reflection formula (sometimes called King's Property):

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$$

Adding the original integral to its reflected version often simplifies the integrand dramatically.

Example 1 (Symmetric Cancellation)**.** *Evaluate* $\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\,dx$.
Let $I = \int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\,dx$. *Applying* $x \mapsto \frac{\pi}{2} - x$ *yields*

$$I = \int_0^{\pi/2} \frac{\sin(\pi/2-x)}{\sin(\pi/2-x)+\cos(\pi/2-x)}\,dx = \int_0^{\pi/2} \frac{\cos x}{\cos x+\sin x}\,dx.$$

Adding these two expressions gives $2I = \int_0^{\pi/2} 1\,dx = \frac{\pi}{2}$, *so* $I = \frac{\pi}{4}$.

In algebraic contexts, involutions such as $x \mapsto \frac{1-x}{1+x}$ map the interval $[0,1]$ onto $[0,1]$, transforming rational or logarithmic integrands into solvable algebraic forms.

1.2 2. Multivariable Symmetrization

When evaluating double integrals $\iint_D f(x,y)\,dxdy$ over symmetric domains D (such as disks $x^2+y^2 \le R^2$ or squares $[0,1]^2$), interchanging x and y preserves the domain of integration. Averaging $f(x,y)$ and $f(y,x)$ yields:

$$\iint_D f(x,y)\,dxdy = \frac{1}{2}\iint_D \big(f(x,y)+f(y,x)\big)\,dxdy.$$

This technique frequently reveals hidden algebraic simplifications or allows transformations to polar coordinates (r,θ).

1.3 3. Bounding Integrals via Extremal Functions

Optimization problems involving integrals $\int_a^b f(x)g(x)\,dx$ with constraints $0 \le f(x) \le 1$ are often maximized by indicator functions $f(x) = \mathbf{1}_E(x)$ that concentrate weight where $g(x)$ is largest.

Theorem 1 (Integral Comparison Principle)**.** *Let $f, g : [a,b] \to \mathbb{R}$ be integrable functions. If g is strictly decreasing on $[a,b]$ and f satisfies $0 \le f(x) \le 1$ with $\int_a^b f(x)\,dx = c$, then $\int_a^b f(x)g(x)\,dx$ is maximized when $f(x) = 1$ on $[a, a+c]$ and $f(x) = 0$ on $(a+c, b]$.*

For derivative-constrained problems, the Fundamental Theorem of Calculus $f(x) = \int_0^x f'(t)\,dt$ enables point-wise bounds $|f(x)| \le Bx$ when $|f'(t)| \le B$.

1.4 4. Divergence of Improper Integrals

To prove that an improper integral $\int_0^\infty g(x)\,dx$ diverges when $g(x) > 0$:

- Partition $[0,\infty)$ into unit intervals $[k, k+1]$.
- Express the integral as an infinite series of integrals $\sum_{k=0}^{\infty} \int_k^{k+1} g(x)\,dx$.
- Apply monotonicity or mean value bounds on each interval to bound the terms below by a divergent series (such as $\sum \frac{1}{k}$).

2 Selected Putnam Problems

Here are five classical Putnam problems on integrals, ordered by difficulty.

Problem 1 (Putnam 2005, Problem A5)

Evaluate

$$\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx.$$

Problem 2 (Putnam 2014, Problem B2)

Let f be an integrable function defined on $[1,3]$ such that $|f(x)| \le 1$ for all $x \in [1,3]$ and $\int_1^3 f(x)\,dx = 0$. Show that

$$\int_1^3 \frac{f(x)}{x}\,dx \le \ln\left(\frac{4}{3}\right).$$

Problem 3 (Putnam 2007, Problem B2)

Suppose that $f : [0,1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)\,dx = 0$. Prove that for every $\alpha \in (0,1)$,

$$\left|\int_0^\alpha f(x)\,dx\right| \le \frac{1}{8}\max_{0\le x\le 1}|f'(x)|.$$

Problem 4 (Putnam 2021, Problem A4)

Let

$$I(R) = \iint_{x^2+y^2\le R^2} \left(\frac{1+2x^2}{1+x^4+6x^2y^2+y^4} - \frac{1+y^2}{2+x^4+y^4}\right) dx\,dy.$$

Find $\lim_{R\to\infty} I(R)$, or show that this limit does not exist.

Problem 5 (Putnam 2010, Problem A6)

Let $f : [0,\infty) \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that

$$\int_0^\infty \frac{f(x) - f(x+1)}{f(x)}\,dx$$

diverges.

3 Solutions and Commentary

Solution to Problem 1 (Putnam 2005, A5)

Proof. **First Solution (Trigonometric Substitution)**: Substitute $x = \tan\theta$, so $dx = \sec^2\theta\, d\theta$. The limits $x = 0$ and $x = 1$ correspond to $\theta = 0$ and $\theta = \pi/4$:

$$I = \int_0^{\pi/4} \frac{\ln(1+\tan\theta)}{\tan^2\theta + 1} \sec^2\theta\, d\theta = \int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta.$$

Using the identity $\tan(\pi/4 - \theta) = \frac{1-\tan\theta}{1+\tan\theta}$, we have:

$$1 + \tan\left(\frac{\pi}{4} - \theta\right) = 1 + \frac{1-\tan\theta}{1+\tan\theta} = \frac{2}{1+\tan\theta}.$$

Applying the substitution $\theta \mapsto \frac{\pi}{4} - \theta$:

$$I = \int_0^{\pi/4} \ln\left(1 + \tan\left(\frac{\pi}{4} - \theta\right)\right) d\theta = \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan\theta}\right) d\theta = \int_0^{\pi/4} \big(\ln 2 - \ln(1+\tan\theta)\big) d\theta.$$

Thus $I = \frac{\pi}{4}\ln 2 - I \implies 2I = \frac{\pi \ln 2}{4} \implies I = \frac{\pi \ln 2}{8}$.

Second Solution (Algebraic Involution): Substitute $x = \frac{1-u}{1+u}$, so $dx = \frac{-2}{(1+u)^2}\, du$. Then $1 + x = \frac{2}{1+u}$ and $x^2 + 1 = \frac{2(1+u^2)}{(1+u)^2}$. The integral transforms to:

$$I = \int_1^0 \frac{\ln\left(\frac{2}{1+u}\right)}{\frac{2(1+u^2)}{(1+u)^2}} \cdot \frac{-2}{(1+u)^2}\, du = \int_0^1 \frac{\ln 2 - \ln(1+u)}{1+u^2}\, du = \ln 2 \int_0^1 \frac{du}{1+u^2} - I.$$

Since $\int_0^1 \frac{du}{1+u^2} = \frac{\pi}{4}$, we obtain $I = \frac{\pi \ln 2}{4} - I \implies I = \frac{\pi \ln 2}{8}$. □

Solution to Problem 2 (Putnam 2014, B2)

Proof. Let $g(x) = 1$ for $x \in [1,2]$ and $g(x) = -1$ for $x \in (2,3]$. Define $h(x) = g(x) - f(x)$. Since $\int_1^3 f(x)\, dx = 0$ and $\int_1^3 g(x)\, dx = \int_1^2 1\, dx - \int_2^3 1\, dx = 0$, we have:

$$\int_1^3 h(x)\, dx = \int_1^3 g(x)\, dx - \int_1^3 f(x)\, dx = 0.$$

Observe the sign of $h(x)$:

- On $[1,2]$, $g(x) = 1 \implies h(x) = 1 - f(x) \geq 0$ (since $|f(x)| \leq 1$).
- On $(2,3]$, $g(x) = -1 \implies h(x) = -1 - f(x) \leq 0$ (since $|f(x)| \leq 1$).

Now consider the difference in integrals:

$$\int_1^3 \frac{h(x)}{x}\, dx = \int_1^2 \frac{h(x)}{x}\, dx + \int_2^3 \frac{h(x)}{x}\, dx.$$

For $x \in [1,2]$, $\frac{1}{x} \geq \frac{1}{2}$, so $\frac{h(x)}{x} \geq \frac{h(x)}{2}$ because $h(x) \geq 0$. For $x \in (2,3]$, $\frac{1}{x} \leq \frac{1}{2}$, so $\frac{h(x)}{x} \geq \frac{h(x)}{2}$ because $h(x) \leq 0$. Summing these bounds gives:

$$\int_1^3 \frac{h(x)}{x}\, dx \geq \int_1^2 \frac{h(x)}{2}\, dx + \int_2^3 \frac{h(x)}{2}\, dx = \frac{1}{2}\int_1^3 h(x)\, dx = 0.$$

Therefore:

$$\int_1^3 \frac{f(x)}{x}\, dx \leq \int_1^3 \frac{g(x)}{x}\, dx = \int_1^2 \frac{dx}{x} - \int_2^3 \frac{dx}{x} = \ln 2 - (\ln 3 - \ln 2) = \ln\left(\frac{4}{3}\right).$$

Equality is achieved when $f(x) = g(x)$, which satisfies all constraints. □

Solution to Problem 3 (Putnam 2007, B2)

Proof. Let $B = \max_{0\le x\le 1} |f'(x)|$. Let $g(x) = \int_0^x f(y)\,dy$. By hypothesis, $g(0) = g(1) = 0$. The maximum value of $|g(x)|$ on $[0,1]$ occurs at a critical point $\alpha \in (0,1)$ where $g'(\alpha) = f(\alpha) = 0$.

Since $\int_0^\alpha f(x)\,dx = -\int_\alpha^1 f(x)\,dx$, by switching the orientation $x \mapsto 1-x$ or reversing the sign of f if necessary, we may assume without loss of generality that $\alpha \le 1/2$ and $\int_0^\alpha f(x)\,dx \ge 0$.

Since $f(\alpha) = 0$ and $f'(x) \ge -B$ for all x, integrating the derivative bound gives:

$$f(x) \le B(\alpha - x) \quad \text{for all } 0 \le x \le \alpha.$$

Integrating $f(x)$ over $[0,\alpha]$:

$$\int_0^\alpha f(x)\,dx \le \int_0^\alpha B(\alpha - x)\,dx = \left[-\frac{1}{2}B(\alpha-x)^2\right]_0^\alpha = \frac{\alpha^2}{2}B.$$

Since $\alpha \le 1/2$, we have $\frac{\alpha^2}{2}B \le \frac{(1/2)^2}{2}B = \frac{1}{8}B$. Taking the absolute value yields the desired result:

$$\left|\int_0^\alpha f(x)\,dx\right| \le \frac{1}{8}\max_{0\le x\le 1}|f'(x)|.$$

□

Solution to Problem 4 (Putnam 2021, A4)

Proof. Interchanging the variables x and y in $I(R)$ gives:

$$I(R) = \iint_{x^2+y^2\le R^2} \left(\frac{1+2y^2}{1+y^4+6x^2y^2+x^4} - \frac{1+x^2}{2+y^4+x^4}\right) dx\,dy.$$

Averaging the two expressions for $I(R)$ yields:

$$I(R) = \iint_{x^2+y^2\le R^2} \big(f(x,y) - g(x,y)\big)\,dxdy,$$

where $f(x,y) = \frac{1+x^2+y^2}{1+x^4+6x^2y^2+y^4}$ and $g(x,y) = \frac{1+(x^2+y^2)/2}{2+x^4+y^4}$.

Notice the key structural relation: $f(x,y) = 2g(x+y, x-y)$. Under the change of variables $u = x+y, v = x-y$, the region $x^2+y^2 \le R^2$ becomes $u^2+v^2 \le 2R^2$, and the Jacobian is $J = 1/2$. Thus:

$$\iint_{x^2+y^2\le R^2} f(x,y)\,dxdy = \iint_{u^2+v^2\le 2R^2} g(u,v)\,dudv.$$

Subtracting the $g(x,y)$ integral over $x^2+y^2 \le R^2$ leaves the integral of $g(x,y)$ over the annulus $R^2 \le x^2+y^2 \le 2R^2$:

$$I(R) = \iint_{R^2\le x^2+y^2\le 2R^2} g(x,y)\,dxdy.$$

Converting to polar coordinates $x = r\cos\theta, y = r\sin\theta$:

$$I(R) = \int_R^{\sqrt{2}R}\int_0^{2\pi} \frac{1+r^2/2}{2+r^4(1-\frac{1}{2}\sin^2 2\theta)}\,r\,d\theta\,dr.$$

Rescaling $r = Rt$:

$$I(R) = \int_1^{\sqrt{2}}\int_0^{2\pi} \frac{1+R^2t^2/2}{2+R^4t^4(1-\frac{1}{2}\sin^2 2\theta)}\,R^2t\,d\theta\,dt.$$

Taking the limit $R\to\infty$ inside the integral:

$$\lim_{R\to\infty} I(R) = \int_1^{\sqrt{2}} \frac{dt}{t}\int_0^{2\pi} \frac{1/2}{1-\frac{1}{2}\sin^2 2\theta}\,d\theta = \ln(\sqrt{2})\int_0^{2\pi}\frac{d\theta}{2-\sin^2 2\theta}.$$

Evaluating the inner integral yields $\int_0^{2\pi}\frac{d\theta}{2-\sin^2 2\theta} = \sqrt{2}\pi$.

Hence the limit exists and equals $\frac{\sqrt{2}}{2}\pi\ln 2$. □

Solution to Problem 5 (Putnam 2010, A6)

First Proof (Unit Interval Summation):

Proof. Rewrite the integrand as $1 - \frac{f(x+1)}{f(x)}$. For any non-negative integers $a < b$:

$$\int_a^b \frac{f(x) - f(x+1)}{f(x)}\,dx = \sum_{k=a}^{b-1} \int_0^1 \frac{f(x+k) - f(x+k+1)}{f(x+k)}\,dx.$$

Since f is strictly decreasing, $f(x+k) \le f(x+a)$ for $x \in [0,1]$. Thus:

$$\int_0^1 \frac{f(x+k) - f(x+k+1)}{f(x+k)}\,dx \ge \int_0^1 \frac{f(x+k) - f(x+k+1)}{f(x+a)}\,dx.$$

Summing over $k = a, \ldots, b-1$ telescopes the numerators:

$$\int_a^b \frac{f(x) - f(x+1)}{f(x)}\,dx \ge \int_0^1 \frac{f(x+a) - f(x+b)}{f(x+a)}\,dx.$$

Since $f(x) \to 0$ as $x \to \infty$, given any integer a, we can choose $b = l(a) > a$ sufficiently large such that $f(l(a)) < \frac{1}{2} f(a+1)$. Then for all $x \in [0,1]$:

$$\frac{f(x+a) - f(x+l(a))}{f(x+a)} = 1 - \frac{f(x+l(a))}{f(x+a)} \ge 1 - \frac{f(l(a))}{f(a+1)} > \frac{1}{2}.$$

Construct an increasing sequence of integers $a_0 = 0, a_{n+1} = l(a_n)$. Then:

$$\int_0^\infty \frac{f(x) - f(x+1)}{f(x)}\,dx = \sum_{n=0}^{\infty} \int_{a_n}^{a_{n+1}} \frac{f(x) - f(x+1)}{f(x)}\,dx > \sum_{n=0}^{\infty} \int_0^1 \frac{1}{2}\,dx = \sum_{n=0}^{\infty} \frac{1}{2} = \infty.$$

Hence the integral diverges. □

Second Proof (Contradiction via Squeeze Theorem):

Proof. If the integral converges, then $\lim_{a\to\infty} \int_a^\infty \frac{f(x)-f(x+1)}{f(x)}\,dx = 0$. For any $a \ge 0$, since f is strictly decreasing:

$$0 < 1 - \frac{f(a+1)}{f(a)} < \frac{1}{f(a)} \int_a^{a+1} f(x)\,dx = \frac{1}{f(a)} \int_a^\infty \big(f(x) - f(x+1)\big)\,dx \le \int_a^\infty \frac{f(x) - f(x+1)}{f(x)}\,dx.$$

As $a \to \infty$, the right side tends to 0, forcing $\lim_{a\to\infty} \frac{f(a+1)}{f(a)} = 1$. However, convergence of the integral also requires the integrand $\left(1 - \frac{f(x+1)}{f(x)}\right)$ to tend to 0, which contradicts the previous lower bound for large a. Thus the integral must diverge. □