Putnam Practice Session 10: Addendum

Additional Practice Problems: Matrix Rank, Orthogonality, and Games

This addendum supplements the Session 10 handout on linear algebra and matrix rank. Problems 6 and 7 below explore rank bounds via row orthogonality and strategic rank deficiency in finite games.

Problem 6 (Putnam 2005, Problem A4)

Let H be an $n \times n$ matrix all of whose entries are ± 1 and whose rows are mutually orthogonal. Suppose H has an $a \times b$ submatrix whose entries are all equal to 1. Prove that

$$ab \leq n.$$

Focus & Strategy Hints:

- *Vector Norms & Dot Products*: Let $r_1, \ldots, r_a$ denote the a rows containing the $a \times b$ block of 1s. Consider the vector sum $v = r_1 + \cdots + r_a \in \mathbb{R}^n$. Compute the squared Euclidean length $\|v\|^2 = v \cdot v = \sum_{i=1}^{a} \sum_{j=1}^{a} r_i \cdot r_j$ using the given mutual row orthogonality ($r_i \cdot r_j = n\delta_{ij}$). Then bound $\|v\|^2$ from below by evaluating the first b coordinates of v.
- *Positive Semi-Definite Matrices*: Alternatively, isolate the submatrix M of size $a \times (n - b)$ consisting of the remaining entries in these a rows, and examine the eigenvalues and trace of the Gram matrix MM^T.

Problem 7 (Putnam 2002, Problem A4)

In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's.
Player 0 wins if the determinant is 0 and Player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

Focus & Strategy Hints:

- *Degeneracy Conditions*: Recall from Section 1 that a matrix is singular ($\det = 0$) if it contains an entire row of zeros, an entire column of zeros, or a 2×2 submatrix of zeros.
- *Strategic Blocking*: Can Player 0 guarantee the creation of at least one of these rank-deficient configurations? Use symmetry (for example, responding to Player 1's opening move by taking the center position A_{22}) to restrict Player 1's options.