Putnam Practice Session 9

Single-Variable and Multivariable Integration

Overview & Pedagogical Goals

Integration is an essential cornerstone of university-level competition mathematics, bridging single-variable calculus, multivariable geometry, and real analysis (corresponding roughly to Section 3.2 of *Putnam and Beyond*). Mastery of integral calculus on the Putnam exam requires a blend of clever algebraic transformations and analytical rigor. Core methodologies include:

1. **Symmetric Substitutions and Interval Invariance**: Exploiting functional symmetries $x \mapsto a+b-x$ or $x \mapsto \frac{1-x}{1+x}$ to collapse non-trivial integrands into manageable constant multiples.
2. **Multivariable Domain Symmetrization**: Swapping variables in double integrals over circular or square domains to simplify complex algebraic expressions before converting to polar or spherical coordinates.
3. **Integral Bounding and Extremal Configurations**: Establishing upper and lower bounds using derivative bounds, step-function approximations, and classical integral inequalities (Cauchy–Schwarz, Hölder, Jensen).
4. **Improper Integrals and Divergence Analysis**: Decomposing unbounded domains into unit intervals to bound series of integrals via telescoping or squeeze arguments.

---

1 Study Guide & Core Principles

1.1 1. Symmetry and Interval Involutions

A central tool in single-variable definite integration is the reflection formula (sometimes called King's Property):

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$$

Adding the original integral to its reflected version often simplifies the integrand dramatically.

Example 1 (Symmetric Cancellation)**.** *Evaluate* $\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\,dx$.
Let $I = \int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\,dx$. *Applying* $x \mapsto \frac{\pi}{2} - x$ *yields*

$$I = \int_0^{\pi/2} \frac{\sin(\pi/2-x)}{\sin(\pi/2-x)+\cos(\pi/2-x)}\,dx = \int_0^{\pi/2} \frac{\cos x}{\cos x+\sin x}\,dx.$$

Adding these two expressions gives $2I = \int_0^{\pi/2} 1\,dx = \frac{\pi}{2}$, *so* $I = \frac{\pi}{4}$.

In algebraic contexts, involutions such as $x \mapsto \frac{1-x}{1+x}$ map the interval $[0,1]$ onto $[0,1]$, transforming rational or logarithmic integrands into solvable algebraic forms.

1.2 2. Multivariable Symmetrization

When evaluating double integrals $\iint_D f(x,y)\,dxdy$ over symmetric domains D (such as disks $x^2 + y^2 \leq R^2$ or squares $[0,1]^2$), interchanging x and y preserves the domain of integration. Averaging $f(x,y)$ and $f(y,x)$ yields:

$$\iint_D f(x,y)\,dxdy = \frac{1}{2}\iint_D \big(f(x,y) + f(y,x)\big)\,dxdy.$$

This technique frequently reveals hidden algebraic simplifications or allows transformations to polar coordinates (r,θ).

1.3 3. Bounding Integrals via Extremal Functions

Optimization problems involving integrals $\int_a^b f(x)g(x)\,dx$ with constraints $0 \leq f(x) \leq 1$ are often maximized by indicator functions $f(x) = \mathbf{1}_E(x)$ that concentrate weight where $g(x)$ is largest.

Theorem 1 (Integral Comparison Principle)**.** *Let $f, g : [a,b] \to \mathbb{R}$ be integrable functions. If g is strictly decreasing on $[a,b]$ and f satisfies $0 \leq f(x) \leq 1$ with $\int_a^b f(x)\,dx = c$, then $\int_a^b f(x)g(x)\,dx$ is maximized when $f(x) = 1$ on $[a, a+c]$ and $f(x) = 0$ on $(a+c, b]$.*

For derivative-constrained problems, the Fundamental Theorem of Calculus $f(x) = \int_0^x f'(t)\,dt$ enables point-wise bounds $|f(x)| \leq Bx$ when $|f'(t)| \leq B$.

1.4 4. Divergence of Improper Integrals

To prove that an improper integral $\int_0^\infty g(x)\,dx$ diverges when $g(x) > 0$:

- Partition $[0,\infty)$ into unit intervals $[k, k+1]$.
- Express the integral as an infinite series of integrals $\sum_{k=0}^\infty \int_k^{k+1} g(x)\,dx$.
- Apply monotonicity or mean value bounds on each interval to bound the terms below by a divergent series (such as $\sum \frac{1}{k}$).

2 Selected Putnam Problems

Here are five classical Putnam problems on integrals, ordered by difficulty.

Problem 1 (Putnam 2005, Problem A5)

Evaluate

$$\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx.$$

Problem 2 (Putnam 2014, Problem B2)

Let f be an integrable function defined on $[1,3]$ such that $|f(x)| \leq 1$ for all $x \in [1,3]$ and $\int_1^3 f(x)\,dx = 0$. Show that

$$\int_1^3 \frac{f(x)}{x}\,dx \leq \ln\left(\frac{4}{3}\right).$$

Problem 3 (Putnam 2007, Problem B2)

Suppose that $f : [0,1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)\,dx = 0$. Prove that for every $\alpha \in (0,1)$,

$$\left|\int_0^\alpha f(x)\,dx\right| \leq \frac{1}{8} \max_{0\leq x\leq 1} |f'(x)|.$$

SHORT VERSION ENDS HERE!

The "short" version of this document ends here, I will distribute it by email. The "full" version includes two more problems and full solutions. I'll release it during (or soon after) our meeting.