41st VTRMC, 2019, Solutions

1. Let M denote the minimal value of $f(n)$. Clearly $M \leq 2+7+7+1 = 17$. We will show that $M = 17$, so assume by way of contradiction that $M < 17$. Choose $n \in \mathbb{N}$ with $f(n) = M$, and write n in reverse order as $1a_1 \dots a_d$ where $a_d \neq 0$ (so n is a $(d+1)$-digit number). We have $f(n) \equiv 2771^n \equiv (-1)^n \mod 9$. First assume that n is odd, so $f(n) \equiv -1 \mod 9$, so we must have $a_1 + \cdots + a_d = 7$. We also have $1 - a_1 + a_2 - \cdots \equiv 2771^n \equiv -1 \mod 11$, so $-a_1 + a_2 - a_3 + \cdots = -2$. Adding these two equations, we obtain $2a_2 + 2a_4 + \cdots = 5$, a contradiction because the left hand side is an even integer and the right hand side is an odd integer. Now assume that n is an even integer. Then we have $f(n) \equiv 1 \mod 9$ and therefore $a_1 + \cdots + a_d = 9$. Also $1 - a_1 + a_2 - \cdots \equiv 1 \mod 11$ and therefore $-a_1 + a_2 - a_3 + \cdots = 0$. Adding the last two equations, we obtain $2a_2 + 2a_4 + \cdots = 9$, again a contradiction and the result follows.

2. Since $BX/XA = 9$, we see that $AX = AB/10$ and we deduce that the area of AXC is 1/10 of the area of ABC, because they have the same height. Using the fact that the area of XYC is 9/100 of the area of ABC, we find that the area XYB is 81/100 of the area of ABC. Therefore the area of XBY is 9/10 of the area of XBC. Let H be the point on AB such that $\angle AHC = 90^\circ$. Since XBY and XBC have the same base, we see that $MY = (9/10)CH$. Now MBY and HBC are similar, consequently

$$HB = (10/9)MB = (10/9)\cdot(1/2)\cdot XB = (10/9)\cdot(1/2)\cdot(9/10)AB = (1/2)AB.$$

   Therefore $AC = BC$ and hence $BC = 20$.

3. Define $g(x) = \int_0^x (1-t)f(t)\,dt$. Then $g(0) = 0$ and

$$\begin{aligned} g(1) &= \int_0^1 f(x)\,dx - \int_0^1 xf(x)\,dx \\ &= \sum_{d=0}^{n} \frac{a_d}{d+1} - \sum_{d=0}^{n} \frac{a_d}{d+2} \\ &= \sum_{d=0}^{n} \frac{a_d}{(d+2)(d+1)} = 0. \end{aligned}$$

   By Rolle's theorem, there exists $q \in (0,1)$ such that $g'(q) = 0$, that is $(1-q)f(q) = 0$. Since $q \neq 1$, we deduce that $f(q) = 0$ as required.

4. Let $I = \int_0^1 \frac{x^2}{x+\sqrt{1-x^2}}\,dx$. We make the substitution $x = \sin t$. Then $dx = dt\cos t$ and we see that $I = \int_0^{\pi/2} \frac{\sin^2 t\cos t}{\sin t+\cos t}\,dt$. Also by making the substitution $x = \cos t$, we see that $I = \int_0^{\pi/2} \frac{\cos^2 t\sin t}{\sin t+\cos t}\,dt$ and we deduce that

$$2I = \int_0^{\pi/2} \frac{\sin^2 t\cos t + \cos^2 t\sin t}{\sin t + \cos t}\,dt.$$

Since $\sin^2 t\cos t + \cos^2 t\sin t = \sin t\cos t(\sin t + \cos t)$, we find that $2I = \int_0^{\pi/2} \sin t\cos t\,dt$. Therefore $4I = \int_0^{\pi/2} \sin 2t\,dt$ and we conclude that $I = 1/4$.

5. We make the substitution $t = 1/x$. Let y' and y'' denote the first and second derivatives of y with respect to t, respectively.Then $dy/dx = -t^2 y$ and $d^2y/dx^2 = 2t^3y' + t^4y''$ and by substituting back into the original equation, we obtain $y'' + (2t^{-1} - 2)y' + (1 - 2t^{-1})y = 0$. It is easy to see that $y = e^t$ is a solution to this equation. We now use reduction of order to obtain a second solution, so let $y = f(t)e^t$ be another solution, where f is to be determined. Then $f'' + 2t^{-1}f' = 0$, which has the solution $f = t^{-1}$. We deduce that $e^{1/x}$ and $xe^{1/x}$ are solutions to the original equation. Since these solutions are clearly linearly independent, it follows that the general solution to the original equation is $y = C_1e^{1/x} + C_2xe^{1/x}$, where C_1 and C_2 are arbitrary constants.

6. For each $s \in S$, there exist $m,n,p,q \in \mathbb{N}$ and $a,b \in \{\pm 1\}$ such that $s \in (am/n, bp/q)$ and $S \cap (am/n, bp/q) = \{s\}$. Then we may define $f(s) = 2^{a+1}3^{b+1}5^m7^n11^p13^q$.

7. For $d \in \mathbb{N}$, the number of d-digit integers in S is 9^d, because we have 9 choices for each digit, and all these integers are $\geq 10^{d-1}$. Therefore the series is bounded by

$$\sum_{d\in\mathbb{N}} 9^d \left(10^{d-1}\right)^{-99/100} = \sum_{d\in\mathbb{N}} 9^d 10^{-99(d-1)/100}.$$

This is a geometric series with ratio between successive terms $9 \cdot 10^{-99/100}$; we show that this ratio is < 1. Rearranging, we find that we need to prove $10^{99}/9^{99} > 9$, equivalently $(1 + 1/9)^{99} > 9$, which is true by the binomial series. It follows that the geometric series is convergent, and we conclude by the comparison test that the original series is convergent.